International Journal of Pure and Applied Mathematics
Volume 47 No. 3 2008, 343-351

LORENTZIAN METRIC INDUCED FROM A BACKGROUND RIEMANNIAN METRIC

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Abstract: In this paper we consider Lorentzian metric induced from a Riemannian metric by a unit vector field. First we show that the Levi-Cita connections of the two metrics are equal if and only if the vector field is parallel. Next we examine the conformal vector fields and Jacobi vector fields for those metrics. Finally, we assume that the vector field is the Reeb vector field of a K-contact manifold and show that if a vector field is conformal with respect to both metrics, then it is Killing.

AMS Subject Classification: 53B30, 83C15, 53D10
Key Words: Lorentzian metric, Riemannian metric, Levi-Cita connections, parallel vector field, Reeb vector field

1. Introduction

Let M denote a smooth manifold, and g denote a Riemannian metric on M. Suppose that U is a unit vector field on (M, g). As described in Hawking and Ellis [4] let us define a non-degenerate metric $\bar{g}$ by

Received: March 18, 2008

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$$\bar{g}(X,Y) = g(X,Y) - 2g(U,X)g(U,Y) \tag{1}$$

for arbitrary smooth vector fields X, Y on M. Then $\bar{g}(U,U) = -1$, and if Y is orthogonal to U with respect to g, then it is orthogonal to U also with respect to $\bar{g}$. Furthermore, if X, Y are orthogonal to U with respect to g, then $\bar{g}(X,Y) = g(X,Y)$. Thus an orthonormal basis for g is also an orthonormal basis for $\bar{g}$. Hence $\bar{g}$ is a Lorentzian metric. Let us denote the Levi-Civita connections of g and $\bar{g}$ by ∇ and $\bar{\nabla}$ respectively. If we set $S(X,Y) = \bar{\nabla}_X Y - \nabla_X Y$, then S is a tensor of type (1,3) and is symmetric in X, Y. Differentiating equation (1) along an arbitrary vector field Z provides

$$\begin{aligned} g(S(Z,X),Y) + g(X,S(Z,Y)) &= 2[g(U,S(Z,X))g(U,Y) \\ &+ g(U,S(Z,Y))g(U,X) - g(\nabla_Z U,X)g(U,Y) - g(\nabla_Z U,Y)g(U,X)]\,. \end{aligned} \tag{2}$$

Permuting X, Y, Z cyclically twice in the above equation we have

$$\begin{aligned} g(S(X,Y),Z) + g(Y,S(X,Z)) &= 2[g(U,S(X,Y))g(U,Z) \\ &+ g(U,S(X,Z))g(U,Y) - g(\nabla_X U,Y)g(U,Z) - g(\nabla_X U,Z)g(U,Y)] \end{aligned} \tag{3}$$

and

$$\begin{aligned} g(S(Y,Z),X) + g(Z,S(Y,X)) &= 2[g(U,S(Y,Z))g(U,X) \\ &+ g((U,S(Y,X))g(U,Z) - g(\nabla_Y U,Z)g(U,X) - g(\nabla_Y U,X)g(U,Z)]\,. \end{aligned} \tag{4}$$

Subtracting (4) from the sum of (2) and (3) we obtain

$$\begin{aligned} g(S(Z,X),Y) &= 2g(U,S(Z,X))g(U,Y) - g(U,Y)(\pounds_U g)(Z,X) \\ &+ 2(du)(Y,Z)g(U,X) - 2(du)(X,Y)g(U,Z)\,, \end{aligned} \tag{5}$$

where $\pounds$ and du stand for the Lie and exterior derivative operators respectively, and u is the 1-form such that $u(X) = g(U,X)$. Substituting U for Y in (5) yields

$$g(S(Z,X),U) = (\pounds_U g)(Z,X) - g(\nabla_U U,Z)g(U,X) - g(\nabla_U U,X)g(U,Z)). \tag{6}$$

The use of equation (6) in (5) shows the following explicit expression for S:

$$\begin{aligned} g(S(Z,X),Y) &= (\pounds_U g)(Z,X)g(U,Y) - 2g(U,X)g(U,Y)g(\nabla_U U,Z) \\ &- 2g(U,Z)g(U,Y)g(\nabla_U U,X) + 2(du)(Y,Z)g(U,X) \\ &- 2(du)(X,Y)g(U,Z)\,. \end{aligned} \tag{7}$$

2. When do g and $\bar{g}$ Determine Same Geodesics

The symmetric connections ∇ and $\bar{\nabla}$ determine the same system of geodesics (stationary paths or paths of free motion) if and only they are equal. Let us examine this condition for them. If $\bar{\nabla} = \nabla$, then equation (6) becomes

$$(\pounds_U g)(Z, X) = g(\nabla_U U, X) g(U, Z) + g(\nabla_U U, Z) g(U, X)$$

Substituting $Z = U$ and noting that U is unit, we get $\nabla_U U = 0$, and hence $\pounds_U g = 0$. Hence, equation (7) reduces to $(du)(Y, Z) g(U, X) = (du)(X, Y) g(U, Z)$. Substituting $Z = U$ in this provides $du = 0$. Consequently, $\nabla U = 0$. Conversely, it is easy to see that $\nabla U = 0$ implies $S = 0$, i.e. $\bar{\nabla} = \nabla$. We state this as the following result.

Theorem 1. *Let M be an n-dimensional smooth connected manifold with a Riemannian metric g and a unit vector field U. Let $\bar{g}$ be the Lorentzian metric defined by equation (1). Then the Levi-Civita connections ∇ and $\bar{\nabla}$ of g and $\bar{g}$ respectively, are equal if and only if U is parallel with respect to either g or $\bar{g}$.*

Remark. The condition that $\bar{\nabla} V = 0$ means that $(M, \bar{g})$ represents locally, a static decomposable spacetime: $ds^2 = -dt^2 + g_{ij} dx^i dx^j$, in terms of local coordinates $(x^0 = t, x^i)$, where g_{ij} is an $(n-1)$-dimensional Riemannian metric.

Secondly, we would like to seek a condition that the wave operator of $\bar{g}$ may become equal to the Laplace operator of g, when they act on a smooth function on M. We set $\bar{\Delta} = \bar{g}^{ab} \bar{\nabla}_a \bar{\nabla}_b$ and $\Delta = g^{ab} \nabla_a \nabla_b$ as those two operators in a local coordinate system (x^a) on M. A straightforward and lengthy calculation using equations (1) and (7) shows that $\operatorname{div} U = \overline{\operatorname{div}}\, U$, and $\bar{\Delta} f = \Delta f + 2 \operatorname{div}((U f) U)$ for an arbitrary smooth function f on M. Thus, $\bar{\Delta} f = \Delta f$ if and only if $\operatorname{div}((U f) U) = 0$. We state this consequence as the following result.

Proposition 1. *$\bar{\Delta} f = \Delta f$ for a smooth function f on M if and only if the vector field $(U f) U$ is divergence-free.*

3. Conformal and Jacobi Vector Fields

We first consider a conformal vector field V on M with respect to the induced Lorentzian metric $\bar{g}$, i.e.

$$\pounds_V \bar{g} = 2 \sigma \bar{g}\,, \tag{8}$$

where σ is the conformal scalar function on M. Taking the Lie-derivative of equation (1) and using (8) gives

$$\begin{aligned}(\pounds_V g)(X,Y) &= 2\sigma(g(X,Y) - 2g(U,X)g(U,Y)) + g(\pounds_V U, Y)g(U,X) \\ &+ (\pounds_V g)(U,Y)g(U,X) + g(\pounds_V U, X)g(U,Y)\,. \end{aligned} \tag{9}$$

We note that the Lie-derivative of $g(U,U) = 1$ along V yields $(\pounds_V g)(U,U) = -2g(\pounds_V U, U)$. Now substituting U for Y in (9) we have

$$\begin{aligned}(\pounds_V g)(X,U) &= 2[(\pounds_V g)(X,U) + g(\pounds_V U, X) \\ &- \sigma g(X,U) - 2g(X,U)g(\pounds_V U, U)]\,. \end{aligned} \tag{10}$$

Substituting U for X in (10) provides $(\pounds_V g)(U,U) = 2\sigma$ and so we get

$$g(\pounds_V U, U) = -\sigma.$$

Thus equation (10) reduces to $(\pounds_V g)(X,U) = -2g(\pounds_V U, X)$. Using this in (9) we obtain

$$\begin{aligned}(\pounds_V g)(X,Y) &= 2\sigma[g(X,Y) - 2g(U,X)g(U,Y)] \\ -2 &\ [g(\pounds_V U, X)g(U,Y) + g(\pounds_V U, Y)g(U,X)]\,. \end{aligned} \tag{11}$$

Hence $(\pounds_V g)(X,Y) = 0$ for any vector fields X, Y orthogonal to U. Contracting the foregoing equation shows $\mathrm{div}\,.V = \frac{m}{2}\sigma$, where m is the dimension of M. Also, contraction of (8) yields $\bar{\mathrm{div}}\,.V = \frac{m}{2}\sigma$.

At this point we seek a condition under which V would be conformal also with respect to g. Let V be conformal also with respect to g, i.e. $\pounds_V g = 2\tau g$ for some function τ on M. Then equation (11) assumes the form

$$\begin{aligned}(\sigma - \tau)g(X,Y) &= g(\pounds_V U, X)g(U,Y) \\ &+ g(\pounds_V U, Y)g(U,X) + 2\sigma g(U,X)g(U,Y)\,. \end{aligned} \tag{12}$$

Taking X, Y orthogonal to U in (12), we find that $\sigma = \tau$. Substituting U for Y in (12) we obtain

$$\pounds_V U = -\sigma U\,. \tag{13}$$

Conversely, if V is conformal with respect to $\bar{g}$ and equation (13) holds, then (11) shows immediately that V is conformal also with respect to g. Summing up, we state this as the following result.

Proposition 2. *Let V be a conformal vector field on M with respect to $\bar{g}$. Then V is conformal also with respect to g if and only if $\pounds_V U = -\sigma U$, i.e. V preserves U up to a scale factor.*

Remark 1. One may easily check that a conformal vector field with respect to g is also conformal with respect to $\bar{g}$ if and only if the equation (13) holds.

Remark 2. The condition (13) for a conformal vector field V with respect to $\bar{g}$ to be conformal also with respect to g is obviously equivalent to $\pounds_V u = \sigma u$. Let us view $(M, \bar{g})$ as a general relativistic space-time with 4-velocity vector U. In general, equation (13) need not hold, i.e., a conformal vector field need not map fluid flow lines (integral curves of U) to fluid flow lines, as shown by Maartens et al [6]. Coley and Tupper [2] defined an inhereting conformal vector field as a conformal vector field V if the fluid flow lines are mapped conformally by V, i.e. if equation (13) holds. Maartens et al [6] noted that an inheriting conformal vector field is an intrinsic conformal vector field of the spacelike hypersurfaces orthogonal to U with intrinsic metric $h = \bar{g} + u \otimes u$. A material curve in a fluid is a curve that consists at all times of the same fluid particles and therefore moves with the fluid as the fluid evolves. Equation (13) implies that the integral curves of the conformal vector field V are material curves. Israel [5] has shown that for a distribution of massless particles in equilibrium, the inverse temperature function vector $V = \frac{1}{T}U$ is an inheriting conformal vector field and therefore closely connected with the relativistic thermodynamics of fluids.

Second part of this section addresses Jacobi vector fields. Let U be a geodesic vector field on (M, g), i.e. $\nabla_U U = 0$, which is equivalent, in view of equation (7), to $\bar{\nabla}_U U = 0$, i.e. U is also geodesic with respect to $\bar{g}$. One may note that this condition holds good for synchronous space-times with metric: $-dt^2 + g_{ij} dx^i dx^j$, where $U = \frac{\partial}{\partial t}$ is the 4-velocity vector, g_{ij} depend on all of the synchronous (Gaussian normal) coordinates $(x^0 = t, x^i)$ (see Misner et al [7], and Coley and Tupper [2]). Now suppose that V is a Jacobi vector field along U, i.e.

$$\nabla_U \nabla_U V = R(U, V)U \,,$$

where R denotes the curvature tensor of g. The Jacobi equation can be cast, in view of the formula (see Duggal and Sharma [3]):

$$(\pounds_V \nabla)(X, Y) = \nabla_X \nabla_Y V - \nabla_{\nabla_X Y} V + R(V, X)Y$$

in the form: $(\pounds_V \nabla)(U, U) = 0$. In view of the identity:

$$\pounds_V \nabla_U U = (\pounds_V \nabla)(U, U) + \nabla_{[V,U]} U + \nabla_U [V, U]$$

the Jacobi equation assumes the form

$$\nabla_{[V,U]} U + \nabla_U [V, U] = 0 \,.$$

Using equation (7) we show, upon a straightforward computation, that the last equation is equivalent to:

$$\bar{\nabla}_{[V,U]} U + \bar{\nabla}_U [V, U] = 0$$

which, in turn, is equivalent to $(\pounds_V \bar{\nabla})(U, U) = 0$, i.e. V is Jacobi also with respect to $\bar{g}$ along U. Thus we have proved the following result.

Proposition 3. *Let g and $\bar{g}$ be defined by equation (1) and the unit vector field U be geodesic. Then a vector field V is Jacobi with respect to g along U if and only if it is Jacobi with respect to $\bar{g}$.*

Remark. The Jacobi equation $\bar{\nabla}_U \bar{\nabla}_U V = \bar{R}(U,V)U$ can be interpreted as Newton's second law with the curvature vector $R(U,V)U$ in the role of tidal force (see O'Neill [8]).

4. Background Metric g as a K-Contact Metric

This section is devoted to Lorentz metrics $\bar{g}$ induced from a special class of Riemannian metrics, namely, K-contact Riemannian metrics on an odd-dimensional manifold. We first review basic contact geometry. A $(2n+1)$-dimensional differentiable manifold M is said to be a contact manifold if it carries a global 1-form u such that $u \wedge (du)^n \neq 0$ anywhere on M. Given a contact form u, there exists a unique vecor field U (called the Reeb vector field) such that $u(U) = 1$ and $(du)(U,X) = 0$ for arbitrary vector field X on M. Hence U is a unit vector field. A Riemannian metric g is said to be an associated metric if there exists a (1,1)-tensor field φ such that

$$(du)(X,Y) = g(X,\varphi Y), u(X) = g(U,X), \varphi^2 = -I + u \otimes U\,. \tag{14}$$

Such an associated metric g is said to be K-contact if U is Killing with respect to g and M with this g is called a K-contact manifold. A K-contact manifold is said to be Sasakian if $(\nabla_X \varphi)Y = g(X,Y)U - u(Y)X$, equivalently; $R(X,Y)U = u(Y)X - u(X)Y$, where R is the curvature tensor of g. For details we refer to Blair [1]. Thus we consider a K-contact metric g such that the Lorentz metric $\bar{g}$ induced from g by U (defined by equation (1)) can represent certain odd $(2n+1)$-dimensional gravities. Such a setting can be justified in view of the superstring theory which tells us that we are living in 11 dimensions (see Shiromigu, Maeda and Sasaki [9]) and the standard model particles are confined to 4-dimensional spacetime. We now proceed to establishing relationship between the curvatures of g and $\bar{g}$. If g is a K-contact metric we know [1] that $\nabla_X U = -\varphi X$. Thus it follows from equation (7) that

$$\bar{\nabla}_Y Z = \nabla_Y Z + 2g(U,Y)\varphi Z + 2g(U,Z)\varphi Y\,. \tag{15}$$

Using this and upon a straightforward computation we find that the curvature tensors $\bar{R}$ and R of $\bar{g}$ and g respectively, are related by

$$\bar{R}(Y,Z)X = R(Y,Z)X + 2g(Y,\varphi X)\varphi Z + 2u(X)(\nabla_Y \varphi)Z - 4u(X)u(Y)Z$$

$$+ 2g(Y,\varphi Z)\varphi X + 2u(Z)(\nabla_Y\varphi)X + 2g(\varphi Z,X)\varphi Y - 2u(X)(\nabla_Z\varphi)Y + 4u(X)u(Z)Y + 2g(\varphi Z,Y)\varphi X - 2u(Y)(\nabla_Z\varphi)X\,. \quad (16)$$

Here, we take a local orthonormal basis $(e_i, e_{2n+1} = U)$ $(i = 1, \dots, 2n)$ for g and note that this is also a local orthonormal basis of $\bar{g}$. Let Ric and r denote the Ricci tensor and scalar curvature of g respectively, and the corresponding over-barred expressions pertain to $\bar{g}$. Computing

$$\bar{Ric}(X,Y) = \sum_{a=1}^{2n} \bar{g}(\bar{R}(e_i,X)Y,e_i) - \bar{g}(\bar{R}(U,X)Y,U)$$

using equation (16) and noting

$$(Ric)(X,Y) = \sum_{i=1}^{2n} g(R(e_i,X)Y,e_i) + g(R(U,X)Y,U)$$

we obtain the following relationship between the Ricci tensors of $\bar{g}$ and g:

$$\bar{Ric}(X,Y) = Ric(X,Y) + 4g(X,Y) - 8(n+1)u(X)u(Y)\,. \quad (17)$$

Finally, we consider a vector field V which is conformal with respect to the K-contact metric g as well as the induced Lorentzian metric $\bar{g}$. Under this hypothesis, we know from Proposition 2 that

$$\pounds_V U = -\sigma U\,. \quad (18)$$

Let us recall the following integrability condition (see Yano [10]) for a conformal vector field on an m-dimensional Riemannian (also true for semi-Riemannian) manifold (M, g):

$$(\pounds_V Ric)(X,Y) = -(m-2)g(\pounds_X D\sigma, Y) + (\Delta\sigma)g(X,Y)\,.$$

Lie-differentiating (17) along V and using the above integrability condition we obtain

$$(1-2n)(\bar{g}(\bar{\nabla}_X \bar{D}\sigma, Y) - g(\nabla_X D\sigma, Y)) + (\bar{\Delta}\sigma)\bar{g}(X,Y) = (8\sigma + \Delta\sigma)g(X,Y) - 16(n+1)\sigma g(U,X)g(U,Y)\,. \quad (19)$$

Now, using equation (1) in $X\sigma = \bar{g}(\bar{D}\sigma, X) = g(D\sigma, X)$ shows $\bar{D}\sigma = D\sigma - 2(U\sigma)U$. Differentiating it with respect to $\bar{g}$ along arbitrary vector field X and using (15) provides

$$\bar{\nabla}_X \bar{D}\sigma = \nabla_X D\sigma - 2X(U\sigma)U + 2g(U,X)\varphi D\sigma\,.$$

Using this we compute: $\bar{\Delta}\sigma = \bar{g}(\bar{\nabla}_U \bar{D}\sigma, U) - \bar{g}(\bar{\nabla}_{e_a}\bar{D}\sigma, e_a) = 2g(\nabla_U D\sigma, U) - \Delta\sigma$. Using this in (19) and substituting $Y = U$ (noting $\varphi U = 0$) we find

$$(1-2n)[X(g(U,D\sigma) - g(\nabla_X D\sigma)] = [4\sigma + g(\nabla_U D\sigma, U) + \Delta\sigma - 16\sigma(n+1)]g(U,X)\,.$$

The formula: $\nabla_X U = -\varphi X$ transforms the above equation into

$$(2n-1)g(X,\varphi D\sigma) + [4\sigma + g(\nabla_U D\sigma, U) + \Delta\sigma - 16\sigma(n+1)]g(U,X) = 0\,. \quad (20)$$

Substituting U for X in the above equation gives

$$4\sigma + g(\nabla_U D\sigma, U) + \Delta\sigma - 16\sigma(n+1) = 0\,. \quad (21)$$

Hence (20) reduces to $\varphi D\sigma = 0$. This entails $d\sigma = (U\sigma)u$. Applying d and using the Poincare Lemma: $d^2 = 0$, shows $0 = d(U\sigma) \wedge u + (U\sigma)du$. Taking its wedge product with u gives $(U\sigma)(du) \wedge u = 0$. But $(du) \wedge u \neq 0$ anywhere on M (otherwise the definition of the contact structure would be violated). Hence $U\sigma = 0$ everywhere on M. Consequently, $D\sigma = 0$, i.e. σ is constant on M. Appealing to equation (21) we obtain $\sigma = 0$. Hence V is Killing with respect to both g. We state this as the following result.

Theorem 2. *Let g be a K-contact Riemannian metric with Reeb vector field U, and $\bar{g}$ be the Lorentzian metric defined by equation (1). If a vector field V is conformal with respect to both g and $\bar{g}$, then V is Killing.*

Remark. The conclusion of Theorem 2 can be extended by noting in view of equation (18) and (14) and the commutativity of Lie and exterior derivations that, like the Reeb vector field U, the vector field V leaves all the structure tensors φ, η, ξ, g invariant, i.e. is an infinitesimal automorphism of the K-contact metric structure. Thus Theorem 2 provides an insight into the conformal geometry of K-contact metrics tied up with Lorentzian metrics induced by the Reeb vector field through equation (1).

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