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ON γ-CONVERGENCY AND ψ-CONVERGENCY IN GENERALIZED TOPOLOGICAL SPACES

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Abstract: The aim of this article is to investigate γ-convergency and ψ-convergency in generalized topologies.

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1. Introduction

Á. Császár [3], [4], [5] defined generalized open sets in generalized topological spaces. Semi-convergency and pre-convergency were introduced and studied in [1] and [7], respectively. Convergency which is one of the most important concepts in general topology is examined in generalized topologies in this paper. Two concepts named γ-convergency and ψ convergency are introduced. Furthermore, μ-convergency which is a more generalized one is introduced.

2. Preliminaries

Let X be a non-empty set and $\gamma : \wp(X) \to \wp(X)$ be a monotonic mapping

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(i.e. $A \subset B$ implies $\gamma A \subset \gamma B$), see [2], [3], [4]. The collection of all mappings having the property monotonic is denoted by $\Gamma(X)$ (see [2]). A set $A \subset X$ is said to be γ-open iff $A \subset \gamma A$ (see [2]). A collection g of subsets of X is said to be generalized topology (briefly, GT) on X iff $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $G = \bigcup_{i\in I} G_i \in g$ (see [2]). The collection of all GT's on X is denoted by $\mathcal{B}(X)$. It is obvious that γ-open sets constitute a GT. If τ is a topology on X in usual sense and we write c for cl, i for int with respect to τ the c and i are all elements of $\Gamma(X)$.

A subset of a topological space (X, τ) is semi-open [5] (rep. pre-open, see [6]) if $A \subset (cl(intA))$ (resp. $A \subset int(clA)$), where $intA$ denotes the interior of A and clA denotes the closure of A. A sequence x_n is said to be semi-converges [1] (resp. pre-converges, see [7]) to a point x in X if x_n is eventually in every semi-open (resp. pre-open) set containing x.

Let $\psi : X \to \wp(\wp(X))$ satisfy $x \in V$ for $V \in \psi(x)$. Then we say that $V \in \psi(x)$ is a generalized neighbourhood (briefly, GN) of $x \in X$ and ψ is a generalized neighbourhood system (briefly, GNS) on X (see [3]). Let $\Psi(X)$ denote the collection of all GNS's on X. In [3], Á. Császár gave a way for obtaining GT's as follows; let $g \in \mathcal{B}(X)$ and $\gamma \in \Gamma(X)$ be such that $A \subset \gamma A$ for $A \in X$. Then we have $V \in \Psi(\gamma, g)(x)$ for $x \in X$ iff $V = \gamma G$ for some $G \in g$ such that $x \in G$. It is clear that $\psi(\gamma, g) \in \Psi(X)$.

Let g and g' be two generalized topologies on X and X' respectively. $f : X \to X'$ is said to be as (g, g')-continuous iff $G' \in g'$ then $f^{-1}(G') \in g$ (see [3]).

3. γ-Convergency and ψ-Convergency

Definition 2.1. Let X be a non-empty set and $\gamma \in \Gamma(X)$. A sequence (x_n) is said to γ-converge to a point x in X if (x_n) is eventually in every γ open set containing x. Denoted by $(x_n) \to^{\gamma} x$.

Remark 2.2. If γ is chosen as ic, γ-convergency coincides with pre-convergency; if $\gamma = ci$ γ-convergency coincides with semi-convergency.

Remark 2.3. Let $\psi \in \Psi(X)$. According to Lemma 1.2 in [3], we know that ψ generates a GT g on X, denoted by $g = g_\psi$. Since g_ψ is a GT on X, then there is a $\gamma \in \Gamma(X)$ such that g_ψ is the collection of all γ-open sets. Hence we can define g_ψ-convergency with respect to Definition 2.1.

Definition 2.4. Let X be a non-empty set and $\gamma \in \Gamma(X)$. A sequence (x_n) is said to ψ-converge to a point x in X if (x_n) is eventually in every set V in $\psi(x)$. Denoted by $(x_n) \to^{\psi} x$.

Lemma 2.5. *Let (x_n) be a sequence in the set X, ψ be a GNS and $g = g_\psi$ is a GT generated by ψ. If $(x_n) \to^{\psi} x$, then $(x_n) \to^{g_\psi} x$.*

Proof. $(x_n) \to^{\psi} x$ implies that (x_n) is eventually in every set V in $\psi(x)$. Since ψ construct a GT as $g = g_\psi$, then $G \in g$ iff $G \subset X$ satisfies: $x \in G$, then there is $V \in \psi(x)$ such that $V \subset G$. Hence (x_n) is eventually in every G. □

The converse of Lemma 2.5 need not to be hold in general.

Example 2.6. Let $X = \{a, b, c\}$, $g = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $\psi = \psi(c_g, g)$ and for $k \in Z$ (Z is a set of all integers number)

$$(x_n)=\begin{cases} c, & n = 2k \\ a, & n = 2k + 1 \end{cases}$$

be a sequence in X.

We have $\psi(b) = \{X, \{b, c\}\}$ and $g_\psi = \{\emptyset, X\}$. It is obvious that (x_n) is eventually in X but not eventually in $\{b, c\}$. Hence, (x_n) g_ψ-converges to b while (x_n) does not ψ-converges to b.

4. Sequentially Continuity in Generalized Topologies and μ-Convergency

Definition 3.1. A function $f : X \to Y$ is said to be sequentially (g, g')-continuous if for each point x in X and each sequence (x_n) in X g-converging to x, then $(f(x_n))$ in Y g'-converging to $f(x)$.

Theorem 3.2. *Let $f : X \to Y$ be a (g, g')-continuous function. Then f is sequentially (g, g')-continuous.*

Proof. Let $x \in X$ and (x_n) be a sequence in X g-converging to x. Then (x_n) is eventually in every γ-open set V containing x. Since f is (g, g')-continuous, we can take $V = f^{-1}(N)$ for γ'-open set N in Y. Then $(x_n) \in f^{-1}(N)$. This implies $f(x_n) \in f(f^{-1}(N)) \subset N$ which means $(f(x_n))$ is eventually in every g'-open set N containing $f(x)$. □

If for every $n \in I$ ($I = \{1, 2, \ldots, n, \ldots\}$) there exists a γ-open set U such that $x \in U \subset E_n$, then we denote the collection of all E_n for every $n \in I$ by $\varepsilon(x)$, i.e. $\varepsilon(x) = \{E_n : n \in I\}$.

Let g be a GT on X. From Lemma 1.1 in [2], for $x \in X$ we can construct a class $\varepsilon(x)$ whose elements are g-open sets. Hence we denote this as $\varepsilon(x) = \varepsilon_g(x)$.

Definition 3.3. Let X be a set and g be a GT on X. Then X is called as generalized first countable space (g first countable space) if for each x in X and $\varepsilon(x) = \{E_n | n \in I\}$, the elements of $\varepsilon(x)$ can be written as

$$E_1 \supset E_2 \supset \ldots \supset E_n \supset \ldots.$$

Lemma 3.4. *Let g be a GT on X. For all $x \in X$, there is a class $\varepsilon(x)$ whose elements are also elements of g.*

Proof. If g is a GT on X, then there is a $\gamma : \wp(X) \to \wp(X)$ such that g is the collection of all γ-open sets (see [2]). Also we know that the elements of $\varepsilon(x)$ are γ'-open sets, for some γ'. We can choose $\gamma' = \gamma$ and they become elements of g. □

Let us write $\varepsilon(x) = \varepsilon_g(x)$ in this manner.

Definition 3.5. A function f is said to be sequentially (ψ, ψ')-continuous if for each sequence (x_n) in X ψ-converging to x, $(f(x_n))$ ψ'-converges to $f(x)$.

Theorem 3.6. *Let X be a g first countable space and $f : X \to Y$ be a sequentially $(\psi_g, \psi_{g'})$-continuous function. Then f is $(\psi_g, \psi_{g'})$-continuous function.*

Proof. Let x in X and $\varepsilon(x) = \{V_n : n \in I\}$ which satisfies the property

$$V_1 \supset V_2 \supset \ldots \supset V_n \supset \ldots.$$

Assume that f is not $(\psi_g, \psi_{g'})$-continuous function in x. Then for a $V' \in \psi_g(x)$ and all $V \in \psi_g(x)$, $f(V) \not\subset V'$ is hold.

In this manner, we can write $f^{-1}(f(V')) \not\subset f^{-1}(V)$. This may imply that $V \not\subset f^{-1}(V')$. It is obvious that $\varepsilon(x) = \varepsilon_g(x)$ is contained by $\psi_g(x)$. Hence we can say that for each $n \in I$, $V \not\subset f^{-1}(V')$. Then for each $n \in I$, there exists a x_n in V_n such that $x_n \notin f^{-1}(V')$. This implies $f(x_n) \notin V'$. This is a contradiction. Hence f is $(\psi_g, \psi_{g'})$-continuous. □

Definition 3.7. (see [4]) Define a mapping $\lambda : \wp(X) \to \wp(X)$. λ is called as an envelope operation satisfying the following conditions:

(3.7.1) $A \subset \lambda A$ for $A \subset X$.

(3.7.2) For $A, B \subset X$, $A \subset B$ implies $\lambda A \subset \lambda B$.

(3.7.3) For $A \subset X$, $\lambda\lambda A = \lambda A$.

Definition 3.8. (see [4]) Let $\mu \subset \wp(X)$ be arbitrary. For $A \subset X$, $x \in \kappa A$ iff $K \in \mu$ satisfying $x \in K$ fulfills $K \cap A \neq \emptyset$. We write $\kappa = \kappa A$.

Lemma 3.9. *$\kappa = \kappa(\mu)$ is an envelope on X for an arbitrary $\mu \subset \wp(X)$.*

In [4], the separation axioms were formulated by replacing open sets by an arbitrary family of subsets of the space X.

Definition 3.10. (see [4]) Assume $\mu \subset \wp(X)$. (T_2) x, $y \in X$, $x \neq y$ imply the existence of K, $K' \in \mu$ such that $x \in K$, $y \in K'$ and $K \cap K' = \emptyset$.

Definition 3.11. x_0 is a ψ-cluster point of (x_n) if for each $U \in \psi(x_0)$ there is a $m \in I$ such that $x_n \in U$ for some n bigger than m.

Definition 3.12. x_0 is a γ-cluster point of (x_n) if for each γ-open set U that contains x_0, there is a $m \in I$ such that $x_n \in U$ for some n bigger than m.

As expressed Remark 2.2, we can introduce g_ψ-cluster point.

Theorem 3.13. *Let (x_n) be a sequence in the set X, ψ be a GNS and $g = g_\psi$ be a GT generated by ψ. If x_0 is a ψ-cluster point for (x_n) then it is a g_ψ-cluster point for (x_n).*

Proof. The proof is similar with the proof of Lemma 2.5. □

Theorem 3.14. *Let (x_n) be a sequence in X, $x_0 \in X$ and $\kappa = \kappa(\psi(x_0))$. x_0 is a ψ-cluster point of (x_n) if and only if $x_0 \in \kappa\{x_n, x_{n+1}, \ldots\}$ for all $n \in I$.*

Proof. ($\Rightarrow$:) Since x_0 is a ψ-cluster point of (x_n), for all $V \in \psi(x_0)$ there exists a $m \in I$ such that $x_n \in V$ for some n bigger than m. Hence $\{x_n, x_{n+1}, \ldots\} \cap V \neq \emptyset$, $n > m$. This implies $x_0 \in \kappa\{x_n, x_{n+1}, \ldots\}$ for all $n \in I$.

($\Leftarrow$:) Since $x_0 \in \kappa\{x_n, x_{n+1}, \ldots\}$ for all $n \in I$; $V \cap \{x_n, x_{n+1}, \ldots\} \neq \emptyset$ is hold for all $V \in \psi(x_0)$ and for all $n \in I$. This implies x_0 is a cluster point of (x_n). □

Definition 3.15. Assume $\mu \subset \wp(X)$. A sequence (x_n) is said to μ-converge to a point x in X if (x_n) is eventually in every set V in μ containing x and denoted by $(x_n) \to^\mu x$.

Theorem 3.16. *Let $\mu \subset \wp(X)$. If X satisfies (T_2) then (x_n) μ-converges to one point in X.*

Proof. Let (x_n) μ-converges to both x and y. Since (x_n) μ-converges to x, it is eventually in every set U in μ containing x and since (x_n) μ-converges to y it is eventually in every set V in μ containing y. This implies (x_n) is eventually in $U \cap V$. Hence $U \cap V \neq \emptyset$ and this is a contradiction. □

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