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OPTIMAL CANDIDATES LOCATION IN MULTICANDIDATE SPATIAL THEORY OF VOTING

S. Gikiri Thuo

Department of Mathematics
Florida A&M University
Tallahassee, FL 32307, USA
e-mail: gikiri.thuo@famu.edu

Abstract: This paper analyzes the proximity spatial models for cumulative voting. We will identify conditions necessary for a symmetric equilibrium to exist when voters' ideal points have a standard normal distribution. We will show, through a proposition and proving it, that under a cumulative voting heuristic, candidates tend to adopt centrifugal positions, away from the median voter. With this in mind, we will place two candidates away from the median, on the opposite sides and investigate conditions necessary for a symmetric equilibrium to exist. We will accomplish this by placing a third candidate within an ϵ neighborhood of either one of the other two candidates. From this analysis, we will develop an equation whose solutions provide the only possibility for a symmetric equilibrium to exist. Using optimization techniques, see Kushner et al [12] and Thuo [20], we will approximate those solutions by a nice elementary function whose properties we know.

AMS Subject Classification: 91B12
Key Words: spatial models, optimization, cumulative voting, equilibrium

1. Introduction

In spatial models the ideals of individuals in a population are represented by points in R^n, where the coordinates in the i-th dimension represents the position of the citizen on a particular issue or interest. In the proximity class of spatial models, see Adams [1], Enelow et al [8], and McKelvey et al [14],

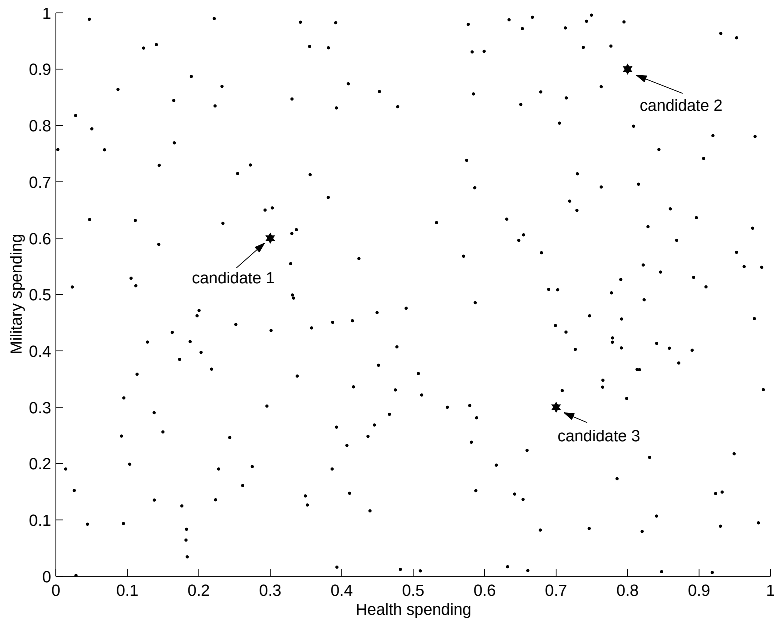

Figure 1: A spatial model

a voter chooses the candidate closest to his or her ideal point often, though not necessarily, under the Euclidean distance metric, see Royden [17]. Most of the extensive literature on spatial theory has been developed for elections of a single candidate. Relatively little is written on spatial models for at-large elections, where $n > 1$ candidates are elected. Typically, at-large elections in multimember districts as shown in Gerber et al [9], are held using the plurality voting method. The cumulative voting method, which is used in this paper and described in Cooper [5], Guinier [10] and Thuo [19] is a different multiple-vote system. Voters in a k-seat constituency are given k votes each and the option of cumulation means that they can cast all of their votes for a single candidate (known as plumping), or split them however they wish (usually in integer quantities, though not always) among two or more candidates. The winning candidates are determined by a simple plurality vote rule; the top k candidates are awarded the k available seats. This voting method has been used in a variety of contexts, such as state legislative elections in Illinois, school board elections and corporate board elections. The effect of this plumping strategy is a much greater chance of minority representation in the elected body than often occurs either in single-member districts or in multimember districts elected by plurality. We begin our analysis by providing a graphical representation of a 2 dimensional spatial model as shown by Figure 1 and then introducing our cumulative voting heuristic where $\beta > 1$.

1.1. Cumulative Voting Heuristic

Assume we have 3 candidates vying for 2 seats and voters are distributed according to some distribution function F, i.e. $v \sim F(v, \theta)$. Our cumulative voting heuristic, see Cooper [5] and Thuo [19] for a detailed analysis, assumes that a voter plumps his votes, awarding both to the nearest candidate if that candidate is much closer to the voter than the second candidate or if the second nearest candidate is not much closer to the voter than the third candidate. Specifically we plump if:

$$d(v, c_1) < \frac{1}{\beta} d(v, c_2)\,, \tag{1}$$

or

$$d(v, c_2) > \frac{1}{\beta} d(v, c_3)\,, \tag{2}$$

and otherwise split votes if:

$$d(v, c_1) \geq \frac{1}{\beta} d(v, c_2)\,, \tag{3}$$

and

$$d(v, c_2) \leq \frac{1}{\beta} d(v, c_3)\,, \tag{4}$$

where $\beta > 1$.

2. Problem Setting

We first investigate intervals, on which voters plump or split their votes, see Cooper [5], Kolliopoulos et al [11] and Thuo [19]. We assume that we have three candidates arrayed on a single predictive dimension, R^1 or a subset of R^1 and that voters' ideal points have a standard normal distribution, i.e. $v \sim N(0, 1)$. The candidate located on the left of the other candidates is denoted by L, the candidate located on the right of the other candidates is denoted by R and the candidate located in between L and R is denoted by M. The total number of votes that candidates L, M and R receive is denoted by V_L, V_M and V_R,

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respectively. Consider a voter $v \in (-\infty, \frac{1}{2}(L+M))$, where $\frac{1}{2}(L+M)$ is the midpoint of L and M. Voter v has L as his nearest candidate and M as his second-nearest candidate. Likewise, the midpoints of L and R and of M and R, denoted by $\frac{1}{2}(L+R)$ and $\frac{1}{2}(M+R)$, respectively, are significant for determining intervals of preference order for a voter v in the predictive dimension space. Figure 2 shows an illustrative example of candidates L, M and R locations and their respective midpoints. Although the figure indicates $M > \frac{1}{2}(L+R)$, this inequality is not necessarily true. However, all other inequalities described by the figure do hold true.

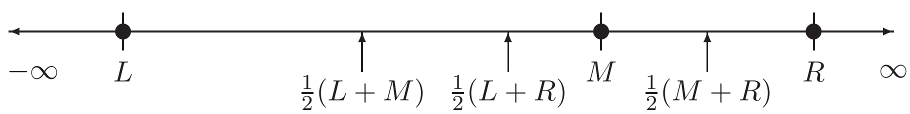

Figure 2: Candidates L, M, and R and intervals of voters' preference orders

By (1), a voter v in the interval $(-\infty, L)$ plumps his two votes for candidate L, his nearest candidate, if

$$\begin{array}{rrcl} & L-v & < & \frac{1}{\beta}(M-v) \\ \Leftrightarrow & \beta L - \beta v & < & M - v \\ \Leftrightarrow & (\beta-1)v & > & \beta L - M \\ \Leftrightarrow & v & > & \frac{\beta L - L + L - M}{\beta - 1} \\ \Leftrightarrow & v & > & L - \frac{M-L}{\beta-1}\,. \end{array}$$

Also by (1), a voter v plumps votes for L in $(L, \frac{1}{2}(L+M))$ if

$$v - L < \frac{1}{\beta}(M-v) \Leftrightarrow v < L + \frac{M-L}{\beta+1}\,.$$

By (2), voter v plumps votes for L in $(-\infty, \frac{1}{2}(L+M))$ if

$$M - v > \frac{1}{\beta}(R-v) \Leftrightarrow v < M - \frac{R-M}{\beta-1}\,.$$

In summary, v plumps his two votes for L in $v \in (-\infty, \frac{1}{2}(L+M))$ if

$$L - \frac{M-L}{\beta-1} < v < L + \frac{M-L}{\beta+1}\,, \tag{5}$$

or if

$$v < M - \frac{R-M}{\beta - 1}\,, \tag{6}$$

otherwise, v splits between L and M. Analyzing similarly, for $v \in (\frac{1}{2}(L + M), \frac{1}{2}(L + R))$, v plumps for M if

$$M - \frac{M-L}{\beta+1} < v < M + \frac{M-L}{\beta-1}\,, \tag{7}$$

or if

$$v > L + \frac{R-L}{\beta+1}\,, \tag{8}$$

splitting between M and L otherwise. For $v \in (\frac{1}{2}(L+R), \frac{1}{2}(M+R))$, v plumps for M if

$$M - \frac{R-M}{\beta-1} < v < M + \frac{R-M}{\beta+1}\,, \tag{9}$$

or if

$$v < R - \frac{R-L}{\beta+1}\,, \tag{10}$$

splitting between M and R otherwise. Lastly, for $v \in (\frac{1}{2}(M+R), \infty)$, v plumps for R if

$$R - \frac{R-M}{\beta+1} < v < R + \frac{R-M}{\beta-1}\,, \tag{11}$$

or if

$$v > M + \frac{M-L}{\beta-1}\,, \tag{12}$$

splitting between R and M otherwise. We wish to establish symmetric equilibrium position for the candidates. That is to say, are there positions in this single predictive dimension where two of the three candidates can situate themselves such that neither can be outvoted by the third candidate? If we allow the third candidate to share an identical location with one of the other two candidates, then the third candidate would have an equal probability of winning a seat with the candidate whose position he shares. To search for the two equilibrium position we begin by establishing, through a proposition and proving it, that under our cumulative voting heuristic candidates tend to adopt centrifugal, see Cox [7] and Margar et al [13], positions away from the median voter.

Theorem 1. *In a two seat, three candidate race in our cumulative voting heuristic there is no centrist equilibrium with all three candidate located at the median voter.*

Proof. Suppose candidates L, M and R were located at the median voters' ideal point as shown in Figure 3. Then each candidate has an equal probability of winning a seat because they all share the same position. It will be helpful in the analysis to normalize the vote total to two, the number of votes alloted to each voter. Thus, the candidates' normalized shares of votes add up to two: i.e., $V_L + V_M + V_R = 2$. Here, with the three candidates sharing an identical position, each candidate has an equal probability of receiving any voters' vote. Thus, their normalized shares of the total vote are $V_L = V_R = V_M = \frac{2}{3}$.

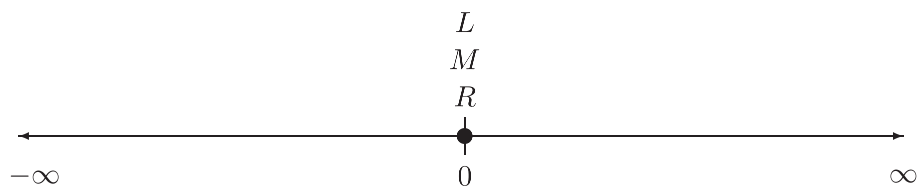

Figure 3: All candidates located at the median

Suppose now candidate L makes an infinitesimally small hop to the left, moving from $L = 0$ to $L = -\epsilon$ ($\epsilon > 0$) then, as Figure 4 illustrates, all voters to the left of him have him as their nearest candidate. This results to the following vote tallies.

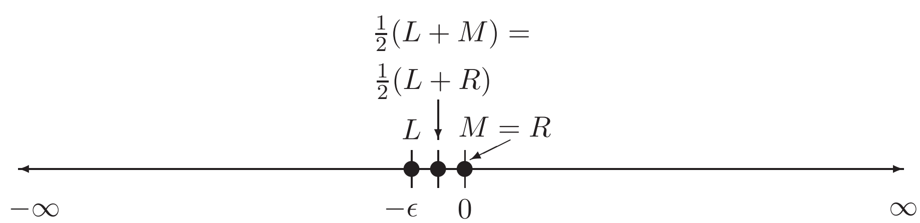

Figure 4: Candidate L hops to the immediate left of M and R

On the interval $(-\infty, \frac{1}{2}(L+M)) = (-\infty, \frac{1}{2}(L+R))$, a voter v plumps his two votes for candidate L by (1) if

$$L - \frac{M-L}{\beta-1} < v < L + \frac{M-L}{\beta+1}, \tag{13}$$

and by (2) if

$$v < M - \frac{R-M}{\beta-1}, \tag{14}$$

otherwise, v splits his votes between L and M or R. Substituting $-\epsilon$ for L and

0 for M and R we get that $v \in (-\infty, \frac{-\epsilon}{2})$ plumps his votes for L if

$$-\epsilon - \frac{\epsilon}{\beta - 1} < v < -\epsilon + \frac{\epsilon}{\beta + 1},$$

or if

$$v < 0.$$

If we let $\epsilon \to 0$, then $(-\infty, \frac{-\epsilon}{2}) \to (-\infty, 0)$, so any voter $v \in (-\infty, 0)$ plumps votes for L by (2). Thus the vote total for L is

$$V_L = 2\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 2\Phi(0) = 1,$$

where Φ is the standard normal cumulative distribution function (c.d.f.). This means that by hopping from 0 to $-\epsilon$, L garnered voters to the left of him to plump votes for him by (2), receiving half of the total votes, guaranteeing him one of the two contested seats. M and R on the other hand are left to share the remaining half of the grand total votes. They both have an equal chance of winning the other seat. L was, therefore, able to improve his electoral chances by unilaterally defecting from the median. The other two candidates, M and R, faces the same incentives. This proves that $L = M = R = 0$ is not an equilibrium position. □

What this is telling us is that, if a equilibrium exists under our cumulative voting heuristic, then candidates will be encouraged to take centrifugal (noncentrist) positions.

3. Third Candidate within an ϵ Neighborhood of r

Suppose the third candidate is located at an infinitesimal distance to the left of r as shown in Figure 5 below. When will this candidate win?

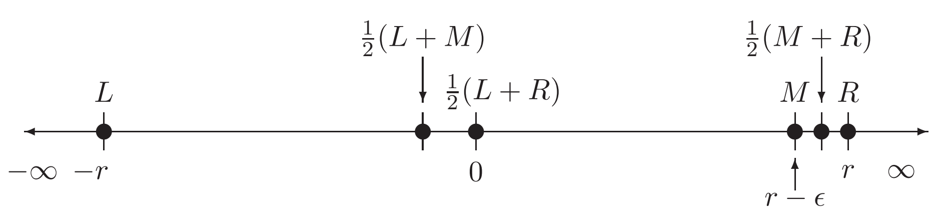

Figure 5: Third candidate to the immediate left of r

First we analyze the leftmost interval, $(-\infty, \frac{1}{2}(L+M))$, where L is the closest candidate to voters in that region. A voter $v \in (-\infty, \frac{1}{2}(L+M))$ plumps his two votes for candidate L by (1) if

$$L-\frac{M-L}{\beta-1}<v<L+\frac{M-L}{\beta+1}\,,$$

and by (2) if

$$v<M-\frac{R-M}{\beta-1}\,,$$

otherwise v splits his two votes between L and M. Substituting L, M and R by $-r$, $r-\epsilon$ and r respectively, we get that v plumps his votes for L if

$$-r-\frac{2r-\epsilon}{\beta-1}<v<-r+\frac{2r-\epsilon}{\beta+1}\,,$$

or if

$$v<r-\epsilon+\frac{\epsilon}{\beta-1}\,.$$

If we let $\epsilon\to 0$, then a voter $v\in(-\infty,\frac{1}{2}(L+M))=(-\infty,0)$ plumps votes for L if

$$-r-\frac{2r}{\beta-1}<v<-r+\frac{2r}{\beta+1}\,, \tag{15}$$

or if

$$v<r\,. \tag{16}$$

Next we consider the interval $(\frac{1}{2}(L+M),\frac{1}{2}(L+R))=(-\frac{\epsilon}{2},0)$. This region diminishes as $\epsilon\to 0$. Since this interval does not yield any contribution to the vote tally, there is no need for further calculations.

For the next interval, $(\frac{1}{2}(L+R),\frac{1}{2}(M+R))=(0,r-\frac{\epsilon}{2})$, a voter v plumps his two votes for M his nearest candidate, if

$$r-\epsilon-\frac{\epsilon}{\beta-1}<v<r-\epsilon+\frac{\epsilon}{\beta+1}\,,$$

or if

$$v<r-\frac{2r}{\beta+1}\,,$$

otherwise v splits his two votes between M and R.

If we again let $\epsilon\to 0$, the inequality due to (1) diminishes and so we plump votes for M in the interval $(0,r)$ only if

$$v<r-\frac{2r}{\beta+1}\,, \tag{17}$$

and split votes between M and R if

$$r-\frac{2r}{\beta+1}\le v<r\,. \tag{18}$$

Finally we consider the rightmost interval $(\frac{1}{2}(M+R),\infty) = (r-\frac{\epsilon}{2},\infty)$. A voter v, in this interval, plumps his two votes for R if

$$r - \frac{\epsilon}{\beta+1} < v < r + \frac{\epsilon}{\beta-1}\,,$$

or if

$$v > r - \epsilon + \frac{2r-\epsilon}{\beta-1}\,,$$

and split his votes between M and R otherwise. Letting $\epsilon \to 0$, the inequality due to (1) diminishes and hence v will plump votes for R in this region if

$$v > r + \frac{2r}{\beta-1}\,, \tag{19}$$

and split votes between M and R if

$$r < v \le r + \frac{2r}{\beta-1}\,. \tag{20}$$

We now wish to tally the votes. Since the voters' ideal points are normally distributed with $\mathcal{N}(0,1)$, we can think of the vote tallies, V_L, V_M and V_R for L, M and R, respectively, as the area under the standard normal density function, see Casella [4], for each subinterval, multiplied by the total number of votes that a candidate receives from voters in that subinterval. For example, V_L is the area under the curve in the interval $(-\infty,0)$ multiplied by 2, because of inequality (6) and the fact that $r>0$. Therefore

$$V_L = 2\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = 2\Phi(0) = 1\,.$$

Since $V_L = 1 > \frac{2}{3}$, L wins one of the seats. Recall, $V_L + V_M + V_R = 2$, which means either V_M and/or V_R is less than $\frac{2}{3}$ and therefore less than V_L.

From inequalities (7) to (12), we determine that

$$\begin{aligned} V_M = {} & 2\int_{0}^{r-\frac{2r}{\beta+1}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx + \int_{r-\frac{2r}{\beta+1}}^{r} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx \\ & + \int_{r}^{r+\frac{2r}{\beta-1}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = \Phi(r-\frac{2r}{\beta+1}) + \Phi(r+\frac{2r}{\beta-1}) - 1\,, \end{aligned}$$

and

$$\begin{aligned} V_R = {} & 2\int_{r+\frac{2r}{\beta-1}}^{r} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx + \int_{r-\frac{2r}{\beta+1}}^{r} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx \\ & = 2 - \Phi(r-\frac{2r}{\beta+1}) - \Phi(r+\frac{2r}{\beta-1})\,. \end{aligned}$$

Therefore

$$\begin{aligned} V_M > V_R \quad &\Leftrightarrow \quad \Phi(r - \frac{2r}{\beta+1}) + \Phi(r + \frac{2r}{\beta-1}) - 1 \\ &\qquad > 2 - \Phi(r - \frac{2r}{\beta+1}) - \Phi(r + \frac{2r}{\beta-1}) \\ &\Leftrightarrow \quad 2\Phi(r + \frac{2r}{\beta-1}) + 2\Phi(r - \frac{2r}{\beta+1}) > 3 \\ &\Leftrightarrow \quad \Phi(r + \frac{2r}{\beta-1}) + \Phi(r - \frac{2r}{\beta+1}) > \frac{3}{2} \end{aligned}$$

and similarly,

$$V_M < V_R \quad \Leftrightarrow \quad \Phi(r + \frac{2r}{\beta-1}) + \Phi(r - \frac{2r}{\beta+1}) < \frac{3}{2}\,.$$

This means that if we put a third candidate immediately to the left of r, then that candidate will win one of the two seats (together with L) if $r \mid \beta$ (i.e., r given β) is chosen such that $\Phi(r + \frac{2r}{\beta-1}) + \Phi(r - \frac{2r}{\beta+1}) > \frac{3}{2}$.

Suppose now we place the third candidate at an infinitesimal distance to the right of r as illustrated by Figure 6. Similar calculations as those developed

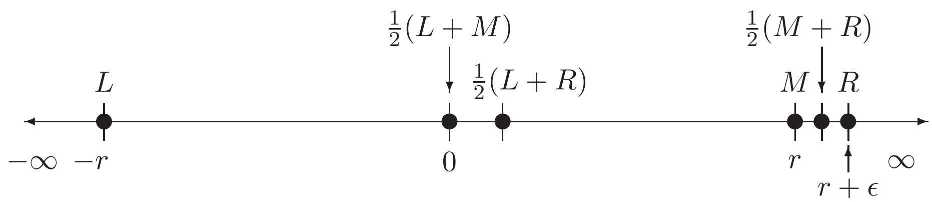

Figure 6: Third candidate to the immediate right of r

for Figure 5 hold. Therefore

$$\begin{aligned} V_L &= 2\Phi(0) = 1\,, \\ V_M &= \Phi(r - \frac{2r}{\beta+1}) + \Phi(r + \frac{2r}{\beta-1}) - 1\,, \\ V_R &= 2 - \Phi(r - \frac{2r}{\beta+1}) - \Phi(r + \frac{2r}{\beta-1})\,. \end{aligned}$$

Again L wins one of the two seats since $V_L > \frac{2}{3}$. Also

$$V_M > V_R \quad \Leftrightarrow \quad \Phi(r + \frac{2r}{\beta-1}) + \Phi(r - \frac{2r}{\beta+1}) > \frac{3}{2}\,,$$

and

$$V_M < V_R \quad \Leftrightarrow \quad \Phi(r + \frac{2r}{\beta-1}) + \Phi(r - \frac{2r}{\beta+1}) < \frac{3}{2}\,.$$

Therefore R, the third candidate, will win the second seat if

$$\Phi(r+\frac{2r}{\beta-1})+\Phi(r-\frac{2r}{\beta+1})<\frac{3}{2}.$$

This implies that if R is positioned within an ϵ neighborhood of r, then candidates L M, and R are not in a Nash equilibrium state, see Nash [16], if

$$\Phi(r+\frac{2r}{\beta-1})+\Phi(r-\frac{2r}{\beta+1})<\frac{3}{2},$$

or if

$$\Phi(r+\frac{2r}{\beta-1})+\Phi(r-\frac{2r}{\beta+1})>\frac{3}{2}.$$

Therefore, it follows that, the only possible way to have a symmetric Nash equilibrium state with candidates at $-r$ and r is if

$$\Phi(r+\frac{2r}{\beta-1})+\Phi(r-\frac{2r}{\beta+1})=\frac{3}{2}.$$

Let us now investigate what happens if, instead, the third candidate is positoned in an ϵ - neighborhood of $-r$ as illustrated by Figure 7 and Figure 8.

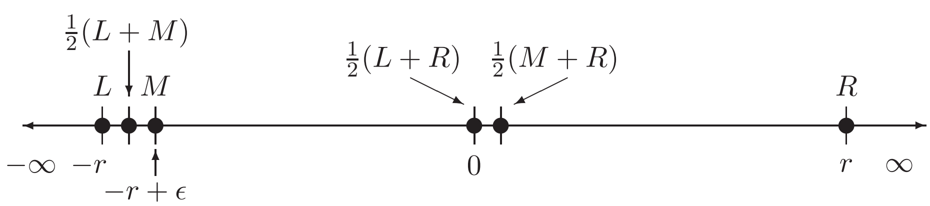

Figure 7: Third candidate to the immediate right of $-r$

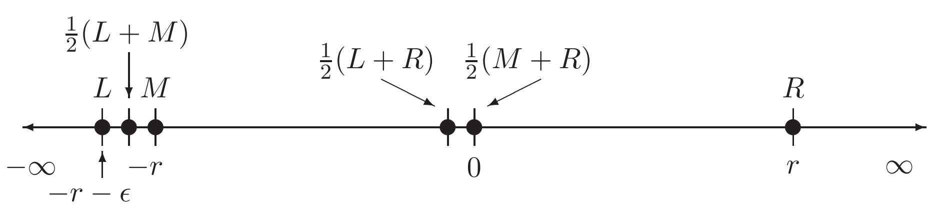

Figure 8: Third candidate to the immediate left of $-r$

Remark 1. If $X \sim \mathcal{N}(0,1)$ then $\Phi(-x)=1-\Phi(x)$.

Using Remark 1 and Figure 5 and Figure 6 we obtain the following results for figure 8 and Figure 7, respectively.

$$\begin{aligned} V_R &= 2(1-\Phi(0)) = 2(1-\frac{1}{2}) = 1\,, \\ V_M &= 1-\Phi(-r+\frac{2r}{\beta+1}) + 1-\Phi(-r-\frac{2r}{\beta-1}) - 1 \\ &= 1-\Phi(-r+\frac{2r}{\beta+1}) - \Phi(-r-\frac{2r}{\beta-1})\,, \\ V_L &= 2-(1-\Phi(-r+\frac{2r}{\beta+1})) + (1-\Phi(-r-\frac{2r}{\beta-1}) \\ &= \Phi(-r+\frac{2r}{\beta+1}) + \Phi(-r-\frac{2r}{\beta-1})\,. \end{aligned}$$

Therefore, as we expected, candidate R is guaranteed of a seat since $V_R > \frac{2}{3}$. On the other hand

$$V_L > V_M \quad\Leftrightarrow\quad \Phi(-r-\frac{2r}{\beta-1}) + \Phi(-r+\frac{2r}{\beta+1}) > \frac{1}{2}\,,$$

and

$$V_L < V_M \quad\Leftrightarrow\quad \Phi(-r-\frac{2r}{\beta-1}) + \Phi(-r+\frac{2r}{\beta+1}) < \frac{1}{2}\,.$$

Following the same logic as established for Figure 5 and Figure 6, we conclude that if equilibrium exists, then we must have

$$\Phi(-r-\frac{2r}{\beta-1}) + \Phi(-r+\frac{2r}{\beta+1}) = \frac{1}{2}\,, \tag{21}$$

which, by Remark 1, is equivalent to

$$\Phi(r+\frac{2r}{\beta-1}) + \Phi(r-\frac{2r}{\beta+1}) = \frac{3}{2}\,. \tag{22}$$

Therefore it suffices to focus our analysis with the third candidate positioned on $[0,\infty)$ rather than $(-\infty,\infty)$. Solving for values of $r \mid \beta$ in equation (22) we obtain the results illustrated by Figure 9.

3.1. Properties of $r = f(\beta)$

Now let us investigate Figure 9 in more detail.

Definition 3.1.1. Let $f(\beta)$ be such that $r = f(\beta)$, where $f(\beta)$ is the unique value of r satisfying

$$\Phi(r+\frac{2r}{\beta-1}) + \Phi(r-\frac{2r}{\beta+1}) = \frac{3}{2} \tag{23}$$

for a given value of β. First we wish to examine the asymptotic properties of

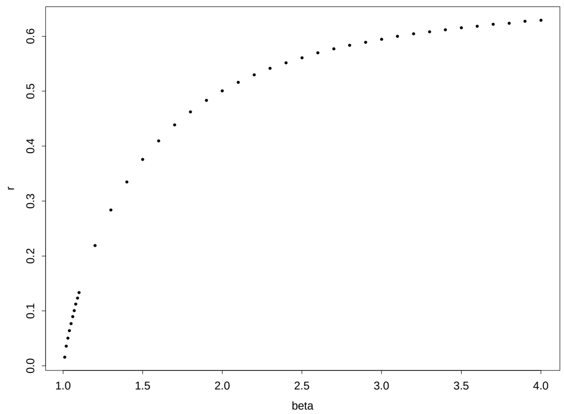

Figure 9: Equilibrium positions for $r \mid \beta$

$f(\beta)$, where $\beta \in (1, \infty)$. Thus by (23), as $\beta \to \infty$ we have

$$\Phi(r) + \Phi(r) = \frac{3}{2} \Rightarrow \quad 2 \cdot \Phi(r) = \frac{3}{2} \Rightarrow \quad r = \Phi^{-1}\left(\frac{3}{4}\right).$$

Also as $\beta \to 1$, 23 shows

$$\Phi(\infty) + \Phi(0) = \frac{3}{2} \Rightarrow \quad 1 + \frac{1}{2} = \frac{3}{2},$$

which does not illuminate the limit of r as $\beta \to 1$. However, computational evidence, see Bordley [3], Cortona et al [6], Thuo [20], and Stoer [18], suggests that r gets closer and closer to 0 as β gets closer and closer to 1. Computational evidence, as shown by figure 9, suggests that $f(\beta)$ has an infinite slope (vertical slope) and, of course, analytically we know that $f(\beta)$ has a horizontal slope at $\Phi^{-1}\left(\frac{3}{4}\right)$. From the above properties the arctan function is a good choice for an elementary function whose values approximate those determined by (9). This is because the arctan function has a horizontal asymptote that we can make $\Phi^{-1}\left(\frac{3}{4}\right)$ and on the other hand as $\beta \to 1$, $\arctan(\beta - 1)$ has a vertical slope. We define our approximating function as follows

$$\frac{\Phi^{-1}\left(\frac{3}{4}\right)}{\frac{\pi}{2}} \arctan(b \cdot (\beta - 1)).$$

We use the method of least squares approximation, see Casella [4] and Stoer [18], to solve the value of b that best approximates our function. The values of b that minimizes the squares of the relative differences given by the Newton's method is 2.6435. The graph of $\frac{\Phi^{-1}\left(\frac{3}{4}\right)}{\frac{\pi}{2}} \arctan(2.6435 \cdot (\beta - 1))$ is shown by

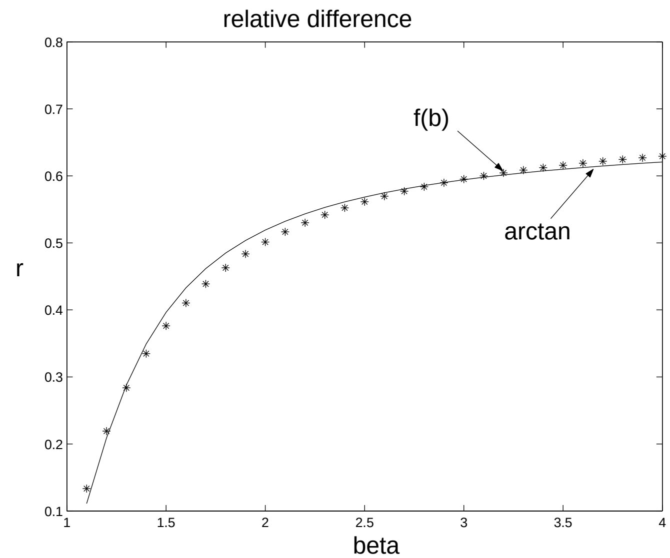

Figure 10: Graph of f(β) and the approximating arctan function that minimizes the relative difference

Figure 10.

4. Conclusion

We have extended the spatial theory of voting, which has been extensively developed for elections of a single candidate, to include at-large elections. Using our cumulative voting heuristic we have been able to identify optimal candidate equilibrium positions when voters' ideal points have a standard normal distribution. We accomplished this by positioning two candidates on a single dimensional policy space and allowing a third candidate to locate a position, where he would outvote either of the two candidates. We let the third candidate assume a position within an ϵ neighborhood of either of the other two candidates and developed an equation whose solutions provide the only possibility for an equilibrium to exist. Using optimization techniques such as those derived in Thuo [20], we came up with a nice elementary function that approximates this equation.

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