International Journal of Pure and Applied Mathematics
Volume 47 No. 4 2008, 581-588

DUAL INTEGRAL EQUATIONS APPLICATION IN NONLINEAR THERMOELASTIC PROBLEMS

A.V. Balueva[1 §], M. Matczynski[2]

[1]Department of Mathematics
Gainesville State College
P.O. Box 1358, Gainesville, Georgia, 30503, USA

[2]Department of Mathematics
Kielce University of Technology
7, Al. Tysiaclecia Panstwa Polskiego, Kielce, 25-314, POLAND

Abstract: Considering durability of structures with cracks it is necessary to take into account a possibility of crack surfaces contact. Indeed, under arbitrary loading of structures there is no guarantee, that the cracks will be completely opened. The complete or partial closure can take place if the material is under compression [2]. The crack surfaces contacting leads to changing the stress and deformations within the structure and influences on the conditions of the crack growth and on life-time of structures.

Problem of partial opening of a penny-shaped crack due to the heat sources is considered. Analytical results are obtained by means of Hankel transforms and corresponding dual integral equations. The closed form solutions for heat flux across the crack surfaces and for the aperture of the crack are obtained. The solution is illustrated by several numerical results. The crack openings as functions of the distance between the heat sources and the crack for different initial openings of the crack are shown.

AMS Subject Classification: 74B20
Key Words: thermal stresses, elasticity, fracture

1. Introduction

Let an infinite elastic body contain a penny-shaped crack occupying the region

Received: September 1, 2008

[§]Correspondence author

$r \leq a$ in the plane $z = 0$ referred to the cylindrical coordinates system $(r, \phi,\ z)$. It is assumed that the crack surfaces are free of load and the arbitrary initial crack opening $\bar{\omega}(r)$, as well as the corresponding stresses $\bar{\sigma}_{ij}$, within the body are given. Let two concentrated heat sources of a constant intensity W are located on z-axis symmetrically with respect to the crack plane at the points P $= \pm$ x produce stresses σ^*_{ij} and displacements (u^*, ω^*). In addition, we assume that the crack surfaces are kept at the constant temperature $T^* = 0$. From the superposition principle, the total stresses and displacements can be described as

$$\bar{\bar{\sigma}}_{ij} = \bar{\sigma}_{ij} + \sigma^*_{ij}\ , \qquad \bar{\bar{u}} = \bar{u} + u^*\ , \qquad \bar{\bar{\omega}} = \bar{\omega} + \omega^*\,. \tag{1}$$

With increase of compressive thermoelastic stresses, "overlapping" of crack surfaces can take place and a zone of "negative" displacement can occur, e. g. $\bar{\bar{\omega}} \leq 0$. To avoid this phenomenon, the restrictions on the crack surfaces displacements have to be applied, namely $\bar{\bar{\omega}} \geq 0$ for $0 \leq r \leq a$.

If the crack surfaces come into contact the boundary conditions for the problem on the partially closed crack are as follows

$$\begin{aligned} &T^* = 0\ , \quad \bar{\bar{\sigma}}_{zz} = 0\ , \quad \bar{\bar{\omega}} \geq 0 \quad \text{for} \quad x \leq r \leq a\ , \quad z = 0, \\ &\frac{\partial\, T^*}{\partial\, z} = 0\ , \quad \bar{\bar{\sigma}}_{zz} = 0\ , \quad \bar{\bar{\omega}} = 0 \quad \text{for} \quad 0 \leq r \leq x\ , \quad z = 0, \end{aligned} \tag{2}$$

where T^* describes the temperature within the body while the regions $x \leq r \leq a$ and $0 \leq r \leq x$ represent the opening and contact zone, respectively.

It was proved by Ju and Rowlands [1] that such a boundary problem can be solved iteratively and the process of iterations converges to the exact solution for the contact domain.

To determine stresses σ^*_{ij} and displacements (u^*, ω^*) caused by two heat sources located symmetrically with respect to the crack plane the following boundary value problem has to be solved [3]:

$$\begin{aligned} &\frac{\partial^2 T^*}{\partial\, r^2} + \frac{1}{r}\frac{\partial\, T^*}{\partial\, r} + \frac{\partial^2 T^*}{\partial\, z^2} = -\frac{W}{\lambda}[\delta(z-\xi) + \delta(z+\xi)], \\ &T^*(r,0) = 0 \quad \text{for} \quad 0 \leq r \leq a\ , \quad \frac{\partial T^*(r,0)}{\partial\, z} = 0 \quad \text{for} \quad r \geq a, \end{aligned} \tag{3}$$

$$\begin{aligned} &2(1-\nu)\left(\frac{\partial^2 u^*}{\partial\, r^2} + \frac{1}{r}\frac{\partial\, u^*}{\partial\, r} - \frac{u^*}{r}\right) + (1-2\nu)\frac{\partial^2 u^*}{\partial\, z^2} + \frac{\partial^2 \omega^*}{\partial\, r\, \partial\, z} = 2(1+\nu)\alpha\frac{\partial\, T^*}{\partial\, r}, \\ &(1-2\nu)\left(\frac{\partial^2 \omega^*}{\partial\, r^2} + \frac{1}{r}\frac{\partial\, \omega^*}{\partial\, r}\right) + 2(1-\nu)\frac{\partial^2 \omega^*}{\partial\, z^2} + \frac{\partial}{\partial\, z}\left(\frac{\partial u^*}{\partial r} + \frac{u^*}{r}\right) \\ &\qquad = 2(1+\nu)\alpha\frac{\partial\, T^*}{\partial\, z}, \end{aligned} \tag{4}$$

$$\sigma^*_{zz}(r,\,0)=0 \quad \text{for} \quad 0\le r\le a\ , \qquad \sigma^*_{rz}(r,\,0)=0 \quad \text{for} \quad r\ge 0,$$
$$\omega^*(r,\,0)=0 \quad \text{for} \quad r\ge a,$$

where α,λ,ν are the coefficients of thermal expansion, heat conductivity and Poisson's ratio, respectively.

2. Temperature Distribution

Distribution of temperature $T^*(r,z)$ within the body containing a crack is represented as

$$T^*(r,z)=T^0(r,z)+T(r,z)\,, \tag{5}$$

where $T^{\,0}$ describes the temperature in the body without crack and caused by the heat sources located at the points $P\ (\pm\xi)$ while T stands for the temperature caused by the application of the prescribed temperature on the crack surfaces and according to Nowacki [3]

$$\begin{aligned} &T^0\,(r,z)=\frac{W}{4\pi\lambda}\left(\frac{1}{R_1}+\frac{1}{R_2}\right),\\ &R_1=\sqrt{r^2+(z-\xi)^2},\quad R_2=\sqrt{r^2+(z+\xi)^2}\,. \end{aligned} \tag{6}$$

To define the temperature $T(r,\,z)$, the following boundary value problem has to be solved:

$$\begin{aligned} &\frac{\partial^2 T}{\partial\, r^2}+\frac{1}{r}\frac{\partial\, T}{\partial\, r}+\frac{\partial^2 T}{\partial\, z^2}=0,\\ &T(r,\,0)=-T^0(r,\,0)=-\frac{1}{2\pi}g(r) \qquad \text{for} \qquad 0\le r\le a,\\ &\frac{\partial\, T(r,0)}{\partial\, z}=0 \quad \text{for} \quad r\ge a, \end{aligned} \tag{7}$$

where

$$g(r)=\frac{W}{\lambda}\frac{1}{\sqrt{r^2+\xi^2}}\,. \tag{8}$$

The boundary value problem (7)-(8) is reduced to the following dual integral equations [3]:

$$\begin{aligned} &\int_0^\infty \Phi(u)J_0(ur)du=\frac{1}{2\pi}g(r) \qquad \text{for} \quad 0\le r\le a,\\ &\int_0^\infty u\Phi(u)J_0(ur)du=0 \qquad \text{for} \quad r\ge a, \end{aligned} \tag{9}$$

and the heat flux across the crack surfaces is given by

$$q(r) = \int_0^\infty u\Phi(u) J_0(ur) du \qquad \text{for} \quad 0 \le r \le a, \tag{10}$$

where $J_0(r)$ is the Bessel function of the first kind. Solution of the dual integral equations (9) can be represented in the form [3]:

$$\Phi(u) = \frac{1}{\pi^2} \int_0^a \cos(ut) \left(\frac{d}{dt} \int_0^t \frac{s\, g(s)}{\sqrt{t^2 - s^2}} ds \right) dt. \tag{11}$$

From (8), (10) and (11) the heat flux across the crack surfaces for $0 \le r \le a$ is finally given by

$$q(r) = \frac{W}{\lambda \pi^2} \frac{\xi}{\xi^2 + r^2} \times \left[\frac{1}{\sqrt{a^2 - r^2}} + \frac{1}{\sqrt{\xi^2 + r^2}} \left(\pi - \arctan \sqrt{\frac{a^2 - r^2}{r^2 + \xi^2}} \right) \right]. \tag{12}$$

3. Opening of the Crack

From the superposition principle, stresses σ_{ij}^* and displacements $(u^*,\ \omega^*)$ corresponding to problem (4) can be represented as

$$\begin{aligned} \sigma_{ij}^* &= \sigma_{ij}^0 + \sigma_{ij}^{(1)} + \sigma_{ij}\,, \\ u^* &= u^0 + u^{(1)} + u \quad , \quad \omega^* = \omega^0 + \omega^{(1)}\,. \end{aligned} \tag{13}$$

Here, σ_{ij}^0, and (u^0, ω^0) describe stresses and displacements caused by the temperature T^0 in the body without crack; $\sigma_{ij}^{(1)}$, and $u^{(1)}, \omega^{(1)}$ stand for stresses and displacements within the body containing the crack, which is loaded by stresses equal to evaluated from the previous step but with opposite sign. Notations σ_{ij}, and (u, ω) denote stresses and displacements in the body with the crack with the surfaces, which are free of mechanical load but heated with the prescribed temperature given by equation $(7)_2$.

Stresses σ_{ij}^0 can be read as [3]

$$\begin{aligned} \frac{\sigma_{zz}^0(r,z)}{2\,\mu} &= A\ \left[\left(r^2 R_1^{-2} - 2 \right) R_1^{-1} + \left(r^2 R_2^{-2} - 2 \right) R_2^{-1} \right], \\ \frac{\sigma_{rz}^0(r,z)}{2\,\mu} &= A\, r\ \left[(\xi - z)\ R_1^{-3} - (\xi + z)\ R_2^{-3} \right], \end{aligned} \tag{14}$$

where functions R_1 and R_2 are described by equations $(6)_{2-3}$, μ is a Lamé constant and

$$A = \frac{mW}{8\pi\lambda}\,, \quad m = \frac{1+\nu}{1-\nu}\,\alpha.$$

According to the solution given by Sneddon [4], the opening of the crack caused by the vertical load applied to its surfaces is defined for $0 \le \rho \le 1$ by

$$\omega^{(1)}(\rho) = -\frac{1-\nu}{\pi\,\mu}\int_{\rho}^{1}\frac{1}{\sqrt{t^2-\rho^2}}\left(\int_0^t \frac{s\,f(s)}{\sqrt{t^2-s^2}}ds\right)dt, \tag{15}$$

where according to equation $(14)_1$

$$\begin{aligned} &f(\rho) = -\sigma^0_{zz}(\rho,0) = \frac{4\mu A}{a}\frac{1}{\sqrt{\rho^2+c^2}}\left(1+\frac{c^2}{\rho^2+c^2}\right), \\ &\rho = \frac{r}{a} \quad , \quad c = \frac{\xi}{a}. \end{aligned} \tag{16}$$

From equation (16)1 and equation (15), the crack opening $\omega^{(1)}(\rho)$ can be derived as follows

$$\begin{aligned} \omega^{(1)}(\rho) = &-A_0\left(\frac{1}{4}+\frac{1}{\pi}\arcsin\frac{1-c^2}{1+c^2}\right)\times\ln(1+\sqrt{1-\rho^2}) \\ &-\left(\frac{1}{4}+\frac{1}{2\pi}\arcsin\frac{\rho^2-c^2}{\rho^2+c^2}\right)\ln\rho+\frac{2c}{\sqrt{\rho^2+c^2}}\arctan\sqrt{\frac{1-\rho^2}{c^2+\rho^2}}-F(\rho,c)\,, \end{aligned} \tag{17}$$

where

$$A_0 = 4A(1-\nu), \qquad F(\rho,c) = \frac{c}{2\pi}\int_{\rho}^{1}\frac{\ln(t+\sqrt{t^2-\rho^2}}{t^2+c^2}dt. \tag{18}$$

Problem of finding stresses σ_{ij} and displacements (u,ω) in the body containing the crack opened due applied temperature on its surfaces is reduced to the solution of the following dual integral equations [4]:

$$\begin{aligned} &\textstyle\int_0^\infty \eta\psi(\eta)J_0(\rho\eta)d\eta = h(\rho) \qquad \text{for} \quad 0 \le \rho \le 1\,, \\ &\textstyle\int_0^\infty \psi(\eta)J_0(\rho\eta)d\eta = 0 \qquad \text{for} \quad \rho \ge 1 \end{aligned} \tag{19}$$

where

$$h(\rho) = (1+\nu)\,\alpha\,a^2\int_0^\infty Q(\eta)J_0(\rho\eta)d\eta, \qquad Q(\eta) = \int_0^1 \rho q(\rho)J_0(\rho\eta)d\rho, \tag{20}$$

and $q(\rho)$ representing the heat flux across crack surfaces is given by equation (12).

The solution of the dual integral equations (19) has the following form [4]

$$\psi(\eta) = \frac{2}{\pi}\int_0^1 \sin(\eta s)\left(\int_0^s \frac{x\,h(x)}{\sqrt{s^2-x^2}}dx\right)ds \tag{21}$$

and the corresponding crack opening ω is read as

$$\omega = \int_0^\infty \psi(\eta)J_0(\rho\eta)d\eta \qquad \text{for} \quad 0 \le \rho \le 1. \tag{22}$$

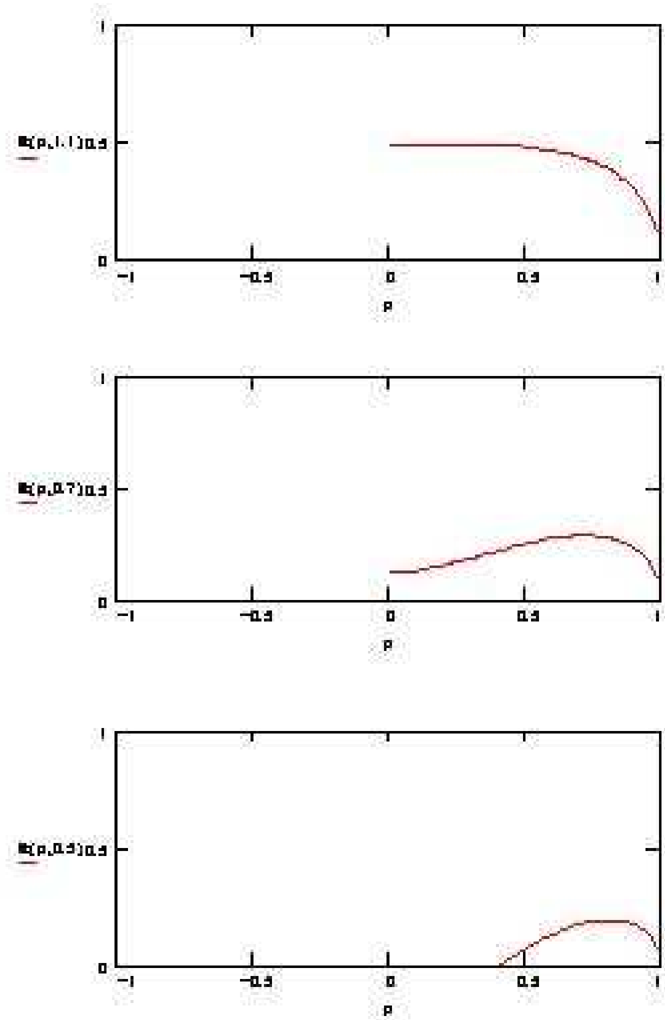

Figure 1: The total crack apertures $R(\rho,\ c)$ for the distances from the heat sources up to the crack surfaces c = 1.1; 0.7; 0.5

From equations (20)-(22) the crack opening $\omega(\rho)$ can be given as follows

$$\omega(\rho) = 2A_0 \Bigg[B(c)\ln(1+\sqrt{1-\rho^2}) + C(c)\ln\rho + \ln\frac{c+\sqrt{1+c^2}}{c+\sqrt{\rho^2+c^2}} + \frac{c}{\sqrt{1+c^2}}L(\rho,c) \Bigg], \quad (23)$$

where

$$B(c) = 1 - \frac{c}{\sqrt{1+c^2}} - \frac{1}{\pi}\arctan\frac{1}{c}\,, \tag{24}$$

$$C(c) = \frac{1}{\pi}\left[\arctan\frac{1}{c} - \frac{c}{\sqrt{1+c^2}}\ln\frac{\sqrt{1+c^2}+1}{\sqrt{1+c^2}-1}\right],$$

$$L(\rho,c) \;=\; \frac{1}{\pi}\int_\rho^1 \frac{1}{t}\ln\frac{\sqrt{1+c^2}-\sqrt{1-t^2}}{\sqrt{1+c^2}+\sqrt{1-t^2}}\,dt \;+\; \frac{1}{\pi}\int_\rho^1 \frac{1}{t}\arctan\sqrt{\frac{1-t^2}{c^2+t^2}}\,dt\,.$$

Finally the total crack's opening $\omega^*(\rho)$ is given by

$$\omega^*(\rho) = \omega^0(\rho) + \omega(\rho)\,, \tag{25}$$

where $\omega^0(\rho)$ and $\omega(\rho)$ are given by equation (17) and equation (24), respectively.

4. Numerical Illustration of Results

The numerical results are represented in form of the graphs of thermal displacements of the surfaces of the crack, ω^*, for various distances from the thermal sources up to the crack surfaces (Figure 1). Gradual closing of the cracks with various initial openings is investigated upon approaching the heat sources to their surfaces. For this purpose three cracks with initial opening $\omega = b\,(1-\rho^2)^{0.5\beta}$ are considered with $b = 1.2$; $\beta = 0.5$; 1; 2.

Numerical results show that for the cracks with the initial opening with $\beta = 0.5$; 1, their closure occurs in the center (Figure 1), whereas for the cracks with the initial opening with $\beta = 2$, which corresponds to the case of smooth overlapping of the crack surfaces on its contour, the cracks closure is located near to their contour.

References

[1] S.H. Ju, R.E. Rowlands, Thermoelastic determination of crack-tip coordinates in composites, *Int. J. Solids and Structures*, **44** (2007), 4845-4859.

[2] M. Matczynski, M. Sokolowski, Crack opening and closure under the action of mechanical and thermal loads, *Theor. Appl. Fract. Mech.*, **11** (1989), 187-198.

[3] W. Nowacki, *Thermoelasticity*, PWN, Pergamon Press (1986).

[4] I.N. Sneddon, M. Lowengrub, *Crack Problems in the Classical Theory of Elasticity*, John Willey and Sons, Inc. (1969).