International Journal of Pure and Applied Mathematics
Volume 47 No. 4 2008, 577-580

A SPLITTING CRITERION FOR VECTOR BUNDLES ON CONES OVER RATIONAL NORMAL CURVES

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Abstract: Here we prove a splitting criterion for torsion free sheaves on the projective cone over a rational normal curve.

AMS Subject Classification: 14J60
Key Words: arithmetically Cohen-Macaulay sheaf, ACM vector bundle, projective cone, splitting criterion, rational normal curve

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Let $X \subset \mathbb{P}^n$ an integral m-dimensional variety and F a torsion-free coherent sheaf on X. We will say that G is arithmetically Cohen-Macaulay (or ACM) if $h^i(F(t)) = 0$ for all integers i, t such that $1 \le i \le m-1$. If F is ACM, then $F|X_{reg}$ is locally free, because F has depth m at each point of X_{reg}. Here we prove the following splitting criterion.

Theorem 1. *Fix an integer $n \ge 3$. Let $H \subset \mathbb{P}^n$ be a hyperplane, and $C \subset H$ a linearly normal smooth rational curve. Fix $O \in \mathbb{P}^n \backslash H$. Let X be the degree $n-1$ cone with vertex O and base C. Fix a line $T \subset X$. Let E be a rank r torsion-free sheaf on X. There are integers $a_1 \ge \cdots \ge a_r$ such that $E \oplus_{i=1}^r \mathcal{O}_X(a_i)$ if and only if $h^1(E(t)) = h^1(\mathcal{I}_T \otimes E(t)) = 0$ for all $t \in \mathbb{Z}$ and there is a hyperplane $U \subset \mathbb{P}^n$ such that $O \notin U$ and $E|(U \cap X)$ has splitting type $c_1 \ge \cdots \ge c_r$ with $c_i \equiv 0 \pmod{n-1}$.*

Received: August 28, 2008

Proof of Theorem 1. Since $O \notin U$, the linear pojection from U induces an isomorphism between $U \cap X$ and $H \cap X$. Hence $X \cap U \cong \mathbb{P}^1$. Any vector bundle on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles and it is uniquely determined by its splitting type. Since $X \setminus \{0\}$ is smooth and $\dim(X) = 2$, every reflexive sheaf on X is locally free, except perhaps at O (see [1], Corollary 1.4). Hence the splitting type of $E|(U \cap X)$ is well-defined. Since $C \subset H$ is a rational normal curve, X is rational and arithmetically Cohen-Maculay. Since $h^1(\mathbb{P}^n, \mathcal{I}_{T,\mathbb{P}^n}(t)) = 0$ for all $t \in \mathbb{Z}$, $h^1(\mathcal{I}_T(t)) = 0$ for all $t \in \mathbb{Z}$. Hence $h^1(\mathcal{O}_X(t)) = \mathcal{I}_T(t)) = 0$ for all $t \in \mathbb{Z}$. Thus the "only if" part in the statement of Theorem 1 is true. Now we will prove the "if" part. Since $h^1(E(t)) = 0$ for all $t \ll 0$, Serre cohomological computation of depth gives that E has the S_2 property. Since X is normal, E is reflexive. Let s be the minimal integer t such that $h^0(E(t)) > 0$. Thus $h^0(E(s-1)) = 0$ and $h^0(E(s)) > 0$. Let $M \subset \mathbb{P}^n$ be a hyperplane such that $O \notin M$. Thus $D := M \cap X \cong C \cong \mathbb{P}^1$. Hence $E|D$ is a direct sum of r line bundles. Let $b_1 \geq \cdots \geq b_r$ be the splitting type of $E|D$. For any integer t we have the following exact sequences of sheaves on X:

$$0 \to E(t-1) \to E(t) \to E(t)|D \to 0\,, \tag{1}$$

$$0 \to \mathcal{I}_A \otimes E(t-1) \to \mathcal{I}_A \otimes E(t) \to \mathcal{I}_T \otimes E(t)|D \to 0\,. \tag{2}$$

Let y be the maximal integer such that $1 \leq y \leq r$ and $b_y = b_1$. Since $h^1(E(t)) = 0$ for all t, (1) implies that s is the minimal integer t such that $h^0(D, E(t)|D) > 0$ and $h^0(E(s)) = h^0(D, E(t)|D)$. Hence $-(n-1)s \leq b_1 < -(n-1)(s-1)$. Obviously, $h^0(D, E(s)|D) \geq y(b_1 + (n-1)s + 1)$ and $h^0(D, E(s)|D) = y(b_1 + (n-1)s + 1)$ if and only if either $y = r$ or $b_{y+1} < -(n-1)s$. Note that F_x intersect D scheme-theoretically in an effective degree x divisor. Now we take $M := U$. Hence $b_1 = c_1 \equiv 0 \pmod{n-1}$. Thus $b_1 = -(n-1)s$. Hence $h^0(D, E(s)|D) = y$. Hence $h^0(E(s)) = y$. Let $G \subseteq E$ be the image of the evaluation map $\beta : H^0(E(s)) \otimes \mathcal{O}_X(-s) \to E$. Since G is a subsheaf of E, G is torsion-free. We just saw that $G|D$ is a rank y vector bundle. Since G Hence β is injective and $\mathrm{rank}(G) = y$. Take another hyperplane $M' \subset \mathbb{P}^n$ such that $O \notin M'$. Hence $D' \cong \mathbb{P}^1$. Let $b'_1 \geq \cdots \geq b'_r$ be the splitting type of $E|D'$. Let y' be the maximal integer i such that $1 \leq i \leq r$ and $b'_i = y'$. Since $h^0(D', E(s)|D') = 0$ and $h^0(D', E(s)|D') = h^0(D, E(s)|D) = y$ (use (1) with D' instead of D), $-(n-1)s \leq b'_1 < -(n-1)(s-1)$ and $y \geq y'(b'_1 - (n-1)s + 1)$ with strict inequality unless either $y' = r$ or $b'_{y'+1} < -(n-1)(s-1)$. Since T intersects D scheme-theoretically in an effective degree 1 divisor and $b_1 = -(n-1)s$, $h^0(D, (\mathcal{I}_T \otimes E(s))|D)) = 0$. Hence $h^0(\mathcal{I}_T \otimes E(s)) = 0$. Since $h^1(\mathcal{I}_T \otimes E(s-1)) = 0$ and D' intersects T in an effective degree 1 divisor, we get $h^0(D', (\mathcal{I}_T \otimes E(s))|D')) = 0$. Hence $b'_1 = -(n-1)s$. Thus the support of the torsion subsheaf Tors(E/G) of E/G is disjoint from D'. Varing M we get that Tors(E/G) (if non-zero) is supported by O. Let $\pi : E \to (E/G)/\mathrm{Tors}(E/G)$ be the quotient map. Set $G' := \mathrm{Ker}(\pi)$. Since E is locally free and Tors(E/G) is torsion free, G' is reflexive (see [1], Proposition 1.1). The natural inclusion map $j : G \hookrightarrow G'$ is an isomorphism outside O. Since G is locally free, it is reflexive. Since X is normal, j is an isomorphism (see [1], Proposition 1.6), i.e. E/G has no torsion. We also saw that E/G is a rank $r-y$ vector bundle outside O. If $y = r$, then $E \cong \mathcal{O}_X(b_1/r)^{\oplus r}$ and hence we are done. Thus we may assume $y < r$. Since $h^0(D', F(t)|D') = 0$ for all D' and all $t \leq s-1$ and $h^1(D', F(t)|D') = 0$ for all D' and all $t \geq s-1$, $h^0(D', E(t)|D') = h^0(D', G(t)|D') + h^0(D', (E/G)(t)|D')$ for all D' and all $t \in \mathbb{Z}$. Thus $h^0(D', (E/G)(t)|D') = h^0(D, (E/G)(t)|D)$ for all D' and all $t \in \mathbb{Z}$. Let z be the number of integers j such that $y+1 \leq j \leq r$ and $b_j = b_{y+1}$. Remember that $b_{y+1} = c_{y+1} \cong 0 \pmod{n-1}$. Hence $-b_{y+1}/(n-1)$ is the first integer $s' > s$ such that $h^0(E(s')) > y \cdot h^0(\mathcal{O}_X(s'-s))$. Since E is ACM, (1) gives $h^0(E(s')) = y \cdot h^0(\mathcal{O}_X(s'-s)) + z$. Recall that $h^0(G(s')) = y \cdot h^0(\mathcal{O}_X(s'-s))$. Take a z-dimensional linear subspace W of $H^0(E(s'))$ supplementary to $H^0(G(s'))$. The image W' of W in $H^0((E/G)(s'))$ is a z-dimensional linear subspace whose restriction to D spans the rank z trivial factor of $(E/G)(s')|D$. Since (1) gives $h^0(D', E(t)|D') = h^0(D, E(t)|D)$ for all $t \in \mathbb{Z}$, we also get $b'_j = b_j$ for all $y+1 \leq j \leq y+z$ and that either $y+z = r$ or $b'_{y+z+1} < b_{y+1}$. As in the case of $E(s)$ we obtain that the evaluation map $e_{s'} : H^0(E(s')) \otimes \mathcal{O}_X \to E(s')$ has as image a rank $y+z$ subsheaf G_1 of E such that the torsion of E/G_1 is supported by O. As in the previous case we get that E/G_1 has no torsion. If $y+z = r$, then we are done. If $y+z < r$, then we continue in the same way using that $b_{y+z+1} = c_{y+z+1} \equiv 0 \pmod{n-1}$. At the end we get that E is a direct sum, because $h^1(\mathcal{O}_X(t)) = 0$ for all t, and hence all the extensions obtained split (start from the last one). □

Question 1. Fix integers $g \geq 0$ and $n \geq 3$. Let $H \subset \mathbb{P}^{n+g}$ be a hyperplane and $C \subset H$ a linearly normal smooth genus g curve. Fix $O \in \mathbb{P}^{n+g} \backslash H$ and let X be the cone with vertex O and base C. For any integer x such that $1 \leq x \leq n-2$ let A_x be the set of all effective degree x divisors of C and D_x the set of all cones with vertex O and an element of B_x as a basis. Let E be a vector bundle on X. Assume $h^1(E(t)) = 0$ and $h^1(\mathcal{I}_{D_x}(t)) = 0$ for all $t \in \mathbb{Z}$, all $1 \leq x \leq n-2$, and all $D_x \in B_x$. Is E ia extension of line bundles $\mathcal{O}_X(a)$?

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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