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GENERIC PROJECTIONS OF SINGULAR VARIETIES

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Abstract: Here we use a recent preprint *ArXiv:* math/0806.1928 by Beheshti and Eisenbud to study the fibers of the projection $X \to \mathbb{P}^{n+c}$ of a singular n-dimensional variety $X \subset \mathbb{P}^r$ from a general $(r-n-c-1)$-dimensional linear subspace.

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1. Introduction

Let X, Y closed subschemes of a scheme P defined over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. Set $Z := X \cap Y$ (scheme-theoretic intersection). In [3] R. Beheshti and D. Eisenbud defined the very important coherent O_P-sheaf $Q(X,Y)$ supported by P as the cokernel of the restriction map

$$\mathrm{Hom}(\mathcal{I}_{Z/Y}/\mathcal{I}^2_{Z/Y}, \mathcal{O}_Z) \to \mathrm{Hom}(\mathcal{I}_{X/P}/\mathcal{I}^2_{X/P}, \mathcal{O}_Z).$$

They used $Q(X,Y)$ to prove several theorems on the generic projections of smooth projectively embedded varieties (greatly improving a theorem of Mather (see [4], [2], [1]), the unions of their multisecant lines, and more (see [3]). Our first observation is that their proofs work in the singular cases if we restrict to the non-singular part X_{reg} of the projective variety $X \subset \mathbb{P}^r$ (see Section 2). For every integer x such that $0 \le x < r$ let $G(x,r)$ denote the Grassmannian of all x-dimensional linear subspaces of $\mathbb{P}^r$. For any $U \in G((r-n-c-1),r)$ set $G_U((r-n-c),r) := \{V \in G((r-n-c),r) : U \subset V\} \cong \mathbb{P}^{n+c}$. In this very

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cheap way we will get the following results.

Theorem 1. *Let $X \subset \mathbb{P}^r$ be an integral n-dimensional projective variety. Fix an integer c such that $0 < c < r-n$. Let U be a general element of $G((r-n-c-1),r)$. Let $\pi_U : X \to \mathbb{P}^{n+c}$ be the linear projection from U. Then* $\mathrm{length}(Q(X_{reg},V)/c \le (n/c)+1$ *for every $V \in G_U((r-n-c),r)$.*

Theorem 2. *Let $X \subset \mathbb{P}^r$ be an integral n-dimensional variety. Fix an integer $l \ge 2$. Let $\widetilde{S}_l \subset \mathbb{P}^r$ be the closure of the union of all lines $D \subset \mathbb{P}^r$ such that* $\deg(X_{reg} \cap L) \ge l$ *and* $L \cap \mathrm{Sing}(X) = \emptyset$. *Then* $\dim(\widetilde{S}_l) \le nl/(l-1)+1$.

Obviously, Theorem 1 is only the first step, because one must also to get the contributions of the singular points of X contained in V to get $\deg(X \cap V$. If X has only isolated singularities, then Proposition 2 may be seen as a reasonable extension of [3], Theorem 1.5. Theorem 2 is weak, because in the definition of $\widetilde{S}_l$ we only counted the closure of the union of the lines L such that $\deg(X_{reg} \cap L) \ge l$ and $L \cap \mathrm{Sing}(X) = \emptyset$: we ignore all lines intersecting $\mathrm{Sing}(X)$. However, if $l \gg \dim(mboxSing(X))$ Theorem 2 gives a strong result (see Remark 4). To state our results on the singular part we introduce the following invariants. For any $P \in \mathbb{P}^r$ set $G(x,r,P) := \{V \in G(x,r) : P \in V\}$. Fix $P \in \mathrm{Sing}(X)$. For any integer c such that $1 \le c \le r-n-1$, let $s(X,P,c)$ be the degree of the connected component of $X \cap V$ supported by P, wherem V is a general element of $G((r-n-c),r,P)$. If X has only finitely many singular points, then everything is easy (see Proposition 2). In the general case we introduce the following invariants of the embedded pair (X,P). For any integer $t \ge 0$ let $s(X,P,c,t)$ be the maximal degree of the connected component of $X \cap V$ supported by P, when V is general in an integral subvariety of $G((r-n-c),r,P)$ whose codimension in $G((r-n-c),r,P)$ is at most t. Hence $s(X,P,c,0) = s(X,P,c)$ and $s(X,P,c,t)$ is a non-decreasing function of t.

2. Proofs and Other Remarks

Remark 1. Fix integers $r > n+c > 0$. $G((r-n-c),r))$ is irdducible and of dimension $(n+c+1)(r+1)$. For any $P \in \mathbb{P}^r$ we have $G((r-n-c),r,P) \cong G((r-n-c-1),r-1)$. Hence $G((r-n-c),r,P)$ is irreducible and of codimension $n+c+1$ in $G((r-n-c),r))$. For all $Q \in \mathbb{P}^r \backslash \{P\}$ we have $G((r-n-c),r,P) \cap G((r-n-c),Q) \cong G((r-n-c-2,r-2)$. Hence $G((r-n-c),r,P) \cap G((r-n-c),Q)$ has codimension $n+c$ in $G((r-n-c),r,P)$.

Remark 2. The statement and the proof of [3], Theorem 3.1, are OK if we only assume that X is locally a complete intersection in P in a neighborhood of Z_{red}. Hence we may also use [3], Corollary 3.2, when X is locally a complete intersection in P in a neighborhood of Z_{red}.

Remark 3. Take the set-up of [3], Theorem 4.1, but only assume that X is smooth in a neighborhood of Z_{red}. Part (2) goes verbatim. To see parts (2) and (3), we first use that the smoothness of X in a neighborhhood of Z_{red} gives that the derivation $\delta : \mathcal{I}_Y \to \Omega_{\mathbb{P}^r}$ induces a map $d : \mathcal{I}_X/\mathcal{I}_X^2 \to \Omega_{\mathbb{P}^r}|X$ which is a split injection in a neighborhood of Z_{red}. This is sufficient to carry over the proof of [3], Theorem 4.1, because it uses that d is a split injection only to see that composing with the functor $Hom(---, \mathcal{O}_Z)$ we get an epimorphism. Thus we may also use [3], Corollary 4.2. However, we stress again that in the set-up of Theorem 1, this observation says nothing obout the contribution to $Q(X,$

Proof of Theorem 1. In our set-up $Z := X_{reg} \cap M$, $M \in G_U((r-n-c), r)$ and the sheaf F introduced in the statement of [3], Theorem 4.1, and used in the proof of [3], Theorem 1 does only depend from $X_{reg} \cap M$. □

Proposition 1. *Assume* $2 \cdot \dim(\mathrm{Sing}(X)) \leq n+c-1$. *Fix a general* $U \in G((r-n-c-1), r)$ *and let* $\pi_U : X \to \mathbb{P}^{n+c}$ *the associated linear projection. Then* $\pi_U|\mathrm{Sing}(X)$ *is injective. If* P *is a general point of any irreducible component of* $\mathrm{Sing}(X)$, *then the unique* $M \in G_U((r-n-c), r)$ *containing* P *intersects* X *in a scheme whose connected component supported by* P *has degree* $s(X, p, c)$.

Proof. Since every irreducible component of the join W of $\mathrm{Sing}(X)$ with itself has dimension at most $2 \cdot \dim(\mathrm{Sing}(X)) + 1$, the generality of U implies $U \cap W = \emptyset$. Hence $\pi_U|\mathrm{Sing}(X)$ is injective. The last assertion follows from the definition of the integer $s(X, P, c)$, because for fixed P we may take U general and hence M may be considered as a general element of $G((r-n-c), r, P)$. □

As a particular case of Proposition 1 we get the following result, except the last assertion.

Proposition 2. *Assume that* X *has only isolated singularity. Fix a general* $U \in G((r-n-c-1), r)$ *and take any* $M \in G_U((r-n-c), r)$. *intersecting* $\mathrm{Sing}(X)$. *Then* $M \cap \mathrm{Sing}(X)$ *is a unique point,* P, *and the connected component of* $X \cap M$ *supported by* P *has degree* $s(X, P, c)$. *If* $r < 2n+c+2$, *then* $(X \cap M)_{red} = \{P\}$.

Proof. By Proposition 1 it is sufficient to check the last assertion. For each $P \in \mathrm{Sing}(X)$ the join $[X; P]$ of X and P has dimension at most $n+1$. Hence if $r-n-c-1 < n+1$ the generality of U implies $U \cap [X; P] = \emptyset$ for all $P \in \mathrm{Sing}(X)$, proving the last assertion. □

Proof of Theorem 2. Here the proofs in [3], §6, works verbatim, because we only use lines L such that $L \cap \mathrm{Sing}(X) = \emptyset$. □

Remark 4. Assume $\mathrm{Sing}(X) \neq \emptyset$ and let x be the maximal dimension of its irreducible components. Let $D(X)$ be the set of all lines $L \subset \mathbb{P}^r$ such that $L \cap \mathrm{Sing}(X) \neq \emptyset$ and $L \cap X_{reg} \neq \emptyset$. Every irreducible components of the join $[X; \mathrm{Sing}(X)]$ has dimension at most $x + n + 1$. The definition of joins gives $D(X) \subseteq [X; \mathrm{Sing}(X)]$. Thus $\dim(D(X)) \leq x + n + 1$. Thus if $l \gg x$ the statement of Theorem 1 gives a nice geometrical result.

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References

[1] A. Alzati, E. Ballico, G. Ottaviani, The theorem of mather for singular varieties, *Geom. Dedicata*, **85**, No-s: 1-3 (2001), 113-117.

[2] A. Alzati, G. Ottaviani, The theorem of mather on generic projections in the setting of algebraic geometry, *Manuscripta Math.*, **74**, No. 4 (1992), 391-412.

[3] R. Beheshti, D. Eisenbud, Fibers of generic projections, *ArXiv:* math/0806.1928.

[4] J.N. Mather, Generic projections, *Ann. of Math.*, **98**, No. 2 (1973), 226-245.