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STABILITY OF QUARTIC FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN SPACE

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Abstract: In this paper, we investigate the Hyers-Ulam stability for a new quartic functional equation

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)+f\left(\frac{x-y}{2}+z\right)+f\left(\frac{x-y}{2}-z\right)$$
$$+f(x)+f(y)-2f(z)=\frac{1}{8}\big\{f(x+y)+f(x-y)+4[f(y+z)+f(y-z)+f(x+z)+f(x-z)]\big\}$$

in non-Archimedean normed space.

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1. Introduction

In 1940, before the auidance of the mathematics club of the University of Wisconsin S.M. Ulam presented a list of unsolved problems. One of these problem

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can be considered as the starting point of a new line of investigations: the stability problem. Suppose that a group G and a metric group H are given. For any $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G \to H$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $a : G \to H$ exits with $d(f(x), a(x)) < \epsilon$ for all x in G?. These kind of questions form the material of the stability theory. In 1941, D.H. Hyers [15] answered Ulam's problem for the case of approximately additive functions under the assumption that G and H are Banach spaces. Taking this fact into account, the additive functional equation $f(x+y) = f(x)+f(y)$ is said to have Hyers-Ulam stability on (G, H). The result of Hyers was further generalized by Th.M. Rassias [29]. In the generalized version of the theorem of Hyers Th.M. Rassias permitted the Cauchy difference to become unbounded. He proved the following theorem by using a direct method.

Theorem 1.1. (see [29]) *If a function $f : G \to H$ between Banach spaces satisfies inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\,(||x||^p + ||y||^p)$$

for some $\theta \geq 0, 0 \leq p < 1$ and for all $x, y \in G$, then there exits a unique additive function $a : G \to H$ such that

$$||f(x) - a(x)|| \leq \frac{2\theta}{2-2^p}||x||^p$$

for any x in G. More over, if $f(tx)$ is continuous in t for each fixed $x \in G$ then a is linear.

The theorem of Th.M. Rassias was later extended to all $p \neq 1$ and generalized in [11, 12, 31, 38]. The stability phenomenon that was presented by Th.M. Rassias is called the generalized Hyers-Ulam stability. This terminology may also be applied to the cases of other functional equations.

The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

Hence the functional equation (1. 1) is called a quadratic functional equation or the Euler-Lagrange functional equation and every solution of the quadratic equation (1. 1) is called a quadratic function. The functional equation (1. 1) is a familiar equation and this equation was dealt by F. Skof [39], P.W. Cholewa [6], S. Czerwik [9] and J.M. Rassias [27]. S.M. Jung and P.K. Sahoo [20] investigated the Hyers-Ulam stability of the quadratic functional equation of pexider type

$$f_1(x+y) + f_2(x-y) = 2f_3(x) + 2f_4(y). \tag{1.2}$$

The different forms of quadratic functional equations and its Hyers-Ulam-Rassias stability were discussed by J.H. Bae and K.W. Jun [17], I.S. Chang, E.H. Lee and H.M. Kim [7], S.M. Jung [19], A. Najati and M.B. Moghini [25], J.M. Rassias [28] and K. Ravi and M. Arun kumar [34, 35].

A functional equation is said to be cubic functional equation if it satisfies the cubic function $f(x) = ax^3$. The cubic functional equations and its Hyers-Ulam-Rassias stability was discussed by I.S. Chang, Y.S. Jung [22], H.Y. Chu, D.S. Kang [8], A. Najati [24] and K.W. Jun and H.W. Kim [21].

The stability of the quartic functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y) \tag{1.3}$$

was discussed by S.H. Lee, S.M. Im, I.S. Hwang [37]. The quartic functional equation (1. 3) and its Hyers-Ulam stability on orthogonal space was discussed by C.G. Park [26]. The various types of functional equations and its stability on different spaces like Metric spaces, Banach spaces, Lipschitz's spaces were discussed by many authors in the last two-decades.

Recently M.S. Moslehian and Th.M. Rassias [30] discussed the Hyers-Ulam stability of the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \tag{1.4}$$

and the quadratic functional equation (1. 1) in non-Archimedean normed spaces.

Motivated by this observation, in this paper the authors investigate the Hyers-Ulam stability for a new class of quartic functional equation

$$\begin{aligned} f\left(\frac{x+y}{2}+z\right) + f\left(\frac{x+y}{2}-z\right) + f\left(\frac{x-y}{2}+z\right) + f\left(\frac{x-y}{2}-z\right) \\ + f(x) + f(y) - 2f(z) = \frac{1}{8}\{f(x+y) + f(x-y) + 4[f(y+z) \\ + f(y-z) + f(x+z) + f(x-z)]\} \end{aligned} \tag{1.5}$$

in non-Archimedean space.

Before we present our main results, we shall introduce the non-Archimedean space and preliminary results.

By a non-Archimedean field we mean a field K equipped with a function (valuation)$|.|$ from K into $[0,\infty)$ such that $|r| = 0$ if and only if r=0, $|rs| = |r||s|$ and $|r+s| \leq \max\{|r|, |s|\}$ for all $r, s \in K$. Clearly $|1| = |-1|$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimean non-trivial valuation $|.|$. A function $||.|| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $||x||$=0 if and only if $x = 0$;

(ii) $||rx|| = |r|||x|| \qquad (x, y \in X)$;

(iii) the strong triangle inequality (ultrametric); namely,

$$||x + y|| \leq \max\{||x||, ||y||\} \qquad (x, y \in X).$$

Then $(X, ||.||)$ is called a non-Archimedean space. Due to the fact that

$$||x_n - x_m|| \leq \max\{||x_{j+1} - x_j|| : m \leq j \leq n-1\} \qquad (n > m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

In 1897, Hensel [14] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. Fix a prime number p. For any nonzero rational number x, there exits a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$ which is called the p-adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k \geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $\left|\sum_{k \geq n_x}^{\infty} a_k p^k\right|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field; see for example [13] and [36].

In Section 2, we prove the equivalence of the functional equation (1. 3) to the functional equation (1. 5) and in Section 3, we discuss the Hyers-Ulam stability in non-Archimedean space, in particular in the field of p-adic numbers.

2. Solution of Functional Equation (1. 5)

Let $\mathbb{N}, \mathbb{R}$ denote the set of all natural numbers and the set of all nonnegative real numbers respectively. Let E_1 and E_2 be two real vector spaces. We here present the general solution of equation (1. 5). We begin with the following lemma.

Lemma 2.1. *A function $f : E_1 \to E_2$ satisfies the functional equation (1.*

3) then

$$\begin{aligned} &f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z) \\ &= 2[f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(z+x)+f(z-x)] \\ &\quad -4[f(x)+f(y)+f(z)]. \end{aligned} \tag{2.1}$$

Proof. The proof can be modelled as that of Theorem 2. 1 of [37] and hence the details are omitted. □

Theorem 2.2. *A function $f : E_1 \to E_2$ satisfies the functional equation (1. 3) if and only if $f : E_1 \to E_2$ satisfies the functional equation (1. 5). Therefore, every solution of functional equation (1. 5) is also a quartic function.*

Proof. Let $f : E_1 \to E_2$ satisfies the functional equation (1. 5). Putting $x = y = z = 0$ in (1. 5), we get $f(0) = 0$. Set $x = y = 0$ in (1. 5) to get $f(-z) = f(z)$. Therefore f is even function. Replacing y, z, by x in (1. 5), we get $f(2x) = 16f(x)$ for all $x \in E_1$. Putting $y = 0$ and $z = x$ in (1. 5), we get $f(3x) = 81f(x)$ for all $x \in E_1$. Replacing x, y, z by $2x$ in (1. 5), we get $f(4x) = 256f(x)$. By induction, we get $f(kx) = k^4 f(x)$ for all positive integer k. Putting $y = 0$ in (1. 5), we get

$$\begin{aligned} &f\left(\frac{x}{2}+z\right)+f\left(\frac{x}{2}-z\right)+f\left(\frac{x}{2}+z\right)+f\left(\frac{x}{2}-z\right)+f(x)-2f(z) \\ &\qquad = \frac{1}{8}\left\{2f(x)+4[f(z)+f(-z)+f(x+z)+f(x-z)]\right\}. \end{aligned} \tag{2.2}$$

Replace x by $2x$ in (2. 2), we get

$$\begin{aligned} &2f(x+z)+2f(x-z)+f(2x)-2f(z) \\ &\qquad = \frac{f(2x)}{4}+f(z)+\frac{1}{2}[f(2x+z)+f(2x-z)]. \end{aligned} \tag{2.3}$$

On simplifying (2. 3) we arrive the equation (1. 3). Let $f : E_1 \to E_2$ satisfies the functional equation (1. 3). Putting $x = y = 0$ in (1. 3), we get $f(0) = 0$. Putting $x = 0$ in (1. 3), we have $f(y) = f(-y)$. Putting $y = 0$ and $y = x$ in (1. 3), we obtain that $f(2x) = 16f(x)$ and $f(3x) = 81f(x)$ respectively. Which leads to $f(kx) = k^4 f(x)$ for all $x \in E_1$ and $k \in \mathbb{N}$. Replace x by $\frac{x+y}{2}$ and y by z in (1. 3), we get

$$\begin{aligned} &4f\left(\frac{x+y}{2}+z\right)+4f\left(\frac{x+y}{2}-z\right)+24f\left(\frac{x+y}{2}\right)-6f(z) \\ &\qquad = f(x+y+z)+f(x+y-z). \end{aligned} \tag{2.4}$$

Replace y by $-y$ in (2. 4), we get

$$4f\left(\frac{x-y}{2}+z\right)+4f\left(\frac{x-y}{2}-z\right)+24f\left(\frac{x-y}{2}\right)-6f(z)$$
$$=f(x-y+z)+f(x-y-z). \quad (2.5)$$

Adding (2. 4) and (2. 5), we obtain

$$4f\left(\frac{x+y}{2}+z\right)+4f\left(\frac{x+y}{2}-z\right)+4f\left(\frac{x-y}{2}+z\right)+4f\left(\frac{x-y}{2}-z\right)$$
$$-12f(z)+\frac{3}{2}[f(x+y)+f(x-y)]=f(x+y+z)+f(x+y-z)$$
$$+f(x-y+z)+f(-x+y+z). \quad (2.6)$$

From Lemma 2. 1 and (2. 6), we obtain

$$4f\left(\frac{x+y}{2}+z\right)+4f\left(\frac{x+y}{2}-z\right)+4f\left(\frac{x-y}{2}+z\right)+4f\left(\frac{x-y}{2}-z\right)$$
$$-12f(z)+\frac{3}{2}[f(x+y)+f(x-y)]=2[f(x+y)+f(x-y)+f(y+z)+f(y-z)$$
$$+f(z+x)+f(z-x)]-4[f(x)+f(y)+f(z)] \quad (2.7)$$

which gives the equation (1. 5). □

3. Stability of Quartic Functional Equation (1. 5)

Throughout this section, we assume G is an additive group and X is a complete non-Archimedean space. We define

$$Df(x,y,z)=f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)+f\left(\frac{x-y}{2}+z\right)$$
$$+f\left(\frac{x-y}{2}-z\right)+f(x)+f(y)-2f(z)$$
$$-\frac{1}{8}\left\{f(x+y)+f(x-y)+4[f(y+z)+f(y-z)+f(x+z)+f(x-z)]\right\}.$$

Theorem 3.1. *Let $\psi : G\times G\times G\to[0,\infty)$ be a function such that*

$$\lim_{n\to\infty}\frac{\psi(2^nx,2^ny,2^nz)}{|16|^n}=0 \qquad (x,y,z\in G) \quad (3.1)$$

and let for each $x\in G$ the limit

$$\lim_{n\to\infty}\max\left\{\frac{\psi(2^jx,2^jx,2^jx)}{|16|^j}:0\le j<n\right\} \quad (3.2)$$

denoted by $\widetilde{\psi}$, exits. Suppose that $f : G \to X$ is a mapping satisfying

$$||Df(x,y,z)|| \leq \psi(x,y,z) \quad (x,y,z \in G). \tag{3.3}$$

Then there exits quartic mapping $Q : G \to X$

$$||f(x) - Q(x)|| \leq \frac{1}{|16|}\widetilde{\psi}(x) \qquad (x \in G). \tag{3.4}$$

Moreover, if

$$\lim_{k\to\infty}\lim_{n\to\infty}\max\left\{\frac{\psi(2^j x, 2^j x, 2^j x)}{|16|^j} : k \leq j < n+k\right\} = 0\,,$$

then Q is the unique quartic mapping satisfying (3. 4).

Proof. Replacing y, z by x in equation (3. 3), we get

$$||f(2x) - 16f(x)|| \leq \psi(x,x,x) \quad (x \in G). \tag{3.5}$$

Replacing x by $2^{n-1}x$ and divide by 16^n in (3. 5), we get

$$\left\|\frac{f(2^n x)}{16^n} - \frac{f(2^{n-1})}{16^{n-1}}\right\| \leq \frac{\psi(2^{n-1}x, 2^{n-1}x, 2^{n-1}x)}{|16|^{n-1}} \qquad (x \in G). \tag{3.6}$$

It follows from (3. 6) and (3. 1) that the sequence $\left\{\frac{f(2^n x)}{16^n}\right\}$ is Cauchy sequence. Since X is complete, the sequence $\left\{\frac{f(2^n x)}{16^n}\right\}$ converges. Now define a mapping $Q : G \to X$ by

$$Q(x) = \lim_{n\to\infty}\frac{f(2^n x)}{16^n}.$$

Using induction one can show that

$$\left\|\frac{f(2^n x)}{16^n} - f(x)\right\| \leq \frac{1}{|16|}\max\left\{\frac{\psi(2^k x, 2^k x, 2^k x)}{|16|^k} : 0 \leq k < n\right\} \tag{3.7}$$

for all $n \in \mathbb{N}$ and for all $x \in G$. By taking n to approach infinity in (3. 7) and using (3. 2), we obtain (3. 4). Replacing x, y and z by $2^n x$, $2^n y$ and $2^n z$ respectively in (3. 3), we get

$$||\frac{1}{16^n}Df(2^n x, 2^n y, 2^n z)|| \leq \frac{\psi(2^n x, 2^n y, 2^n z)}{|16|^n} \qquad (x,y,z \in G).$$

Taking the limit as $n \to \infty$ and using (3. 1) we get

$$Q\left(\frac{x+y}{2}+z\right) + Q\left(\frac{x+y}{2}-z\right) + Q\left(\frac{x-y}{2}+z\right) + Q\left(\frac{x-y}{2}-z\right)$$
$$+ Q(x) + Q(y) - 2Q(z) = \frac{1}{8}\big\{Q(x+y) + Q(x-y) + 4[Q(y+z)$$
$$+ Q(y-z) + Q(x+z) + Q(x-z)]\big\}.$$

If Q' is another quartic mapping satisfying (3. 4), then

$$\begin{aligned} ||Q(x)-Q'(x)|| &= \lim_{k\to\infty} |16|^{-k}||Q(2^kx)-Q'(2^kx)|| \\ &\le \lim_{k\to\infty} |16|^{-k} \max ||Q(2^kx)-f(2^kx)||, ||f(2^kx)-Q'(2^kx)|| \\ &\le \frac{1}{|16|} \lim_{k\to\infty}\lim_{n\to\infty} \max\left\{\frac{\psi(2^jx,2^jx,2^jx)}{|16|^j} : k\le j<n+k\right\} \\ &= 0 \qquad (x\in G). \end{aligned}$$

Therefore $Q = Q'$. This completes the proof of the theorem. □

Corollary 3.2. *Let $\rho : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$\rho(|2|t) \le \rho(|2|)\rho(t) \quad (t\ge 0), \quad \rho(|2|) < |2|. \tag{3.8}$$

Let $\delta > 0$, let G be a normed space and let $f : G \to X$ fulfill the inequality

$$||Df(x,y,z)|| \le \delta[\rho(||x||)+\rho(||y||)+\rho(||z||)]\rho(||x||)\rho(||y||)\rho(||z||) \quad (x,y,z\in G).$$

Then there exits a unique quartic mapping $Q : G \to X$ such that

$$||f(x)-Q(x)|| \le \frac{3\delta\rho(||x||)^4}{|16|}. \tag{3.9}$$

Proof. Define $\psi : G\times G\times G \to [0,\infty)$ by

$$\psi(x,y,z) = \delta[\rho(||x||)+\rho(||y||)+\rho(||z||)]\rho(||x||)\rho(||y||)\rho(||z||).$$

Using the equation (3. 8), we obtain

$$\lim_{n\to\infty} \frac{\psi(2^nx,2^ny,2^nz)}{|16|^n} \le \lim_{n\to\infty} \left(\frac{\rho(|2|)}{|2|}\right)^{4n} \psi(x,y,z) = 0 \qquad (x,y,z\in G).$$

Also

$$\widetilde{\psi}(x) = \lim_{n\to\infty} \max\left\{\frac{\psi(2^jx,2^jx,2^jx)}{|16|^j} : 0\le j<n\right\} = \psi(x,x,x)$$

and hence

$$\lim_{k\to\infty}\lim_{n\to\infty} \max\left\{\frac{\psi(2^jx,2^jx,2^jx)}{|16|^j} : k\le j<n+k\right\} = \lim_{k\to\infty} \psi\frac{(2^kx,2^kx,2^kx)}{|16|^k} = 0.$$

All the conditions of Theorem 3. 1 are satisfies and hence the result follows from Theorem 3. 1. □

Theorem 3.3. *Let $\phi : G\times G\times G \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} |16|^n \phi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) = 0 \qquad (x,y,z\in G) \tag{3.10}$$

and let for each $x\in G$ the limit

$$\lim_{n\to\infty} \max\left\{|16|^j\phi\left(\frac{x}{2^j},\frac{x}{2^j},\frac{x}{2^j}\right) : 1\le j\le n\right\} \tag{3.11}$$

denoted by $\widetilde{\phi}(x)$ *exits. Suppose that* $f : G \to X$ *is a mapping satisfying*

$$||Df(x,y,z)|| \leq \phi(x,y,z) \qquad (x,y,z \in G). \tag{3.12}$$

Then there exits a quartic mapping $Q : G \to X$ *such that*

$$||f(x) - Q(x)|| \leq \frac{1}{|16|}\widetilde{\phi}(x) \qquad (x \in G). \tag{3.13}$$

Moreover if

$$\lim_{k\to\infty}\lim_{n\to\infty} \max\left\{|16|^j \phi\left(\frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}\right) : k+1 \leq j \leq n+k\right\} = 0.$$

Then Q *is the unique quartic mapping satisfying (3. 13).*

Proof. Replacing y, z by x and x by $\frac{x}{2}$ in equation (3. 12), we get

$$\left\|f\left(\frac{x}{2}\right) - \frac{1}{16}f(x)\right\| \leq \frac{1}{|16|}\phi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right). \tag{3.14}$$

Replace x by $\frac{x}{2^{n-1}}$ and multiply by 16^n in (3. 14), we get

$$\left\|16^n f\left(\frac{x}{2^n}\right) - 16^{n-1} f\left(\frac{x}{2^{n-1}}\right)\right\| \leq |16|^{n-1}\phi\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right). \tag{3.15}$$

It follows from (3. 15) and (3. 10) that the sequence $\left\{16^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence in X. Since X is complete, we conclude that $\left\{16^n f\left(\frac{x}{2^n}\right)\right\}$ is convergent. Now define a function $Q : G \to X$ by

$$Q(x) = \lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right).$$

Using induction one can show that

$$\left\|16^n f\left(\frac{x}{2^n}\right) - f(x)\right\| \leq \frac{1}{|16|}\max\left\{|16|^k \phi\left(\frac{x}{2^k}, \frac{x}{2^k}, \frac{x}{2^k}\right) : 1 \leq k \leq n\right\} \tag{3.16}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking $n \to \infty$ in (3. 16), we have

$$||Q(x) - f(x)|| \leq \frac{1}{|16|}\widetilde{\phi}(x).$$

Replacing x, y and z by $\frac{x}{2^n}, \frac{y}{2^n}$ and $\frac{z}{2^n}$ respectively in (3. 12), we obtain

$$||16^n Df\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)|| \leq |16|^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \qquad (x,y,z \in G).$$

Taking the limit as $n \to \infty$ and using (3. 10) we obtain

$$Q\left(\frac{x+y}{2} + z\right) + Q\left(\frac{x+y}{2} - z\right) + Q\left(\frac{x-y}{2} + z\right) + Q\left(\frac{x-y}{2} - z\right)$$
$$+ Q(x) + Q(y) - 2f(z) = \frac{1}{8}\{Q(x+y) + Q(x-y) + 4[Q(y+z)$$
$$+ Q(y-z) + Q(x+z) + Q(x-z)]\}.$$

If Q' is another quartic mapping satisfying (3. 13), then

$$\begin{aligned}||Q(x)-Q'(x)|| &= \lim_{k\to\infty}|16|^k||Q\left(\frac{x}{2^k}\right)-Q'\left(\frac{x}{2^k}\right)|| \\ &\le \lim_{k\to\infty}|16|^k \max ||Q\left(\frac{x}{2^k}\right)-f\left(\frac{x}{2^k}\right)||, ||f\left(\frac{x}{2^k}\right)-Q'\left(\frac{x}{2^k}\right)|| \\ &\le \frac{1}{|16|}\lim_{k\to\infty}\lim_{n\to\infty}\max\left\{|16|^j\phi\left(\frac{x}{2^j},\frac{x}{2^j},\frac{x}{2^j}\right) : k+1\le j<n+k\right\} \\ &= 0 \quad (x\in G).\end{aligned}$$

Therefore $Q=Q'$. This completes the proof of the theorem. □

Corollary 3.4. *Let $\tau:[0,\infty)\to[0,\infty)$ be a function satisfying*

$$\tau\left(\frac{t}{|2|}\right)\le\frac{\tau(t)}{\tau(|2|)} \quad (t\ge 0), \tau(|2|)>|2|. \tag{3.17}$$

Let $\delta>0$, let G be a normed space and let $f:G\to X$ fulfill the inequality

$$||Df(x,y,z)|| \le \delta[\tau(||x||)+\tau(||y||)+\tau(||z||)]\tau(||x||)\tau(||y||)\tau(||z||) \quad (x,y,z\in G).$$

Then there exits a unique quartic mapping $Q:G\to X$ such that

$$||f(x)-Q(x)|| \le \frac{3\delta\tau(||x||)^4}{|16|}. \tag{3.18}$$

Proof. Define $\phi : G\times G\times G\to[0,\infty)$ by

$$\phi(x,y,z)=\delta[\tau(||x||)+\tau(||y||)+\tau(||z||)]\tau(||x||)\tau(||y||)\tau(||z||).$$

Using the inequality (3. 17), we obtain

$$\lim_{n\to\infty}|16|^n\phi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right)\le\lim_{n\to\infty}\left(\frac{\tau(|2|)}{|2|}\right)^{4n}\phi(x,y,z)=0 \quad (x,y,z\in G).$$

Also

$$\widetilde{\phi}(x)=\lim_{n\to\infty}\max\left\{|16|^j\phi\left(\frac{x}{2^j},\frac{x}{2^j},\frac{x}{2^j}\right):1\le j\le n\right\}=|16|\phi(x/2,x/2,x/2)$$

and hence

$$\begin{aligned}&\lim_{k\to\infty}\lim_{n\to\infty}\max\left\{|16|^k\phi\left(\frac{x}{2^j},\frac{x}{2^j},\frac{x}{2^j}\right):k\le j<n+k\right\} \\ &\qquad = \lim_{k\to\infty}|16|^{k+1}\phi\left(\frac{x}{2^{k+1}},\frac{x}{2^{k+1}},\frac{x}{2^{k+1}}\right)=0.\end{aligned}$$

All the conditions of Theorem 3. 3 are satisfies and hence by Theorem 3. 3, we arrive the inequality (3. 18). □

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