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AN ASYMPTOTIC FORMULA FOR A FUNCTION RELATED TO BESSEL FUNCTION OF THE FIRST KIND

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Abstract: We obtain a new asymptotic formula for the Bessel function of the first kind.

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1. Introduction

The Bessel function of the first kind, of order α, is defined by

$$J_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+\alpha}}{2^{2k+\alpha} k! \Gamma(k+\alpha+1)}. \tag{1}$$

For $\mathrm{Re}(\alpha) > -\frac{1}{2}$ we have the integral representation (see [2, p. 962])

$$J_\alpha(z) = \frac{z^\alpha}{\sqrt{\pi} 2^\alpha \Gamma(\alpha + \frac{1}{2})} \int_{-1}^{1} e^{izt} (1-t^2)^{\alpha - \frac{1}{2}} \mathrm{d}t. \tag{2}$$

A very important asymptotic formula for J_α is (see [3, p. 199])

$$J_\alpha(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{2}\alpha - \frac{\pi}{4}\right), \tag{3}$$

as $|z| \to \infty$, $-\pi < \mathrm{Arg}(z) < \pi$. It is interesting to observe that (3) is an equality

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for $\alpha = -\frac{1}{2}$ and $\alpha = \frac{1}{2}$; we have $J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\cos(z)$ and $J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin(z)$. It follows from (3) that

$$J_\alpha^{(m)}(z) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{2}\alpha - \frac{\pi}{4} + m\frac{\pi}{2}\right), \tag{4}$$

for $m = 0, 1, 2, \ldots$. There also exists asymptotic formulas for large values of the parameter α (see [3, Chapter VIII]).

In this paper, we consider the function

$$g_\alpha(z) := 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha}. \tag{5}$$

The function g_α is an entire function of exponential type 1, with $g_\alpha(0) = 1$. It readily follows from (1) that, for any fixed complex number z,

$$\lim_{\alpha\to\infty} g_\alpha(z) = 1. \tag{6}$$

The representation (2) can be rewritten as

$$g_\alpha(z) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{-1}^{1} e^{izt}(1-t^2)^{\alpha-\frac{1}{2}}\mathrm{d}t. \tag{7}$$

We can deduce from (7) that

$$\lim_{\alpha\to\infty} g_\alpha(z\sqrt{\alpha}) = e^{-\frac{z^2}{4}}, \tag{8}$$

for any complex number z.

The function defined by (5) is the basic function used in the recent development of a new [1] fractional calculus.

2. The Asymptotic Formula

The aim of this paper is to prove the following result.

Theorem 2.1. *Let x and α be real numbers with $\alpha \geq -\frac{1}{2}$. We have, as $m \to \infty$,*

$$g_\alpha^{(m)}(mx) \sim \frac{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)}{\sqrt{\pi}m^{\alpha+\frac{1}{2}}}\frac{\cos\left(m(x+\frac{\pi}{2}) - (\alpha+\frac{1}{2})\arctan(x)\right)}{(\sqrt{1+x^2})^{\alpha+\frac{1}{2}}}. \tag{9}$$

Formula (9) can also be written as

$$g_\alpha^{(m)}(mx) \sim \frac{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)}{\sqrt{\pi}m^{\alpha+\frac{1}{2}}}\frac{\mathrm{Re}(i^m e^{imx}(1-ix)^{\alpha+\frac{1}{2}})}{(1+x^2)^{\alpha+\frac{1}{2}}}. \tag{10}$$

Note that (9) is an equality for $\alpha = -\frac{1}{2}$ since $g_{-\frac{1}{2}}(z) = \cos(z)$. For $x = 0$, the result follows from the evaluations $g_\alpha^{(m)}(0) = 0$ if m is odd, $g_\alpha^{(m)}(0) = \frac{(2k)!\Gamma(\alpha+1)}{2^{2k}k!\Gamma(\alpha+k+1)}$ if $m = 2k$ is even and Stirling's formula. Also, if we informally replace x by $\frac{x}{m}$, and let $x \to \infty$, then we get a result equivalent to (4), where $z = x$.

The following lemma will be useful.

Lemma 2.2. *Let a_m, b_m be two sequences such that $\lim_{m\to\infty} a_m = \cos(c)$, $\lim_{m\to\infty} b_m = \sin(c)$, where c is a real number. If A is a real number such that $\cos(Am - c) \neq 0$ for large integers m, then*

$$\lim_{m\to\infty} \frac{\cos(Am)a_m + \sin(Am)b_m}{\cos(Am - c)} = 1. \tag{11}$$

The assertion (11) follows easily by writing $a_m = (a_m - \cos(c)) + \cos(c)$ and $b_m = (b_m - \sin(c)) + \sin(c)$.

Proof. The representation (7) can be written as

$$g_\alpha(z) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^1 \cos(zt)(1-t^2)^{\alpha-\frac{1}{2}}\mathrm{d}t, \tag{12}$$

so that

$$g_\alpha^{(m)}(mx) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^1 t^m \cos\left(mxt + m\frac{\pi}{2}\right)(1-t^2)^{\alpha-\frac{1}{2}}\mathrm{d}t. \tag{13}$$

Let us make the change of variable $t = 1 - \frac{v^2}{m}$. We obtain

$$\begin{aligned} g_\alpha^{(m)}(mx) &= \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \cdot \frac{2^{\alpha+\frac{1}{2}}}{m^{\alpha+\frac{1}{2}}} \qquad (14) \\ &\cdot \int_0^{\sqrt{m}} \left(1-\frac{v^2}{m}\right)^m \cos\left(mx + m\frac{\pi}{2} - xv^2\right) v^{2\alpha}\left(1-\frac{v^2}{2m}\right)^{\alpha-\frac{1}{2}}\mathrm{d}v. \end{aligned}$$

We immediately note that $(1-\frac{v^2}{m})^m \leq e^{-v^2}$ (which follows from $1-u \leq e^{-u}$, $0 \leq u \leq 1$) and that $(1-\frac{v^2}{2m})^{\alpha-\frac{1}{2}} \leq 1$ if $\alpha \geq \frac{1}{2}$ while $(1-\frac{v^2}{2m})^{\alpha-\frac{1}{2}} \leq 2$ if $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$, $0 \leq v \leq \sqrt{m}$. These inequalities will permit to enter the limit as $m \to \infty$ inside the integral sign.

Let us denote by R_m the right-hand member of (9). From (14), we have

$$\begin{aligned} \frac{g_\alpha^{(m)}(mx)}{R_m} &= \frac{2(\sqrt{1+x^2})^{\alpha+\frac{1}{2}}}{\Gamma(\alpha+\frac{1}{2})\cos(Am-c)} \qquad (15) \\ &\cdot \int_0^{\sqrt{m}} v^{2\alpha}\left(1-\frac{v^2}{2m}\right)^{\alpha-\frac{1}{2}}\left(1-\frac{v^2}{m}\right)^m \cos(Am - xv^2)\mathrm{d}v, \end{aligned}$$

where $A := x + \frac{\pi}{2}$ and $c := (\alpha+\frac{1}{2})\arctan(x)$. Now consider the sequences a_m,

b_m defined by

$$a_m = \frac{2(\sqrt{1+x^2})^{\alpha+\frac{1}{2}}}{\Gamma(\alpha+\frac{1}{2})} \int_0^{\sqrt{m}} v^{2\alpha} \left(1 - \frac{v^2}{2m}\right)^{\alpha-\frac{1}{2}} \left(1 - \frac{v^2}{m}\right)^m \cos(xv^2) \mathrm{d}v \quad (16)$$

and

$$b_m = \frac{2(\sqrt{1+x^2})^{\alpha+\frac{1}{2}}}{\Gamma(\alpha+\frac{1}{2})} \int_0^{\sqrt{m}} v^{2\alpha} \left(1 - \frac{v^2}{2m}\right)^{\alpha-\frac{1}{2}} \left(1 - \frac{v^2}{m}\right)^m \sin(xv^2) \mathrm{d}v. \quad (17)$$

We have

$$\begin{aligned} \lim_{m\to\infty} a_m &= \frac{2(\sqrt{1+x^2})^{\alpha+\frac{1}{2}}}{\Gamma(\alpha+\frac{1}{2})} \int_0^{\infty} v^{2\alpha} e^{-v^2} \cos(xv^2) \mathrm{d}v \quad (18) \\ &= \cos\left(\left(\alpha + \frac{1}{2}\right) \arctan(x)\right) = \cos(c) \end{aligned}$$

and

$$\begin{aligned} \lim_{m\to\infty} b_m &= \frac{2(\sqrt{1+x^2})^{\alpha+\frac{1}{2}}}{\Gamma(\alpha+\frac{1}{2})} \int_0^{\infty} v^{2\alpha} e^{-v^2} \sin(xv^2) \mathrm{d}v \quad (19) \\ &= \sin\left(\left(\alpha + \frac{1}{2}\right) \arctan(x)\right) = \sin(c). \end{aligned}$$

The integrals, in the right-hand members of (18) and (19), have been evaluated in terms of the gamma function by the way of [2, p. 382]

$$\int_0^{\infty} v^{2\alpha} e^{-(1-ix)v^2} \mathrm{d}v = \frac{\Gamma(\alpha+\frac{1}{2})}{2(\sqrt{1-ix})^{2\alpha+1}}. \quad (20)$$

We can thus write (15) in the form

$$\frac{g_\alpha^{(m)}(mx)}{R_m} = \frac{\cos(Am)a_m + \sin(Am)b_m}{\cos(Am-c)} \quad (21)$$

and the lemma gives

$$\lim_{m\to\infty} \frac{g_\alpha^{(m)}(mx)}{R_m} = 1. \quad (22)$$

The possibility $\cos(Am - c) = \cos\left((x + \frac{\pi}{2})m - (\alpha + \frac{1}{2})\arctan(x)\right) = 0$ is a limiting case in (9).

3. Conclusion

The series representation

$$g_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\alpha+1) z^{2k}}{2^{2k} k! \Gamma(\alpha+k+1)} \tag{23}$$

and Stirling's formula readily give

$$g_\alpha^{(m)}(z) \sim \frac{2^{\alpha+\frac{1}{2}} \Gamma(\alpha+1)}{\sqrt{\pi} m^{\alpha+\frac{1}{2}}} \cos\left(z + m\frac{\pi}{2}\right) \tag{24}$$

as $m \to \infty$. The asymptotic expansion (24) is valid for all complex numbers z and α if α is not a negative integer.

The same idea starting with (1) gives

$$J_n^{(m)}(z) \sim \sqrt{\frac{2}{\pi m}} \cos\left(z + (m-n)\frac{\pi}{2}\right) \tag{25}$$

as $m \to \infty$. In (25), n is an integer and z a complex number. The asymptotic behavior of $J_\alpha^{(m)}(z)$, as $m \to \infty$, is less clear if α is not an integer.

References

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