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ON A MODIFIED VERSION OF ILDM METHOD AND ITS ASYMPTOTIC ANALYSIS

S. Borok[1], I. Goldfarb[2], V. Gol'dshtein[3 §]

[1]Division of Mathematical Sciences, SPMS
Nanyang Technological University
SINGAPORE
e-mail: borok@ntu.edu.sg

[2,3]Department of Mathematics
Ben-Gurion University of the Negev
P.O. Box 653, Beer-Sheva, ISRAEL
[3]e-mail: vladimir@bgu.ac.il

Abstract: It is known that processes which take place in complex chemical kinetics and combustion systems have very different time scales. It is often desirable to decouple such systems into fast and slow sub-systems for the reduction of their complexity. One of such reduction methods is Intrinsic Low-Dimensional Manifolds (ILDM) method proposed by Maas and Pope. This method successfully locates invariant manifolds of a considered system, but also has a number of disadvantages. One of the main problems of ILDM numerical realization is the existence of so-called "ghost"-manifolds that do not have any connection to the dynamics of the system. It is shown that even for two-dimensional singularly perturbed system, for which the fast-slow decomposition is explicit, the "ghost"-manifolds can appear. In the present paper a modified version of the ILDM-method is discussed. This modification, which we call TILDM approach, has a much better performance in the context of "ghost"-manifolds problem. The asymptotic analysis of the TILDM method explains why one of the main reasons for "ghost"-manifolds appearance is absent for TILDM.

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§Correspondence author

1. Introduction

The decomposition of complex systems into simplest subsystems using different rates of changes for different subprocesses is almost universally used in physical and engineering models. The main obstacle for this decomposition is its "hidden" nature. Two main natural problem arise for these systems. Whether it is possible to check existence of a decomposition? How to evaluate the unclosed hierarchy? The numeric asymptotic method of Intrinsic Low-Dimensional Manifolds (ILDM) is one of the popular methods for a decomposition of chemical kinetics and combustion models.

The paper concerns the ILDM method and its modification TILDM. The ILDM method was suggested in [18]. Since then many studies have been devoted to its mathematical justification, comparison with other methods, applications to different multiscale problems and modifications, see [1]-[4], [6]-[7], [9], [13], [14], [16], [24].

The following principal problems of the ILDM-method can be formulated:

1. The algorithm cannot be applied in domains of phase space, where the leading eigenvalues of the Jacobian matrix of the RHS of the considered system are complex.

In this case the ILDM method does not produce any decomposition of the original system or produces a non correct decomposition even in the case of an explicitly known decomposition.

2. The method cannot treat some zones on the phase plane ("turning zones" (manifolds), i.e. zones, where critical changes in the system behavior occur).

For many practical models a slow invariant manifold is not unique. Its boundaries can be evaluated for zero-order approximations. The boundaries represent so-called "turning submanifolds". Knowledge about shapes and positions of the "turning manifolds" is crucial for a dynamical regimes classifications. For example this knowledge permits to evaluate so-called "explosion limits" in combustion models etc., see for example [10], [11].

3. The numerical applications of the method produce additional solutions ("ghost" manifolds) that do not have any connection to the dynamics of the system.

The presented paper concerns the third problem from this list. We focus on one of the basic causes of the "ghost" manifolds in the case of a singularly perturbed system. It follows immediately from the ILDM equation for a singularly perturbed system, whose slow manifold is not a normally hyperbolic one. Namely, the existence of the "ghost" manifold will be shown at the level of the zeroth approximation of the ILDM. It will be shown that this disadvantage can be overcome using the improved version of the algorithm – TILDM. The analysis of the TILDM-equation will demonstrate that the zeroth approximation of the TILDM coincides exactly with the zeroth approximation of the slow invariant manifold of the considered singularly perturbed system. It will be also shown that the TILDM-method does not have practical difficulties in neighborhoods of the turning zones, in contradiction to the original approach (see problem 2 in the above list).

In our study, the method of invariant manifolds (MIM) is used as a theoretical test tool for ILDM/TILDM correctness and for comparison these two methods.

2. Theoretical Background

2.1. Method of Invariant Manifolds (Basic Facts)

Consider a singularly perturbed system of ordinary differential equations

$$\frac{dy}{dt} = f(y, z, \epsilon)\,, \tag{1}$$

$$\epsilon\frac{dz}{dt} = g(y, z, \epsilon)\,. \tag{2}$$

Here $y \in \Re^{n_s},\ z \in \Re^{n_f},\ (n_s + n_f = n)$ are vectors in Euclidean space, $t \in (t_0, +\infty)$ is a time-like variable, $0 < \epsilon < \epsilon_0 \ll 1$, functions $f:\ \Re^{n_s} \times \Re^{n_f} \to \Re^{n_s},\ g:\ \Re^{n_s} \times \Re^{n_f} \to \Re^{n_f}$ are supposed to be sufficiently smooth for all $y \in \Re^{n_s},\ z \in \Re^{n_f},\ 0 < \epsilon < \epsilon_0$. The values $|f_i(y, z, \epsilon)|,\ |g_i(y, z, \epsilon)|,\ (i = 1, \ldots m;\ j = 1, \ldots n)$ are assumed to be comparable with the unity as $\epsilon \to 0$. Recall that a smooth surface in the phase space $M \in \Re^{n_s} \times \Re^{n_f} \times (-\infty, \infty)$ is called an invariant manifold of the system (1)-(2), if any phase trajectory $(y(t, \epsilon), z(t, \epsilon))$ such that $(y(t_1, \epsilon), z(t_1, \epsilon)) \in M$ belongs to M for any $t > t_1$. If the last condition holds only for $t \in [t_1, T]$, then M is called a local invariant manifold. The manifold's existence leads to the fact that the analysis of the system behaviour can be considerably simplified by reducing the dimension of the system.

We are interested in the invariant manifolds of the dimension n_s (the dimension of the slow variable) that can be represented as a graph of the vector-valued function:

$$z = h(y, \epsilon). \tag{3}$$

The system dynamics on this manifold is described by the equation

$$\frac{dy}{dt} = f(y, h(y, \epsilon), \epsilon). \tag{4}$$

If $y(t, \epsilon)$ is a solution of equation (4), then the pair $(y(t, \epsilon), z(t, \epsilon))$, where $z(t, \epsilon) = h(y(t, \epsilon), \epsilon)$ is a solution of the original system (1)-(2), since it determines a trajectory on the invariant manifold.

A usual approach in the qualitative study of (1)-(2) is to consider first the degenerate system, which is obtained by substituting $\epsilon = 0$ into the system

$$\dot{y} = f(y, z, 0)\,, \tag{5}$$

$$0 = g(y, z, 0)\,, \tag{6}$$

and then to draw conclusions for the qualitative behaviour of the full system for sufficiently small ϵ. Equation (6) determines the slow surface (slow manifold M_0). Recall that the slow invariant manifold M_0 is normally hyperbolic if the eigenvalues of $D_z g(y, z)$ have non zero real parts for all points $(y, z) \in M_0$. The slow surface is the zeroth approximation of the slow invariant manifold. It is assumed that equation (6) has an isolated smooth solution $z = h_0(y)$, and the following locally uniform convergence takes place

$$\lim_{\epsilon \to 0} h(y, \epsilon) = h_0(y)\,.$$

By the famous Tikhonov's theorem, the question of stability of an invariant manifold can be reduced to the study of its zeroth approximation stability. Invariant manifold $z = h(y, \epsilon)$ of the system (1)-(2) is stable, if the real parts of all eigenvalues of the matrix $D_z g(y, h_0(y), 0)$ are negative.

Points of the slow surface determined by (6) are sub-divided into two types: *standard* points and *turning* points. A point (y, z) is a standard point of the slow surface if in some neighborhood of this point the surface can be represented as a graph of the function $z = h_0(y)$ such that $g(y, h_0(y), 0) = 0$. It means that the condition of the Implicit Function Theorem $D_z g(y, h_0(y), 0) \neq 0$ holds and the slow surface has the dimension of the slow variable. Points for which this condition does not hold are turning points of the slow surface. In other words, turning points are defined as solutions of the system that consists of the slow manifold equation (6) and the following equation

$$D_z g(y, z, 0) = 0\,. \tag{7}$$

Problems of existence, uniqueness and stability of invariant manifolds have been studied by many authors (see, for example, [5], [10], [19], [20], [25]).

When the method of slow invariant manifolds is used to solve a specific problem, a central question is an evaluation of the function $z = h(y, \epsilon)$. Exact evaluation is generally impossible, and various approximations are necessary. One possibility is an asymptotic expansion of $z = h(y, \epsilon)$ in powers of a small parameter:

$$z = h(y, \epsilon) = h_0(y) + \epsilon h_1(y) + \ldots \epsilon^k h_k(y) + \ldots . \tag{8}$$

The asymptotics of the slow invariant manifold are given explicitly, for example, in [10], [19].

In particular, functions h_0 and h_1 are

$$h_0 : \quad g(y, h_0(y), 0) = 0\,, \tag{9}$$

$$h_1 : \quad (D_z g)h_1 = (Dh_0)f - g_\epsilon\,, \tag{10}$$

For the planar case the zeroth and the first terms look as

$$h_0 : \quad g(y, h_0(y), 0) = 0\,, \tag{11}$$

$$h_1 : \quad h_1 g_z = -\frac{g_y}{g_z} f - g_\epsilon\,. \tag{12}$$

Remind that all the functions and their derivatives are calculated in $(y, h_0(y), 0)$.

It should be noted that on the turning surfaces and within their close neighborhoods the asymptotic expansions are inapplicable (only the zeroth approximation $h_0(y)$ has sense). For these regions more delicate asymptotic expansions are correct (see, for example [19]). In particular, (12) is not defined if $g_z = 0$. Nevertheless, it does not mean that the manifold is supposed to be a normally hyperbolic one. Our study is not restricted by considering only normally hyperbolic manifolds.

Note that the requirement of normal hyperbolicity means that the slow surface does not have turning points.

2.2. The Idea of the ILDM

The ILDM method was described in a large number of papers (see, for example, [3], [6], [9], [16], [18], [24]). Therefore here the idea of the method will be formulated very briefly.

Consider a system of ordinary differential equations

$$\frac{dZ}{dt} = F(Z)\,. \tag{13}$$

Here Z and $F(Z)$ are n-dimensional vectors. Assume that this system can be represented locally as a multi-scale system for a corresponding choice of a local basis. The last depends on the choice of an arbitrary point Z in the n-dimensional Euclidian space $\Re^n$. In other words in this local basis a separation of variables in accordance with their rates of changes is possible (i.e. the considered system can be rewritten in this local basis for some neighborhood of the point Z as a singularly perturbed system). According to the assumption made, the system can be subdivided locally into fast relaxing and slow or non-relaxing subsystems. This supposes that the fast sub-system has the same dimension n_f $(n_f < n)$ at any point $Z \in \Re^n$.

For typical situations a set of all steady states of the fast subsystem represents an n_s-dimensional slow manifold $(n_s = n - n_f)$ and the aim is to determine its location. The authors of ILDM suggest that the dynamics of the overall system from arbitrary initial condition should decay very quickly onto this n_s-dimensional manifold. The ILDM allows approximate identification of the slow invariant manifolds as a set of separate points (so-called intrinsic low-dimensional manifolds – ILDM). These manifolds can be found in the following manner [18]. Suppose a local basis of the original phase space is formed by the invariant subspaces of the Jacobian matrix M_J of the vector field F at an arbitrary point Z_0. If the set of eigenvalues λ_i can be sub-divided into two groups

$$\max\{\mathrm{Re}\,[\lambda_i], i = 1, \ldots, n_f\} \ll \tau \ll \min\{\mathrm{Re}\,[\lambda_i], i = n_f + 1, \ldots, n\},\ \tau < 0 \quad (14)$$

one can introduce invariant sub-spaces T_f and T_s .T_f is spanned by the eigenvectors corresponding to the eigenvalues with large negative (fast) real parts. In turn, the subspace T_s is spanned by the eigenvectors corresponding to the eigenvalues with small negative or positive (slow) real parts. Therefore, the transition matrix $Q(Z)$ from this local basis to the standard one that is constructed from the eigenvectors of the Jacobian matrix and its inverse $Q^{-1}(Z)$ can read like two block matrixes

$$Q = \left(\begin{array}{cc} Q_f & Q_s \end{array} \right), \quad \tilde{Q} = \left(\begin{array}{c} \tilde{Q}_f \\ \tilde{Q}_s \end{array} \right),$$

where matrixes Q_f and Q_s correspond to the fast and slow subspaces (Q_f is matrix $n \times n_f$ of the fast eigenvectors, Q_s is matrix $n \times n_s$ of the slow eigenvectors, $\tilde{Q}_f$ is a matrix $n_f \times n$ and $\tilde{Q}_s$ is a matrix $n_s \times n$). The parameter τ is a time scale splitting parameter. This splitting parameter determines the dimensions of the slow (n_s) and fast (n_f) subspaces.

According to [18], the ILDMs are defined as solutions of the following equation

$$\tilde{Q}_f(Z)F(Z) = 0\,.$$

This definition means that the fast component of the original vector field $F(Z)$, that corresponds to the fast block J_f of the Jacobian matrix representation, has vanished.

3. "Ghost" ILDM Manifolds

In this section we demonstrate the existence of "ghost" manifolds for singularly perturbed systems at the level of the zeroth approximation of the ILDM. We start with a two-dimensional case, for which the analysis is more transparent. It would be helpful to consider here the evaluation method for ILDM manifolds used in [18] and analyzed in [16].

Consider the system

$$\dot{y} = f(y, z, \epsilon)\,, \tag{15}$$

$$\epsilon \dot{z} = g(y, z, \epsilon)\,. \tag{16}$$

In [16] it is assumed that $g_z < 0$ on the slow manifold. In other words, the slow manifold is assumed to be stable. This implies also that the slow manifold does not have turning points. Such situation is not realistic, see, for instance, [3], [9] and [1]. Therefore, we suppose that g_z can be equal to or greater than zero. In this case $g_z = 0$ is a curve and its intersection with the zero order approximation of slow invariant manifolds $g = 0$ represents turning points (if curves $g = 0$, $g_z = 0$ are transversal, i.e. in a general position).

For $g_z \neq 0$ eigenvalues of the Jacobian matrix are

$$\lambda_f = \epsilon^{-1} g_z + O(1), \quad \lambda_s = f_y - \frac{f_z g_y}{g_z} + O(\epsilon)\,.$$

Asymptotically λ_f is the biggest eigenvalue at any point except points, where $g_z = 0$.

The nonnormalized slow eigenvector is

$$v_s = \begin{pmatrix} \lambda_s - \epsilon^{-1} g_z \\ \epsilon^{-1} g_y \end{pmatrix},$$

and there is a corresponding fast eigenvector v_f. In general situation the vectors v_s and v_f are not orthogonal. The vector v_s spans the slow subspace, v_f – the fast subspace. To avoid problems with the non orthogonal decomposition of the vector field $F = (f,\, \epsilon^{-1} g)^t$ into the fast and slow parts in [16] the ILDM is defined as the locus of all the points in the phase plane, where the vector field

F lies entirely in the slow subspace. It means F is orthogonal to the orthogonal complement of the slow subspace $v_s^{\perp}$:

$$v_s^{\perp} = \left(\begin{array}{cc} \epsilon^{-1} g_y & \epsilon^{-1} g_z - \lambda_s \end{array} \right).$$

The formal ILDM-equation is

$$\epsilon f g_y + g g_z - \epsilon g \lambda_s = 0\,. \tag{17}$$

The set of all solutions of this functional equation represents ILDM manifolds. If conditions of the Implicit Function Theorem is correct for the left-hand side of this functional equation, than the set of all solutions of this equation is a union of 1-dimensional manifolds (curves).

Equation (17) corresponds to a standard practice of ILDM numerical applications (see [18], [3], [9]). Let us study possible asymptotic solutions of this equation. In $\epsilon = 0$ approximation we have

$$g g_z = 0\,. \tag{18}$$

Recall that $g(y, z, 0) = 0$ determines the slow curve of (15)-(16). Hence, we realize that $g_z = 0$ represents additional solutions of the formal ILDM equation and can be one of sources for the appearance of additional ("ghost") solutions of the ILDM-equation. Numerical simulations demonstrate appearance of such "ghost" manifolds. Simple examples can be found in [1], [2], [3].

Perform the same calculation for a general singularly perturbed system (1)-(2). Fix a point $Z = (y, z)$ in a phase domain. Suppose that the set of eigenvalues of the Jacobi matrix of the vector field $F(Z)$ can be subdivided into two groups: eigenvalues with "large" real parts (which have to correspond to the "fast" subsystem) and eigenvalues with "small" real parts (which have to correspond to the "slow" subsystem). One can introduce invariant subspaces T_f of dimension n_f and T_s of dimension n_s. T_f is spanned by eigenvectors corresponding to the first ("fast") group of eigenvalues, and T_s is spanned by eigenvectors corresponding to the second "slow" group of eigenvalues. Choose an orthonormal basic in T_f and an orthonormal basic in T_s. Its union represents a basic (not necessary orthonormal) in $R^{n_s+n_f}$. Let Q be a transformation matrix of this local basis to the standard basis. A transformation matrix Q and its inverse $Q^{-1} = \tilde{Q}$ have the following representation

$$Q = \left(\begin{array}{cc} Q_f & Q_s \end{array} \right) = \left(\begin{array}{cc} Q_{fs} & Q_{ss} \\ Q_{ff} & Q_{sf} \end{array} \right), \tag{19}$$

where Q_{fs} is an $n_s \times n_f$ matrix, Q_{ff} is an $n_f \times n_f$ matrix, Q_{ss} is an $n_s \times n_s$ matrix, Q_{sf} is an $n_f \times n_s$ matrix,

$$\tilde{Q} = \left(\begin{array}{c} \tilde{Q}_f \\ \tilde{Q}_s \end{array} \right) = \left(\begin{array}{cc} \tilde{Q}_{fs} & \tilde{Q}_{ff} \\ \tilde{Q}_{ss} & \tilde{Q}_{sf} \end{array} \right), \tag{20}$$

where $\tilde{Q}_{ff}$ is an $n_f \times n_f$ matrix, $\tilde{Q}_{fs}$ is an $n_f \times n_s$ matrix, $\tilde{Q}_{sf}$ is an $n_s \times n_f$ matrix and $\tilde{Q}_{ss}$ is an $n_s \times n_s$ matrix.

The formal ILDM-equation gets the form

$$\tilde{Q}_{fs} f + \epsilon^{-1} \tilde{Q}_{ff} g = 0 \,. \tag{21}$$

In the zeroth approximation $\epsilon \to 0$ the equation is

$$\tilde{Q}_{ff} g = 0. \tag{22}$$

If $\det \tilde{Q}_{ff} = 0$, then the last equation produces additional solutions ("ghost" manifolds) except the slow manifold $g = 0$. This is one of the causes for "ghost" manifolds appearance.

Thus, (22) (or, equivalently, (18)) show that $\det \tilde{Q}_{ff} = 0$ ($g_z = 0$) represent the "ghost" manifold at the level of the leading term of the ILDM expansion.

4. TILDM

4.1. The Idea of TILDM

In Section 3 one of the causes of "ghost" manifolds appearance was demonstrated. Namely, for the two-dimensional case the equation $g_z = 0$, and for the general case the equation $\det \tilde{Q}_{ff} = 0$ can produce "ghost" manifolds of the same dimension as slow invariant manifolds. In this section it is shown how this problem can be partially overcome using the TILDM method.

Remind that the ILDM-method was invented to decompose locally an original system into fast and slow subsystems, and to locate slow invariant manifolds attracting system's trajectories for general (not SPS) systems of the type (13). Application of the ILDM algorithm for singularly perturbed systems is important only as a test tool for the ILDM correctness.

Consider the system (13), $n = 2$. It is clear that the Jacobian of the original system contains full information concerning the system dynamics in the case of the linear vector field F (13) only. Assuming that F is a linear mapping, we can conclude that it has directions of the maximal and minimal stretch (fast and slow directions, in the context of the present paper). Therefore, it would be ideal, if we could build a transformation, which converts the original variables into these fast and slow variables. The latter can be found using a general theory of linear algebra. An arbitrary linear non-degenerate transformation maps a unit circle with the center in the origin into an ellipse with the center in the origin. We can identify the directions of the fast and slow motions with directions of the large and small semiaxes of the ellipse, respectively. If M is a matrix of an arbitrary linear non-degenerate transformation and M^t is its transpose matrix, then the eigenvalues of the matrix M^t represent the lengths squares of the ellipse's semiaxes and the corresponding eigenvectors coincide with the directions of the semiaxes. Assuming that the Jacobian J of the considered system contains the relevant information regarding the system dynamics, we can conclude that the transformation matrix chosen as J^t provides us with more accurate information regarding the subdivision of the vector field, than the Jacobian J does.

The difference between the algorithms is that the TILDM uses the symmetric matrix $T = J^t$ instead of the Jacobian matrix J. This symmetric matrix has real non negative eigenvalues and orthogonal eigenvectors.

Let us obtain a TILDM equation for two-dimensional SPS (15)-(16) and compare it with corresponding ILDM equation (17). The corresponding symmetric matrix is

$$T = JJ^t = \begin{pmatrix} f_y^2 + f_z^2 & \epsilon^{-1}(f_y g_y + f_z g_z) \\ \epsilon^{-1}(f_y g_y + f_z g_z) & \epsilon^{-2}(g_y^2 + g_z^2) \end{pmatrix}. \tag{23}$$

The eigenvalues are

$$\lambda_f = \epsilon^{-2}(g_y^2 + g_z^2) + O(1), \quad \lambda_s = \frac{(f_z g_y - f_y g_z)^2}{g_y^2 + g_z^2} + O(\epsilon^2).$$

Asymptotically λ_f is the biggest eigenvalue and λ_s is the smallest eigenvalue.

Let us remark that condition $g_y^2 + g_z^2 \neq 0$ is much less restrictive than $g_z = 0$ because it is equivalent to the system of equations $g_y = 0, g_z = 0$ and for any nondegenerate case solutions of $g_y^2 + g_z^2 = 0$ represent a number of isolated points.

The slow nonnormalized eigenvector is

$$v_s = \begin{pmatrix} \lambda_s - \epsilon^{-2}(g_y^2 + g_z^2) \\ \epsilon^{-1}(f_y g_z - f_z g_y) \end{pmatrix}.$$

The corresponding fast eigenvector v_f is orthogonal to it.

Using the algorithm described in [16] a formal TILDM-equation can be written as

$$\epsilon^2 f(f_y g_y + f_z g_z) + g(g_y^2 + g_z^2) - \epsilon^2 g \lambda_s = 0. \tag{24}$$

Let us give some formal remarks about the set of solutions of this equation. For any nondegenerate case solutions of the equation $g_y^2 + g_z^2 = 0$ are isolated points. Therefore we can restrict ourselves to domains U where $g_y^2 + g_z^2 > a^2 > 0$.

Fix such a domain U. Then, the previous equation is equivalent to

$$G := \epsilon^2 \frac{f(f_y g_y + f_z g_z)}{g_y^2 + g_z^2} - \epsilon^2 \frac{g\lambda_s}{g_y^2 + g_z^2} + g = 0\,. \tag{25}$$

For a sufficiently small ϵ by the Implicit Function Theorem the set of solutions is a 1-dimensional manifold S, which is not necessary connected. The manifold S can be subdivided by points $G_z = 0$ to branches that correspond to different slow invariant manifolds (stable and unstable).

Equation (25) has one significant difference from the equation (17): the leading term in the expansion of $G(y, z) = 0$ looks like

$$g(g_y^2 + g_z^2) = 0\,. \tag{26}$$

The first factor gives us $g = 0$. This equation is the equation of the slow curve (the zero order approximation of slow invariant manifolds) and the corresponding factor in equation (18). The second factor gives us $g_y^2 + g_z^2 = 0$. It is equivalent to the following system of equations: $g_y = 0,\ g_z = 0$. Thus, this system of equations is an analogue of the expression $g_z = 0$ in (18). It means that for the function g in the general (nondegenerate) position the "ghost"-*curves* of the ILDM-equation (17) have turned into the "ghost"-*points* of the TILDM-equation (25), i.e. the dimension of possible "ghost"-manifolds is reduced by one.

4.2. Asymptotic Analysis of the TILDM (Planar Case)

In this section we suppose that $g_z^2 + g_y^2 > 0$. Consider the system (15)-(16). The corresponding symmetric matrix is (23). The eigenvalues of the symmetric matrix are

$$\lambda_{s,f} = \frac{1}{2}\left(f_y^2 + f_z^2 + \frac{g_y^2 + g_z^2}{\epsilon^2} \pm \sqrt{D}\right),$$

where

$$D = \left(f_y^2 + f_z^2 + \frac{g_y^2 + g_z^2}{\epsilon^2}\right)^2 - \frac{4}{\epsilon^2}\left((f_y^2 + f_z^2)(g_y^2 + g_z^2) - (f_y g_y + f_z g_z)^2\right)\,.$$

The fast eigenvector can be found as

$$v_f = \begin{pmatrix} f_y g_y + f_z g_z \\ -\epsilon(f_y^2 + f_z^2 - \lambda_f) \end{pmatrix}$$

and the direction $\frac{v_f}{\|v_f\|}$ is asymptotically close to the real fast direction $(\epsilon,\ 1) \approx (0,\ 1)$, where $\|\cdot\|$ denotes the Euclidean norm of a vector. The corresponding slow vector is orthogonal to v_f. The basis of eigenvectors Q is

$$Q = \begin{pmatrix} Q_f & Q_s \end{pmatrix} = \begin{pmatrix} \frac{v_f}{\|v_f\|} & \left(\frac{v_f}{\|v_f\|}\right)^{\perp} \end{pmatrix}.$$

The matrix Q is orthonormal (its vectors are orthogonal and $\det Q = 1$). Its inverse is

$$\tilde{Q} = \begin{pmatrix} \tilde{Q}_f \\ \tilde{Q}_s \end{pmatrix}.$$

By definition, the TILDM equation is defined as follows

$$\tilde{Q}_f F = 0,$$

where $\tilde{Q}_f$ is the fast part of $\tilde{Q}$ and F is the vector field of the system. The TILDM equation looks as

$$\frac{1}{\| v_f \|}\big((f_y g_y + f_z g_z)f - (f_y^2 + f_z^2 - \lambda_f)g\big) = 0\,. \tag{27}$$

Theorem 1. *Suppose a slow invariant manifold $z = h(y)$ of system (15)-(16) exists for $y_0 \le y \le y_1$. Then the first term of the formal asymptotic series*

$$z = \varphi(y, \epsilon) = \varphi_0 + \epsilon\varphi_1 + \dots \tag{28}$$

of the corresponding TILDM manifold coincides with the first term of the corresponding asymptotic series for the slow invariant manifold, and the second term does not coincide, i.e. the functions φ_0 and φ_1 are defined by the expressions

$$\varphi_0 : \; g(y, \varphi_0, 0) = 0, \tag{29}$$

$$\varphi_1 g_z = -g_\epsilon, \tag{30}$$

where all the functions and their derivatives are calculated at $(y, \varphi_0(y), 0)$.

Proof. We expand all the functions from equation (27) into the power series with respect to the small parameter ϵ and equate coefficients of the same powers of ϵ. □

Remark. Except the zero-order term this asymptotic expansion is not convenient for TILDM manifolds in a vicinity of turning points. A convenient asymptotic expansion algorithm can be found in [19].

4.3. Asymptotic Analysis of the TILDM (General Case)

First, we obtain the TILDM-equation. Consider system (1)-(2). Fix an arbitrary point (Y, Z). The Jacobi matrix is

$$J = \begin{pmatrix} D_y f & D_z f \\ \epsilon^{-1} D_y g & \epsilon^{-1} D_z g \end{pmatrix}.$$

The corresponding symmetric matrix is

$$T = JJ^t = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.$$

Here T_{11} is $n_s \times n_s$ matrix with the elements proportional to $O(\epsilon^0)$, T_{12} is $n_s \times n_f$ matrix with the elements proportional to $O(\epsilon^{-1})$, T_{21} is $n_f \times n_s$ matrix with the elements proportional to $O(\epsilon^{-1})$, T_{22} is $n_f \times n_f$ matrix with the elements proportional to $O(\epsilon^{-2})$. For an arbitrary point (Y, Z) matrix T has nonnegative eigenvalues and orthogonal eigenvectors. Suppose that the eigenvalues of T fall into two distinct groups: n_f fast eigenvalues (proportional to $O(\epsilon^{-2})$) and n_s slow ones (proportional to $O(1)$).

In this subsection we suppose that the matrix T_{22} is regular.

There exists an orthonormal basis Q, in which T has a diagonal form. The eigenvalues of T can appear along the diagonal in any desirable order (see [12], [15]).

$$T = QT_dQ^t,$$

where

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \quad Q^t = \begin{pmatrix} Q_{11}^t & Q_{21}^t \\ Q_{12}^t & Q_{22}^t \end{pmatrix}, \quad T_d = \begin{pmatrix} \Lambda_f & 0 \\ 0 & \Lambda_s \end{pmatrix}. \tag{31}$$

The dimensions of the Q_{ij} matrices are as follows: Q_{11} is $n_s \times n_f$, Q_{12} is $n_s \times n_s$, Q_{21} is $n_f \times n_f$, Q_{22} is $n_f \times n_s$. Here

$$\begin{pmatrix} Q_{11} \\ Q_{21} \end{pmatrix}, \quad \begin{pmatrix} Q_{12} \\ Q_{22} \end{pmatrix}$$

are the orthonormal basis of the fast and slow subspaces, respectively; Λ_f is the fast block ($n_f \times n_f$ block of the fast eigenvalues), Λ_s is the slow block ($n_s \times n_s$ block of the slow eigenvalues). The fast eigenvalues are proportional to $O(\epsilon^{-2})$, the slow eigenvalues are proportional to $O(\epsilon^0)$. By definition, the TILDM-equation is

$$Q_{11}^t f + \epsilon^{-1} Q_{21}^t g = 0. \tag{32}$$

Theorem 2. *Suppose a slow invariant manifold $z = h(y)$ of system (1)-(2) exists in a compact smooth domain $U \subseteq \Re^{n_s}$. Then the first term of the formal* *asymptotic series*

$$z = \varphi(y, \epsilon) = \varphi_0(y) + \epsilon\varphi_1(y) + \dots$$

of the corresponding TILDM manifold coincides with the first term of the corresponding asymptotic series for the slow invariant manifold, and the second term does not coincide, i.e. the functions φ_0 *and* φ_1 *are defined by the expressions*

$$\varphi_0 : \; g(y, \varphi_0, 0) = 0, \tag{33}$$

$$(D_z g)\varphi_1 = -g_\epsilon, \tag{34}$$

where all the functions and their derivatives are calculated at $(y, \varphi_0(y), 0)$.

Proof. The proof consists of two steps: 1) calculating the asymptotic expansion of the new basis Q; 2) substituting various asymptotic expansions into the TILDM-equation and equating the coefficients of like powers of ϵ.

1. We find the representation of the new basis Q in the form

$$Q \equiv Q(\epsilon) = \begin{pmatrix} Q_{11}^{(0)} + \epsilon Q_{11}^{(1)} & Q_{12}^{(0)} + \epsilon Q_{12}^{(1)} \\ Q_{21}^{(0)} + \epsilon Q_{21}^{(1)} & Q_{22}^{(0)} + \epsilon Q_{22}^{(1)} \end{pmatrix} + O(\epsilon^2)\,.$$

The matrix Q must satisfy the following two conditions:

$$QQ^t = I_{n_f \times n_s} \tag{35}$$

(orthogonality of the matrix Q). We require the matrix Q to be orthogonal up to the order $O(\epsilon)$.

$$T = QT_dQ^t \tag{36}$$

(Q diagonalizes the symmetric matrix T).

From (35) we obtain four equations for the unknowns $Q_{ij}^{(0)}$, $Q_{ij}^{(1)}$:

$$\left(Q_{11}^{(0)} + \epsilon Q_{11}^{(1)}\right)\left(Q_{11}^{(0)^t} + \epsilon Q_{11}^{(1)^t}\right) + \left(Q_{12}^{(0)} + \epsilon Q_{12}^{(1)}\right)\left(Q_{12}^{(0)^t} + \epsilon Q_{12}^{(1)^t}\right) = I_{n_s \times n_s} \tag{37}$$

$$\left(Q_{11}^{(0)} + \epsilon Q_{11}^{(1)}\right)\left(Q_{21}^{(0)^t} + \epsilon Q_{21}^{(1)^t}\right) + \left(Q_{12}^{(0)} + \epsilon Q_{12}^{(1)}\right)\left(Q_{22}^{(0)^t} + \epsilon Q_{22}^{(1)^t}\right) = 0_{n_s \times n_f} \tag{38}$$

$$\left(Q_{21}^{(0)} + \epsilon Q_{21}^{(1)}\right)\left(Q_{11}^{(0)^t} + \epsilon Q_{11}^{(1)^t}\right) + \left(Q_{22}^{(0)} + \epsilon Q_{22}^{(1)}\right)\left(Q_{12}^{(0)^t} + \epsilon Q_{12}^{(1)^t}\right) = 0_{n_f \times n_s} \tag{39}$$

$$\left(Q_{21}^{(0)} + \epsilon Q_{21}^{(1)}\right)\left(Q_{21}^{(0)^t} + \epsilon Q_{21}^{(1)^t}\right) + \left(Q_{22}^{(0)} + \epsilon Q_{22}^{(1)}\right)\left(Q_{22}^{(0)^t} + \epsilon Q_{22}^{(1)^t}\right) = I_{n_f \times n_f} \tag{40}$$

From (36) it follows

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} \Lambda_f & 0 \\ 0 & \Lambda_s \end{pmatrix} \begin{pmatrix} Q_{11}^t & Q_{21}^t \\ Q_{12}^t & Q_{22}^t \end{pmatrix}$$

$$= \begin{pmatrix} Q_{11}\Lambda_f Q_{11}^t + Q_{12}\Lambda_s Q_{12}^t & Q_{11}\Lambda_f Q_{21}^t + Q_{12}\Lambda_s Q_{22}^t \\ Q_{21}\Lambda_f Q_{11}^t + Q_{22}\Lambda_s Q_{12}^t & Q_{21}\Lambda_f Q_{21}^t + Q_{22}\Lambda_s Q_{22}^t \end{pmatrix}.$$

Hence, at

$$\Lambda_f \equiv \Lambda_f(\epsilon) = \epsilon^{-2}\Lambda_f^{(-2)} + \epsilon^{-1}\Lambda_f^{(-1)} + \Lambda_f^{(0)} + \epsilon\Lambda_f^{(1)} + O(\epsilon^2)$$

$$\Lambda_s \equiv \Lambda_s(\epsilon) = \Lambda_s^{(0)} + \epsilon\Lambda_s^{(1)} + O(\epsilon^2)$$

we obtain

$$\left(Q_{11}^{(0)} + \epsilon Q_{11}^{(1)}\right)\left(\epsilon^{-2}\Lambda_f^{(-2)} + \epsilon^{-1}\Lambda_f^{(-1)} + \Lambda_f^{(0)} + \epsilon\Lambda_f^{(1)} + \dots\right)\left(Q_{11}^{(0)^t} + \epsilon Q_{11}^{(1)^t}\right) + \left(Q_{12}^{(0)} + \epsilon Q_{12}^{(1)}\right)\left(\Lambda_s^{(0)} + \epsilon\Lambda_s^{(1)} + \dots\right)\left(Q_{12}^{(0)^t} + \epsilon Q_{12}^{(1)^t}\right) = T_{11}\,, \quad (41)$$

$$\left(Q_{11}^{(0)} + \epsilon Q_{11}^{(1)}\right)\left(\epsilon^{-2}\Lambda_f^{(-2)} + \epsilon^{-1}\Lambda_f^{(-1)} + \Lambda_f^{(0)} + \epsilon\Lambda_f^{(1)} + \dots\right)\left(Q_{21}^{(0)^t} + \epsilon Q_{21}^{(1)^t}\right) + \left(Q_{12}^{(0)} + \epsilon Q_{12}^{(1)}\right)\left(\Lambda_s^{(0)} + \epsilon\Lambda_s^{(1)} + \dots\right)\left(Q_{22}^{(0)^t} + \epsilon Q_{22}^{(1)^t}\right) = T_{12}\,, \quad (42)$$

$$\left(Q_{21}^{(0)} + \epsilon Q_{21}^{(1)}\right)\left(\epsilon^{-2}\Lambda_f^{(-2)} + \epsilon^{-1}\Lambda_f^{(-1)} + \Lambda_f^{(0)} + \epsilon\Lambda_f^{(1)} + \dots\right)\left(Q_{11}^{(0)^t} + \epsilon Q_{11}^{(1)^t}\right) + \left(Q_{22}^{(0)} + \epsilon Q_{22}^{(1)}\right)\left(\Lambda_s^{(0)} + \epsilon\Lambda_s^{(1)} + \dots\right)\left(Q_{12}^{(0)^t} + \epsilon Q_{12}^{(1)^t}\right) = T_{21}\,, \quad (43)$$

$$\left(Q_{21}^{(0)} + \epsilon Q_{21}^{(1)}\right)\left(\epsilon^{-2}\Lambda_f^{(-2)} + \epsilon^{-1}\Lambda_f^{(-1)} + \Lambda_f^{(0)} + \epsilon\Lambda_f^{(1)} + \dots\right)\left(Q_{21}^{(0)^t} + \epsilon Q_{21}^{(1)^t}\right) + \left(Q_{22}^{(0)} + \epsilon Q_{22}^{(1)}\right)\left(\Lambda_s^{(0)} + \epsilon\Lambda_s^{(1)} + \dots\right)\left(Q_{22}^{(0)^t} + \epsilon Q_{22}^{(1)^t}\right) = T_{22}\,. \quad (44)$$

Block T_{11} is proportional to the unity. It means that in (41) the terms of the orders $O(\epsilon^{-2})$, $O(\epsilon^{-1})$ should be equal to zero, while the terms of the order $O(\epsilon^0)$ should not be equal to zero. Thus, from (41) we obtain

$$Q_{11}^{(0)}\Lambda_f^{(-2)}Q_{11}^{(0)^t} = 0\,, \quad (45)$$

$$Q_{11}^{(0)}\Lambda_f^{(-2)}Q_{11}^{(0)^t} + Q_{11}^{(0)}\Lambda_f^{(-2)}Q_{11}^{(1)^t} + Q_{11}^{(1)}\Lambda_f^{(-1)}Q_{11}^{(0)^t} = 0\,, \quad (46)$$

$$Q_{11}^{(1)}\Lambda_f^{(-2)}Q_{11}^{(1)^t} + Q_{11}^{(1)}\Lambda_f^{(-1)}Q_{11}^{(0)^t} + Q_{11}^{(0)}\Lambda_f^{(-1)}Q_{11}^{(1)^t} + Q_{12}^{(0)}\Lambda_s^{(0)}Q_{12}^{(0)^t} \neq 0\,. \quad (47)$$

In a similar way, from (42) we obtain

$$Q_{11}^{(0)}\Lambda_f^{(-2)}Q_{21}^{(0)^t} = 0\,, \quad (48)$$

$$Q_{11}^{(1)}\Lambda_f^{(-2)}Q_{21}^{(0)^t} + Q_{11}^{(0)}\Lambda_f^{(-2)}Q_{21}^{(1)^t} + Q_{11}^{(0)}\Lambda_f^{(-1)}Q_{21}^{(0)^t} \neq 0\,. \quad (49)$$

From (43) it follows

$$Q_{21}^{(0)}\Lambda_f^{(-2)}Q_{11}^{(0)^t} = 0\,, \quad (50)$$

$$Q_{21}^{(1)}\Lambda_f^{(-2)}Q_{11}^{(0)^t} + Q_{21}^{(0)}\Lambda_f^{(-2)}Q_{11}^{(1)^t} + Q_{21}^{(0)}\Lambda_f^{(-1)}Q_{11}^{(0)^t} \neq 0\,. \quad (51)$$

From (44) it follows

$$Q_{21}^{(0)}\Lambda_f^{(-2)}Q_{21}^{(0)^t} \neq 0\,. \quad (52)$$

The desired matrix Q must satisfy conditions (37)-(40), (45)-(52), i.e.

$$Q(\epsilon) = \begin{pmatrix} -\epsilon Q_{12}^{(0)} Q_{22}^{(1)^t} Q_{21}^{(0)} & Q_{12}^{(0)} \\ Q_{21}^{(0)} & \epsilon Q_{22}^{(1)} \end{pmatrix} + O(\epsilon^2)\,. \tag{53}$$

2. Substitute the asymptotic expansions of the functions f and g, and formula (53) in TILDM-equation (32).

$$- \epsilon Q_{21}^{(0)^t} Q_{22}^{(1)} Q_{12}^{(0)^t} (f + \epsilon((D_z f)\varphi_1 + f_\epsilon) + \dots) \\ + \epsilon^{-1} Q_{21}^{(0)^t} (g + \epsilon((D_z g)\varphi_1 + g_\epsilon) + \dots) = 0\,.$$

The last equation can be simplified by multiplying by $Q_{21}^{(0)}$ (now it is an orthogonal matrix!):

$$- \epsilon Q_{22}^{(1)} Q_{12}^{(0)^t} (f + \epsilon((D_z f)\varphi_1 + f_\epsilon) + \dots) \\ + \epsilon^{-1} (g + \epsilon((D_z g)\varphi_1 + g_\epsilon) + \dots) = 0\,.$$

Use the usual procedure of equating coefficients of the same powers of the small parameter. We obtain the first two terms of the TILDM expansion as follows

$$g(y, \varphi_0(y), 0) = 0\,, \tag{54}$$

$$(D_z g)\varphi_1 + g_\epsilon = 0\,. \tag{55}$$

This completes the proof. □

4.4. Examples

Theoretical Example 1. Consider the following system of differential equations with a small parameter ϵ :

$$\epsilon \dot{x} = -x - \sin(x) - \sin(y)\,, \tag{56}$$

$$\dot{y} = -y = g(y)\,. \tag{57}$$

This example was discussed in [1]. Here we repeat it briefly in order to compare with the modified version.

The slow surface is given by the equation

$$-x - \sin(x) - \sin(y) = 0\,. \tag{58}$$

It is easy to see that the slow surface is normally hyperbolic (does not have turning points) and stable.

Apply the ILDM-method for this system. The eigenvalues of the Jacobian are: $\lambda_1 = (-1 - \cos(x))/\epsilon$ and $\lambda_2 = -1$. At the first glance λ_1 can be accepted as a fast eigenvalue (because it is comparable with ϵ^{-1}). But it is easy to see that there is an infinite number of domains, where the hierarchy is opposite, i.e. $|\lambda_2| > |\lambda_1|$. In these domains the formal application of the ILDM-algorithm represents y as a"fast" variable and "x" as a "slow" one, which contradicts the real fast-slow subdivision. Therefore, the two ILDM equations should be treated together for domains with different hierarchy of the eigenvalues $\lambda_{1,2}$, namely

$$-x - \sin(x) - \sin(y) + \frac{\epsilon y \cos(y)}{-1 + \epsilon - \cos(x)} = 0, \ |\lambda_1| > |\lambda_2| \,, \tag{59}$$

$$y = 0, \ |\lambda_2| > |\lambda_1| \tag{60}$$

The first ILDM-equation (59) consists of two parts: the main part (which is described by the equation of the slow curve) and some small (proportional to ϵ) expression, denominator of which equals zero for $\cos(x) = \epsilon - 1$. It is easy to see that in the same points the eigenvalues of the Jacobian matrix equal each other.

The second ILDM-equation (60) is a result of incorrect fast-slow division by the ILDM-algorithm (this is the "ghost" manifold). Figure 1a depicts the slow surface of the system defined by equation (59) (solid line) and an arbitrary trajectory of the system (dashed line). Figure 1b represents the result obtained by the formal application of the ILDM-method (solid line) and the trajectory of the system (dashed line).

The calculation of ILDMs (solid lines) gives the central curve (which coincides with the slow surface), the x axis (the result of the second ILDM-equation (60)) and a large number of additional objects. The reason of their appearance is the incorrect fast-slow division provided by the formal application of the algorithm.

Thanks to the low dimensionality of the considered system, the analytical result of the ILDM-method application can be performed explicitly. The analytical result (see Figure 1c) is represented by the central curve (coincides with the slow curve) and short segments AB, CD, EF along the x axis, which are the result of incorrect fast-slow division by the method.

In Figure 1b we can see that the trajectory with arbitrary initial conditions approaches the ILDM curve passing through the "ghost" manifolds (fast motion, almost parallel to the x axis). It should be noticed that one of the ILDM-manifolds (the central part of Figure 1b) is very close to the slow surface.

Consider the TILDM running outcome for Theoretical Example 1. The symmetric matrix T looks as

$$T = \begin{pmatrix} \frac{(1+\cos(x))^2 + \cos^2(y)}{\epsilon^2} & \frac{\cos(y)}{\epsilon} \\ \frac{\cos(y)}{\epsilon} & 1 \end{pmatrix}.$$

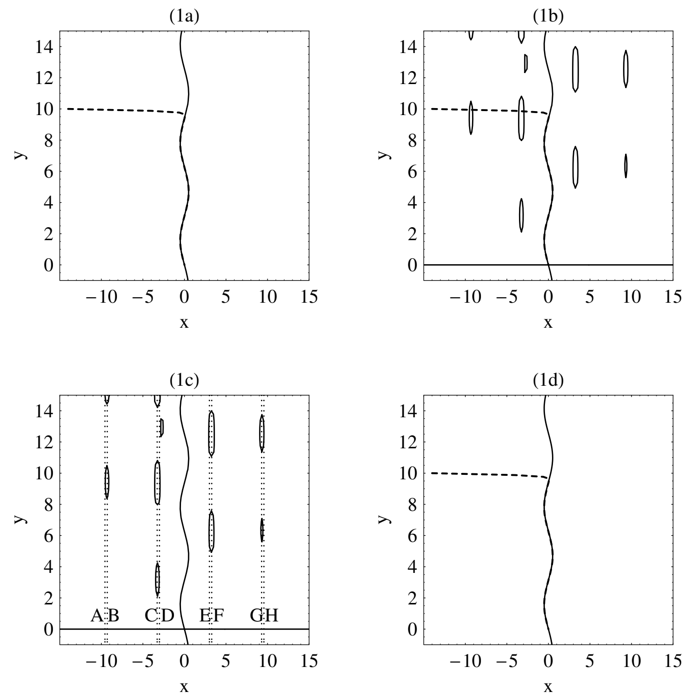

Figure 1: Theoretical Example 1

The matrix has two eigenvalues: $\mu_1 = O(\epsilon^{-2})$ and $\mu_2 = O(1)$. One of eigenvectors of the matrix T looks like $h_1 = \begin{pmatrix} \frac{\epsilon\cos(y)}{\epsilon^2\mu_1-(1+\cos(x))^2-\cos^2(y)} & 1 \end{pmatrix}^t$. The second eigenvector h_2 is orthogonal to it: $h_2 = h_1^t$. The TILDM-equation reads

$$\frac{1}{\| h_1 \|}\left(\frac{-x-\sin(x)-\sin(y)}{\epsilon} + \frac{\epsilon y\cos(y)}{\epsilon^2\mu_1-(1+\cos(x))^2-\cos^2(y)}\right) = 0\,.$$

Figure 1d depicts the TILDM manifold (solid line) and numerical trajectory of the system (dashed line).

In Figure 1d we can see that the TILDM does not produce any artificial objects like the ILDM does. This justifies our previous theoretical conclusion regarding a better ability of the TILDM approach proposed here to determine the slow surface of the multi-scale systems.

Theoretical Example 2. This example will demonstrate the essential perturbations produced by the ILDM algorithm on a unique stable slow manifold. Consider the following system of differential equations with a small parameter

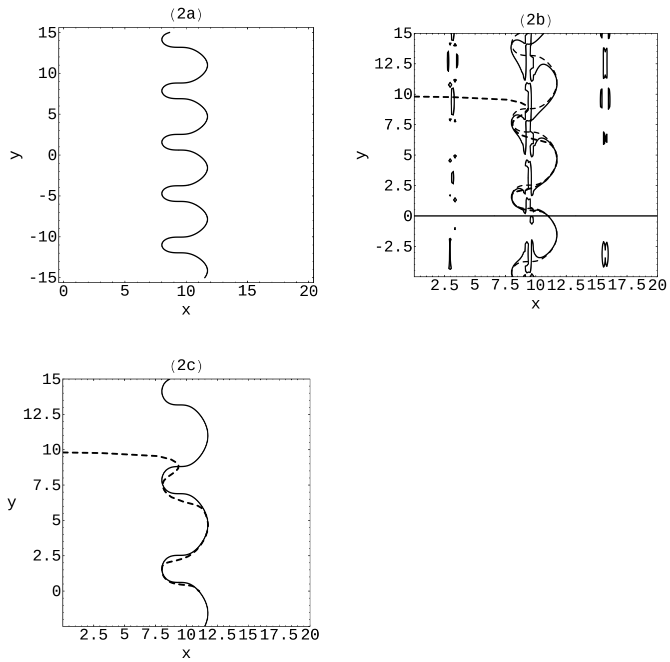

Figure 2: Theoretical Example 2

ϵ

$$\epsilon \dot{x} = -x - \sin(x) - \sin(y) + 10\,, \tag{61}$$

$$\dot{y} = -2y - \sin(y)\,. \tag{62}$$

Figure 2a shows the slow curve for this system, which has the following equation

$$-x - \sin(x) - \sin(y) + 10 = 0\,. \tag{63}$$

The theory of invariant manifold provides us with the conclusion that it is an attractive (stable) manifold.

The Jacobian matrix J is upper triangular with eigenvalues $\lambda_{1,2}$ on its diagonal:

$$J = \begin{pmatrix} -\frac{1+\cos(x)}{\epsilon} & -\frac{\cos(y)}{\epsilon} \\ 0 & -2-\cos(y) \end{pmatrix},\ \lambda_1 = -\frac{1+\cos(x)}{\epsilon},\ \lambda_2 = -\frac{\cos(y)}{\epsilon}.$$

Both eigenvalues are not positive. It is similar to the previous example that at first glance we accept λ_1 as a fast eigenvalue because it is proportional to ϵ^{-1}. However, in infinite number of points $\epsilon(2+\cos(y)) = 1+\cos(x)$ the eigenvalues are equal and in the domains $\epsilon(2+\cos(y)) > 1+\cos(x)$ we have $|\lambda_2| > |\lambda_1|$.

As in the previous example, we obtain two ILDM-equations (it depends on which of the eigenvalues is "fast" in the considered domain) applying the algorithm

$$-x-\sin(x)-\sin(y)+10+\frac{\epsilon\cos(y)(2y+\sin(y))}{\epsilon(2+\cos(y))-1-\cos(x)}=0,\ |\lambda_1|>|\lambda_2|\,, \quad (64)$$

$$-2y-\sin(y)=0,\ |\lambda_2|>|\lambda_1|\,. \quad (65)$$

Figure 2b demonstrates two ILDM manifolds (solid lines), the slow curve (central dashed line) and a system's trajectory (bold dashed line). It is clear that equation (65) does not represent correctly the slow manifold. Figure 2b shows the "ghost" objects, which are located both far from the real slow manifold and on it. This phenomenon did not appear in the previous example.

The symmetric matrix T looks as

$$\begin{pmatrix} \frac{(1+\cos(x))^2+\cos^2(y)}{\epsilon^2} & \frac{\cos(y)(2+\cos(y))}{\epsilon} \\ \frac{\cos(y)(2+\cos(y))}{\epsilon} & (2+\cos(y))^2 \end{pmatrix}.$$

The matrix has the following eigenvalues: $\mu_f = \frac{(1+\cos(x))^2+\cos^2(y)}{\epsilon^2} + O(1)$ and $\mu_s = O(1)$. The corresponding (non-normalized) eigenvectors can be chosen as

$$h_f = \begin{pmatrix} \frac{\epsilon\cos(y)(2+\cos(y))}{\epsilon^2\mu_1-(1+\cos(x))^2-\cos^2(y)} \\ 1 \end{pmatrix}, \quad h_s = h_f^t\,.$$

It can be easily seen that $\frac{h_f}{\|h_f\|}$, $\frac{h_s}{\|h_s\|}$ asymptotically coincide with the real fast $(1,\ 0)$ and slow $(0,\ 1)$ directions of the system. Then, the TILDM equation looks as

$$\frac{1}{\|\ h_f\ \|}\left(\frac{-x-\sin(x)-\sin(y)+10}{\epsilon}+\frac{\epsilon(2y+\sin(y))\cos(y)(2+\cos(y))}{\epsilon^2\mu_f-(1+\cos(x))^2-\cos^2(y)}\right)=0\,.$$

Figure 2c depicts the TILDM (solid line) and the numerical trajectory (dashed line). The comparison of Figures 2b and 2c shows that when we use the improved version of the ILDM-method the "ghost" manifolds do not appear.

Practical Example. Here we consider a mathematical model of the ignition of the cool fuel spray in the hot combustible mixture. A formulation of the physical model and detailed asymptotic analysis of the dynamics of the corresponding system of the governing equations along the lines of the MIM approach can be found in [8], [17]. The finally reduced system of ODEs contains two equations and reads as

$$\gamma\dot{u}=(1-\epsilon_2(\psi-1)(r^3-1))\exp\left(\frac{\mathrm{u}}{1+\beta\mathrm{u}}\right)-\epsilon_1\mathrm{ur}=\mathrm{F}(\mathrm{u},\mathrm{r}),\ \mathrm{u}(0)=0\,, \quad (66)$$

$$\epsilon_2 \dot{r} = -\frac{\epsilon_1 u}{3r} = G(u, r), \; r(0) = 1 \tag{67}$$

Here u is a dimensionless temperature, r is a droplet radius, γ is a reciprocal of the dimensionless adiabatic temperature rise and β is a dimensionless initial temperature. The parameters β, γ are normally small with respect to unity. The parameter γ serves as the singular parameter of the system. The value of the parameter ϵ_2 can essentially vary and it can serve as a singular parameter too. The system (66)-(67) has two small parameters and relation fast-slow depends on the value of the ratio γ/ϵ_2. Let us restrict ourselves by the case of the fast temperature ($\gamma/\epsilon_2 \ll 1$). Assuming this, the slow curve of the system is given by

$$F(u, r) = (1 - \epsilon_2(\psi - 1)(r^3 - 1))\exp\left(\frac{\mathrm{u}}{1 + \beta\mathrm{u}}\right) - \epsilon_1 \mathrm{ur} = 0, \tag{68}$$

where the chosen system parameters for the example presented here look as follows: $\gamma = 0.01, \; \beta = 0, \; \epsilon_1 = 5, \; \epsilon_2 = 1.43, \; \psi = 0.5$. The slow curve is represented on Figure 3a. Here A and C are turning points, ABC is a stable branch of the slow curve, CDA is an unstable one.

The Jacobi matrix of the RHS of the system is

$$J = \begin{pmatrix} \gamma^{-1} F_u & \gamma^{-1} F_r \\ G_u & G_r \end{pmatrix}$$

The line $F_u = 0$ divides the phase plane into parts, where $F_u < 0$ and $F_u > 0$. In each of these domains the ILDM-equation should be obtained, because the sign of the function F_u defines which of the eigenvalues of the Jacobi matrix is leading in the considered domain.

Figure 3b represents the line $F_u = 0$. From the left of this line $F_u < 0$, from the right of this line $F_u > 0$.

In the domain $F_u < 0$ the eigenvalues are

$$\lambda_f = \frac{F_u}{\gamma} + O(1), \; \lambda_s = G_r - \frac{F_r G_u}{F_u} + O(\gamma), \tag{69}$$

i.e. λ_f is very big in absolute value (proportional to γ^{-1}) and negative, while λ_s proportional to γ^0. The result of the ILDM-algorithm application to this domain is shown on Figure 3c.

In the domain $F_u > 0$ one of the eigenvalues is very big in absolute value (proportional to γ^{-1}), but positive, while the second eigenvalue is proportional to the unity. The result of the ILDM-algorithm application to this domain is shown on Figure 3d.

Figure 3e shows the two obtained manifolds (solid lines) together with the

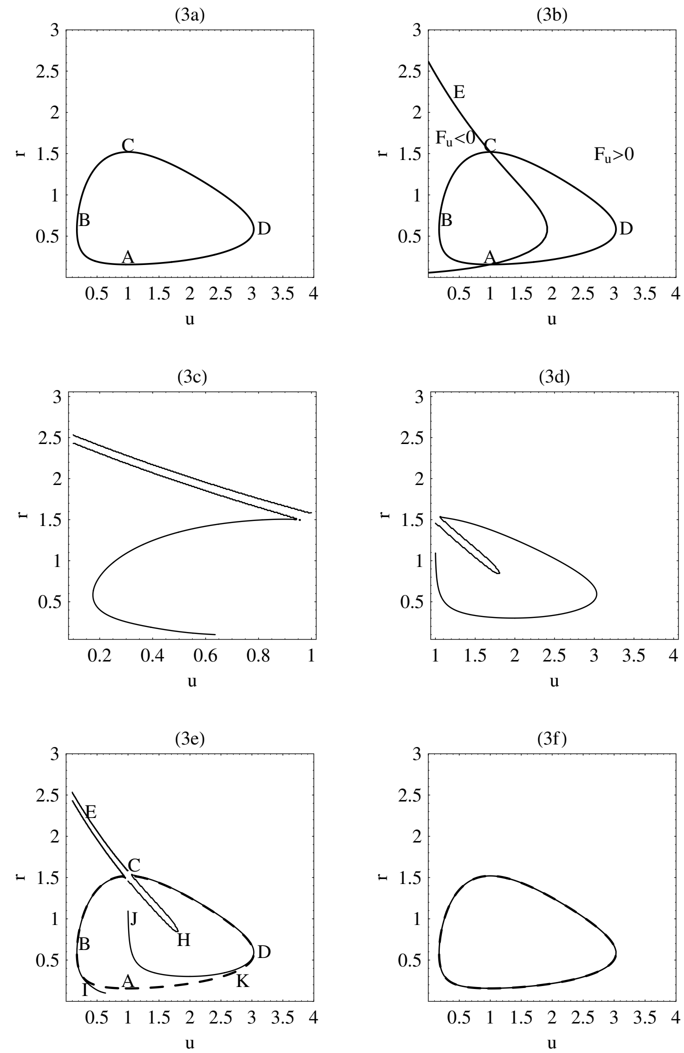

Figure 3: Practical Example

slow curve (dashed line). On the figure we can see that the ILDM-curves CBI and CDK coincide with the slow curve, but after passing the points I and D both ILDM-lines drop from the slow curve. Continuation of the line CBI and the line KJ do not represent a correct manifold of the system. The analysis shows that the curve EH lies in zone of complex eigenvalues of the Jacobi matrix. The ILDM-algorithm can not be applied in this zone.

Let us compare the results of the TILDM-method with the original ILDM-approach. The TILDM is shown on Figure 3f. The figure demonstrates a good agreement between the TILDM (solid) and the slow curve (dashed).

Thus, it is shown that the TILDM-method works better than the ILDM.

5. Conclusions

The present paper represents a natural continuation of the authors' investigation of the Intrinsic Low-Dimensional Manifold method.

As any other algorithm, the ILDM has a number of restrictions. One of them is the appearance of additional solutions ("ghost" manifolds) of the equation describing the ILDM. There are several sources of "ghost" manifolds appearance. One of them is discussed in this paper. Namely, for two-dimensional case, the main term of the ILDM-equation asymptotic expansion is the product gg_z, where $g = 0$ is the slow manifold of the system (correct solution of the ILDM-equation) and g_z is the "ghost" manifold. Notice that these "ghost" manifolds appear in the leading term of the asymptotic expansion.

Recall that in several studies (see, for example, [16], [24]) normally hyperbolic slow manifolds are considered (i.e. the slow manifolds for which $g_z \neq 0$ (for the n-dimensional case the corresponding determinant is not equal to zero)). But g_z can be equal to zero beyond the slow manifold. In this case the "ghost" manifolds appear outside the slow manifolds. A relevant example can be found in [1].

If $g_z = 0$ on the slow manifold, then the "ghost" manifolds appear in the vicinity of the turning point.

It is shown that the TILDM-approach has the following advantages as compared with the ILDM: 1) The zeroth approximation of the TILDM coincides with the zeroth approximation of the slow manifold. In other words, the problem of the "ghost" manifolds at the level of the leading term is overcome; 2) The asymptotic expansion shows that the TILDM does not involve any practical problems in the vicinities of the turning points; 3) The dimension of the "ghost"-manifolds is reduced by one.

The benefits of the TILDM-technique are demonstrated on a number of examples.

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