THE GENERALIZED QUASILINEARIZATION METHOD FOR THREE-POINT BOUNDARY VALUE PROBLEMS ON TIME SCALES

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Abstract: The generalized quasilinearization method is applied to the three-point boundary value problems on time scales, and two sequences could be constructed which converge uniformly to the unique solution and the convergence is quadratic.

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1. Introduction

We consider the three-point boundary value problems (BVPs) on time scales

\[
\begin{aligned}
&x^{Δν}(t) + h(t)f(t,x) = 0, \quad t \in [a,b] \subset T, \\
&αx(ρ(a)) - βx^{Δ}(ρ(a)) = 0, \quad x(σ(b)) - λx(ξ) = 0,
\end{aligned}
\]

(1.1)

where the following assumptions are satisfied:

(H1) \( b > 0, \quad α > 0, \quad β ≥ 0, \quad 0 < λ < 1, \quad ξ \in (ρ(a),σ(b)); \)

(H2) \( h : [a,b] \to (0,+∞) \) is ld-continuous;

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\((H_3)\ f : [a, b] \times (-\infty, +\infty) \rightarrow (-\infty, +\infty)\) is continuous, \(f(t, \cdot) \neq 0\) on any subset of \([a, b]\).

It is well known that the method of quasilinearization \([4]\) provides an excellent tool for obtaining approximate solutions of nonlinear differential equations. The method has been applied to dynamic systems on time scales, see \([5, 8, 2]\). Recently, Wang and Lu \([8]\) discussed BVPs \((1.1)\) with requiring the function involved to be convex/concave. The convexity assumption is relaxed and the method is generalized. B. Ahmad and R. N. Mohapatra et al \([1, 7]\) discussed the two points BVPs.

In the paper, the generalized quasilinearization method is applied to BVPs \((1.1)\) without requiring the function involved to be convex/concave and the sequences of approximate solutions are provided, which converge uniformly and quadratically to the unique solution of BVPs \((1.1)\).

2. Preliminaries

Let \(G(t, s)\) be the Green’s function for BVPs
\[
\begin{align*}
\begin{cases}
\Delta v(t) + m(t) = 0, & t \in [a, b] \subset T, \\
\alpha x(\sigma(a)) - \beta x^\Delta(\rho(a)) = 0, & x(\sigma(b)) = 0.
\end{cases}
\end{align*}
\tag{2.1}
\]

In \([3]\), the Green’s function for BVPs \((2.1)\) is given by
\[
G(t, s) = \frac{1}{D} \left\{ \begin{array}{ll}
(\sigma(b) - t)(\alpha(s - \rho(a)) + \beta), & \rho(a) \leq s < t \leq \sigma(b), \\
(\sigma(b) - s)(\alpha(t - \rho(a)) + \beta), & \rho(a) \leq t < s \leq \sigma(b),
\end{array} \right.
\]
where \(D = \alpha(\sigma(b) - \rho(a)) + \beta\). We note that \(G(t, s) \geq 0\) for \(t \in [a, b]\). If \(x(t)\) is the solution of BVPs \((1.1)\), then
\[
x(t) = \frac{1}{D} \left( \frac{\lambda x(\xi)}{\sigma(b)} (\beta - \alpha \rho(a)) \right) (\sigma(b) - t) + \frac{\lambda x(\xi)}{\sigma(b)} t
+ \frac{1}{D} \int_{\rho(a)}^{\sigma(b)} G(t, s) h(s) f(s, x) \nabla s.
\]

We define the set
\[
\mathbb{D} := \{x \in \mathbb{B} : \Delta x \text{ is continuous on } [\rho(a), \sigma(b)]^k, \ \nabla x^\Delta \text{ is differentiable and } \nabla \Delta x \text{ is continuous on } [\rho(a), \sigma(b)]^k \}.
\]

**Definition 2.1.** A real-valued function \(v(t) \in \mathbb{D}\) on \([\rho(a), \sigma(b)]\) is said to
be a lower solution of BVPs (1.1), if
\[
\begin{cases}
  v^{\Delta \nabla} (t) + h(t)f(t,v) \geq 0, & t \in [\rho(a), \sigma(b)], \\
  \alpha v(\rho(a)) - \beta v^{\Delta}(\rho(a)) \leq 0, & v(\sigma(b)) - \lambda v(\xi) \leq 0
\end{cases}
\]  
(2.2)
and a upper solution of BVPs (1.1), if the reversed inequalities hold.

To state the main result of this paper, we need following Lemmas which are derived from [8].

**Lemma 2.1.** Assume that:

(i) \(v(t)\) and \(w(t)\) are the lower and upper solutions of BVP (1.1), respectively;

(ii) \(f(t,x)\) is strictly decreasing in \(x\) for each \(t \in [\rho(a),\sigma(b)]\).

Then \(v(t) \leq w(t)\) on \([\rho(a),\sigma(b)]\).

**Lemma 2.2.** Assume that:

(i) \(v(t)\) and \(w(t)\) are the lower and upper solutions of BVPs (1.1) respectively, and \(v(t) \leq w(t)\) on \([\rho(a),\sigma(b)]\);

(ii) \(f(t,x)\) is strictly decreasing in \(x\) for each \(t \in [\rho(a),\sigma(b)]\).

Then there exists a unique solution \(x(t)\) of BVPs (1.1), such that \(v(t) \leq x(t) \leq w(t)\) on \([\rho(a),\sigma(b)]\).

**3. Main Results**

Firstly, we define the sector for every \(v, w \in D\) such that
\[
[v, w] := \{x(t) \in D : v(t) \leq x(t) \leq w(t), \ t \in [\rho(a),\sigma(b)]\}.
\]

**Theorem 3.1.** Assume that:

(A1) \(v_0, w_0 \in D\) are lower and upper solutions of BVPs (1.1) respectively, such that \(v_0(t) \leq w_0(t)\) on \([\rho(a),\sigma(b)]\);

(A2) \(f_x(t,x), f_{xx}(t,x)\) are continuous in \(x\) on \([v_0,w_0]\) and ld-continuous in \(t\) on \([\rho(a),\sigma(b)]\), satisfying \(f_x(t,x) < 0\);

(A3) There exists \(\phi(t,x)\) satisfying \(\phi_{xx}(t,x) \geq 0, \ f_x(t,x) + \phi_x(t,x) < 0, \ f_{xx}(t,x) + \phi_{xx}(t,x) \geq 0\), where \(\phi_x(t,x)\) and \(\phi_{xx}(t,x)\) are continuous in \(x\) on \([v_0,w_0]\) and ld-continuous in \(t\) on \([\rho(a),\sigma(b)]\).

Then there exist monotone sequences \(\{v_n\}\) and \(\{w_n\}\) which converge uniformly to the unique solution \(x(t)\) of BVPs (1.1). Moreover, the convergence is quadratic.
Proof. First we note that $f_{xx}(t, x) + \phi_{xx}(t, x) \geq 0$ along with using mean value theorem on $R$ yield the following inequalities

$$f(t, x) \geq f(t, y) + (f_x(t, y) + \phi_x(t, y))(x - y) - (\phi(t, x) - \phi(t, y)), \quad (3.1)$$

for $x \geq y$, where $x, y \in [v_0, w_0], t \in [\rho(a), \sigma(b)]$.

We define

$$F(t, x; v_0, w_0) = f(t, v_0) + (f_x(t, v_0) + \phi_x(t, v_0))(x - v_0) - (\phi(t, x) - \phi(t, v_0)), \quad G(t, x; v_0, w_0) = f(t, w_0) + (f_x(t, v_0) + \phi_x(t, v_0))(x - w_0) - (\phi(t, x) - \phi(t, w_0)),$$

where $x$ is function of $t$ on $[\rho(a), \sigma(b)]$.

We consider

$$\begin{cases} x^{\Delta \nabla} + h(t)F(t, x; v_0, w_0) = 0, & t \in [a, b], \\ \alpha x(\rho(a)) - \beta x^{\Delta}(\rho(a)) = 0, & x(\sigma(b)) - \lambda x(\xi) = 0 \quad (3.2) \end{cases}$$

and

$$\begin{cases} x^{\Delta \nabla} + h(t)G(t, x; v_0, w_0) = 0, & t \in [a, b], \\ \alpha x(\rho(a)) - \beta x^{\Delta}(\rho(a)) = 0, & x(\sigma(b)) - \lambda x(\xi) = 0. \quad (3.3) \end{cases}$$

Now we shall show that $v_0, w_0$ are the lower and upper solutions of BVPs (3.2) respectively. In view of (A1) and (3.1), we have

$$v_0^{\Delta \nabla} + h(t)F(t, v_0; v_0, w_0) \equiv v_0^{\Delta \nabla} + h(t)f(t, v_0) \geq 0,$$

$$w_0^{\Delta \nabla} + h(t)F(t, w_0; v_0, w_0) \leq w_0^{\Delta \nabla} + h(t)f(t, w_0) \leq 0.$$

Hence, by Lemma 2.2, there exists a solution $v_1$ of BVPs (3.2) such that $v_0 \leq v_1 \leq w_0$ on $[\rho(a), \sigma(b)]$. Similarly, there exists a solution $w_1$ of BVPs (3.3) such that $v_0 \leq w_1 \leq w_0$ on $[\rho(a), \sigma(b)]$.

Next we prove $v_1 \leq w_1$ on $[\rho(a), \sigma(b)]$. To prove this, we show that $v_1$ is a lower solution of BVPs (1.1) and $w_1$ is an upper solution. In view of (3.1) and (A3), we have

$$0 = v_1^{\Delta \nabla} + h(t)F(t, v_1; v_0, w_0)$$

$$\leq v_1^{\Delta \nabla} + h(t)(f(t, v_1) - (f_x(t, v_0) + \phi_x(t, v_0))(v_1 - v_0)$$

$$+ (\phi(t, v_1) - \phi(t, v_0)) + (f_x(t, v_0) + \phi_x(t, v_0))(v_1 - v_0)$$

$$- (\phi(t, v_1) - \phi(t, v_0))) = v_1^{\Delta \nabla} + h(t)f(t, v_1),$$

$$0 = w_1^{\Delta \nabla} + h(t)G(t, w_1; v_0, w_0)$$

$$\geq w_1^{\Delta \nabla} + h(t)(f(t, w_1) + (f_x(t, w_1) + \phi_x(t, w_1))(w_0 - w_1)$$

$$- (\phi(t, w_0) - \phi(t, w_1)) + (f_x(t, v_0) + \phi_x(t, v_0))(w_1 - w_0)$$

$$- (\phi(t, w_1) - \phi(t, v_0))) \geq w_1^{\Delta \nabla} + h(t)f(t, w_1).$$
Hence, by Lemma 2.1 we have \( v_1 \leq w_1 \) on \( [\rho(a), \sigma(b)] \). Consequently these results yield
\[
v_0 \leq v_1 \leq w_0 \quad \text{on} \quad [\rho(a), \sigma(b)].
\]

Continuing this process by induction, we can obtain the sequences \( \{v_n\} \) and \( \{w_n\} \) such that
\[
v_0 \leq v_1 \leq \cdots \leq v_n \leq w_n \leq \cdots \leq w_1 \leq w_0 \quad \text{on} \quad [\rho(a), \sigma(b)],
\]
where for each \( n \in \mathbb{N} \), \( v_{n+1} \) and \( w_{n+1} \) satisfy BVPs
\[
\begin{aligned}
  x^{\Delta^{\nu}} + h(t)F(t; x, v_n) &= 0, \quad t \in [a, b], \\
  \alpha x(\rho(a)) - \beta x^{\Delta}(\rho(a)) &= 0, \quad x(\sigma(b)) - \lambda x(\xi) = 0,
\end{aligned}
\]
and
\[
\begin{aligned}
  x^{\Delta^{\nu}} + h(t)G(t; x, v_n) &= 0, \quad t \in [a, b], \\
  \alpha x(\rho(a)) - \beta x^{\Delta}(\rho(a)) &= 0, \quad x(\sigma(b)) - \lambda x(\xi) = 0.
\end{aligned}
\]
Moreover
\[
\begin{aligned}
w_{n+1}(t) &= \frac{1}{D} \left( \frac{\lambda w_{n+1}(\xi)}{\sigma(b)} (\beta - \alpha \rho(a)) \right) (\sigma(b) - t) + \frac{\lambda w_{n+1}(\xi)}{\sigma(b)} t \\
&\quad + \frac{1}{D} \int_{\rho(a)}^{\sigma(b)} G(t, s) h(s) G(s, w_{n+1}; v_n, w_n) \nabla s, \quad t \in [a, b].
\end{aligned}
\]

Using standard argument, the sequences \( \{v_n\} \) and \( \{w_n\} \) converge uniformly to some function \( x \), that is \( \lim_{n \to \infty} v_n = \lim_{n \to \infty} w_n = x \). Note that \( \lim_{n \to \infty} F(t, v_{n+1}; v_n, w_n) = f(t, x) \). Now it is straightforward to see that
\[
x(t) = \frac{1}{D} \left( \frac{\lambda x(\xi)}{\sigma(b)} (\beta - \alpha \rho(a)) \right) (\sigma(b) - t) + \frac{\lambda x(\xi)}{\sigma(b)} t \\
&\quad + \frac{1}{D} \int_{\rho(a)}^{\sigma(b)} G(t, s) a(s) f(s, x) \nabla s, \quad t \in [a, b].
\]
Hence \( x(t) \) is a unique solution of BVPs (1.1) as desired.

Set \( p_{n+1} = x - v_{n+1} \) and \( q_{n+1} = w_{n+1} - x \). Notice that \( p_{n+1} \geq 0 \) and \( q_{n+1} \geq 0 \). By the mean value theorem, there exists \( \eta \), \( v_n \leq \eta \leq w_n \), such that
\[
q_{n+1} = \frac{1}{D} \left( \frac{\lambda (q_{n+1}(\xi))}{\sigma(b)} (\beta - \alpha \rho(a)) \right) (\sigma(b) - t) + \frac{\lambda (q_{n+1}(\xi))}{\sigma(b)} t \\
&\quad + \frac{1}{D} \int_{\rho(a)}^{\sigma(b)} G(t, s) h(s) (G(s, w_{n+1}; v_n, w_n) - f(s, x)) \nabla s \\
\leq \frac{1}{D} \left( \frac{\lambda q_{n+1}(\xi)}{\sigma(b)} (\beta - \alpha \rho(a)) \right) (\sigma(b) - t) + \frac{\lambda q_{n+1}(\xi)}{\sigma(b)} t
\]
Taking into account the following inequality
\[
(\beta - \alpha \rho(a))(\sigma(b) - t) + \frac{t}{\sigma(b)} = \frac{\sigma(b)(\beta - \alpha \rho(a) + \alpha t)}{D \sigma(b)} 
\]
we obtain
\[
\|p_{n+1}\| < \lambda\|p_{n+1}\| + \frac{1}{D} MNL \left( \frac{1}{2} \|p_n\|^2 + \frac{3}{2} \|q_n\|^2 \right),
\]
where
\[
M = \max_{t \in [a,b]} |f_{xx}(t, x) + \phi_{xx}(t, x)|, \quad N = \max_{t \in [a,b]} |h(t)| \quad \text{and} \quad L = \max_{t \in [a,b]} \int_{\rho(a)}^{\sigma(b)} G(t, s) \nabla s.
\]
Hence, we get
\[
\|q_{n+1}\| \leq \frac{MNL}{D(1 - \lambda)} \left( \frac{1}{2} \|p_n\|^2 + \frac{3}{2} \|q_n\|^2 \right).
\]
Similarly we obtain
\[
\|p_{n+1}\| \leq \frac{MNL}{D(1 - \lambda)} \left( \frac{3}{2} \|p_n\|^2 + \frac{1}{2} \|q_n\|^2 \right).
\]

**Corollary 3.1.** Assume that:

(B1) (A1) and (A2) hold;

(B2) There exists \( \phi(t, x) \) satisfying \( \phi_{xx}(t, x) \leq 0, f_{xx}(t, x) + \phi_{xx}(t, x) \leq 0 \), where \( \phi_{x}(t, x) \) and \( \phi_{xx}(t, x) \) are continuous in \( x \) on \([v_0, w_0]\) and ld-continuous in \( t \) on \([\rho(a), \sigma(b)]\).

Then the conclusion of Theorem 3.1 remains valid.

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References


