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THE GENERALIZED QUASILINEARIZATION METHOD FOR THREE-POINT BOUNDARY VALUE PROBLEMS ON TIME SCALES

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Abstract: The generalized quasilinearization method is applied to the three-point boundary value problems on time scales, and two sequences could be constructed which converge uniformly to the unique solution and the convergence is quadratic.

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1. Introduction

We consider the three-point boundary value problems (BVPs) on time scales

$$\begin{cases} x^{\Delta\nabla}(t) + h(t)f(t,x) = 0, \quad t \in [a,b] \subset \mathbb{T}, \\ \alpha x(\rho(a)) - \beta x^{\Delta}(\rho(a)) = 0, \quad x(\sigma(b)) - \lambda x(\xi) = 0\,, \end{cases} \tag{1.1}$$

where the following assumptions are satisfied:

(H_1) $b > 0,\ \alpha > 0,\ \beta \geq 0,\ 0 < \lambda < 1,\ \xi \in (\rho(a), \sigma(b))$;

(H_2) $h : [a,b] \to (0, +\infty)$ is ld-continuous;

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(H_3) $f : [a, b] \times (-\infty, +\infty) \to (-\infty, +\infty)$ is continuous, $f(t, \cdot) \not\equiv 0$ on any subset of $[a, b]$.

It is well known that the method of quasilinearization [4] provides an excellent tool for obtaining approximate solutions of nonlinear differential equations. The method has been applied to dynamic systems on time scales, see [5, 8, 2]. Recently, Wang and Lu [8] discussed BVPs (1.1) with requiring the function involved to be convex/concave. The convexity assumption is relaxed and the method is generalized. B. Ahmad and R. N. Mohapatra et al [1, 7] discussed the two points BVPs.

In the paper, the generalized quasilinearization method is applied to BVPs (1.1) without requiring the function involved to be convex/concave and the sequences of approximate solutions are provided, which converge uniformly and quadratically to the unique solution of BVPs (1.1).

2. Preliminaries

Let $G(t, s)$ be the Green's function for BVPs

$$\begin{cases} x^{\Delta\nabla}(t) + m(t) = 0, \quad t \in [a, b] \subset \mathbb{T}, \\ \alpha x(\rho(a)) - \beta x^{\Delta}(\rho(a)) = 0, \quad x(\sigma(b)) = 0. \end{cases} \tag{2.1}$$

In [3], the Green's function for BVPs (2.1) is given by

$$G(t, s) = \frac{1}{D} \begin{cases} (\sigma(b) - t)(\alpha(s - \rho(a)) + \beta), & \rho(a) \le s < t \le \sigma(b), \\ (\sigma(b) - s)(\alpha(t - \rho(a)) + \beta), & \rho(a) \le t < s \le \sigma(b), \end{cases}$$

where $D = \alpha(\sigma(b) - \rho(a)) + \beta$. We note that $G(t, s) \ge 0$ for $t \in [a, b]$. If $x(t)$ is the solution of BVPs (1.1), then

$$x(t) = \frac{1}{D}\left(\frac{\lambda x(\xi)}{\sigma(b)}(\beta - \alpha\rho(a))\right)(\sigma(b) - t) + \frac{\lambda x(\xi)}{\sigma(b)} t + \frac{1}{D}\int_{\rho(a)}^{\sigma(b)} G(t, s) h(s) f(s, x) \nabla s.$$

We define the set

$$\mathbb{D} := \{x \in \mathbb{B} : \ x^{\Delta} \text{ is continuous on } [\rho(a), \sigma(b)]^k, \ x^{\Delta} \text{ is } \nabla \text{ differentiable and } x^{\Delta\nabla} \text{ is continuous on } [\rho(a), \sigma(b)]_k^k\}.$$

Definition 2.1. A real-valued function $v(t) \in \mathbb{D}$ on $[\rho(a), \sigma(b)]$ is said to be a lower solution of BVPs (1.1), if

$$\begin{cases} v^{\Delta\nabla}(t) + h(t)f(t,v) \geq 0, \quad t \in [\rho(a), \sigma(b)], \\ \alpha v(\rho(a)) - \beta v^{\Delta}(\rho(a)) \leq 0, \quad v(\sigma(b)) - \lambda v(\xi) \leq 0 \end{cases} \tag{2.2}$$

and a upper solution of BVPs (1.1), if the reversed inequalities hold.

To state the main result of this paper, we need following Lemmas which are derived from [8].

Lemma 2.1. *Assume that:*

(i) $v(t)$ and $w(t)$ are the lower and upper solutions of BVP (1.1), respectively;

(ii) $f(t,x)$ is strictly decreasing in x for each $t \in [\rho(a), \sigma(b)]$.

Then $v(t) \leq w(t)$ on $[\rho(a), \sigma(b)]$.

Lemma 2.2. *Assume that:*

(i) $v(t)$ and $w(t)$ are the lower and upper solutions of BVPs (1.1) respectively, and $v(t) \leq w(t)$ on $[\rho(a), \sigma(b)]$;

(ii) $f(t,x)$ is strictly decreasing in x for each $t \in [\rho(a), \sigma(b)]$.

Then there exists a unique solution $x(t)$ of BVPs (1.1), such that $v(t) \leq x(t) \leq w(t)$ on $[\rho(a), \sigma(b)]$.

3. Main Results

Firstly, we define the sector for every $v,\ w \in \mathbb{D}$ such that

$$[v, w] := \{x(t) \in \mathbb{D} : \ v(t) \leq x(t) \leq w(t),\ t \in [\rho(a), \sigma(b)]\}.$$

Theorem 3.1. *Assume that:*

(A_1) $v_0,\ w_0 \in \mathbb{D}$ are lower and upper solutions of BVPs (1.1) respectively, such that $v_0(t) \leq w_0(t)$ on $[\rho(a), \sigma(b)]$;

(A_2) $f_x(t,x),\ f_{xx}(t,x)$ are continuous in x on $[v_0, w_0]$ and ld-continuous in t on $[\rho(a), \sigma(b)]$, satisfying $f_x(t,x) < 0$;

(A_3) There exists $\phi(t,x)$ satisfying $\phi_{xx}(t,x) \geq 0,\ f_x(t,x) + \phi_x(t,x) < 0,\ \ f_{xx}(t,x) + \phi_{xx}(t,x) \geq 0$, where $\phi_x(t,x)$ and $\phi_{xx}(t,x)$ are continuous in x on $[v_0, w_0]$ and ld-continuous in t on $[\rho(a), \sigma(b)]$.

Then there exist monotone sequences $\{v_n\}$ and $\{w_n\}$ which converge uniformly to the unique solution $x(t)$ of BVPs (1.1). Moreover, the convergence is quadratic.

Proof. First we note that $f_{xx}(t,x)+\phi_{xx}(t,x)\geq 0$ along with using mean value theorem on R yield the following inequalities

$$f(t,x)\geq f(t,y)+(f_x(t,y)+\phi_x(t,y))(x-y)-(\phi(t,x)-\phi(t,y)), \tag{3.1}$$

for $x\geq y$, where $x,\ y\in[v_0,w_0]$, $t\in[\rho(a),\sigma(b)]$.

We define

$$F(t,x;v_0,w_0)=f(t,v_0)+(f_x(t,v_0)+\phi_x(t,v_0))(x-v_0)-(\phi(t,x)-\phi(t,v_0)),$$

$$G(t,x;v_0,w_0)=f(t,w_0)+(f_x(t,v_0)+\phi_x(t,v_0))(x-w_0)-(\phi(t,x)-\phi(t,w_0)),$$

where x is function of t on $[\rho(a),\sigma(b)]$.

We consider

$$\begin{cases} x^{\Delta\nabla}+h(t)F(t,x;v_0,w_0)=0, \quad t\in[a,b],\\ \alpha x(\rho(a))-\beta x^{\Delta}(\rho(a))=0, \quad x(\sigma(b))-\lambda x(\xi)=0 \end{cases} \tag{3.2}$$

and

$$\begin{cases} x^{\Delta\nabla}+h(t)G(t,x;v_0,w_0)=0, \quad t\in[a,b],\\ \alpha x(\rho(a))-\beta x^{\Delta}(\rho(a))=0, \quad x(\sigma(b))-\lambda x(\xi)=0. \end{cases} \tag{3.3}$$

Now we shall show that $v_0,\ w_0$ are the lower and upper solutions of BVPs (3.2) respectively. In view of (A_1) and (3.1), we have

$$v_0^{\Delta\nabla}+h(t)F(t,v_0;v_0,w_0)\equiv v_0^{\Delta\nabla}+h(t)f(t,v_0)\geq 0,$$

$$w_0^{\Delta\nabla}+h(t)F(t,w_0;v_0,w_0)\leq w_0^{\Delta\nabla}+h(t)f(t,w_0)\leq 0.$$

Hence, by Lemma 2.2, there exists a solution v_1 of BVPs (3.2) such that $v_0\leq v_1\leq w_0$ on $[\rho(a),\sigma(b)]$. Similarly, there exists a solution w_1 of BVPs (3.3) such that $v_0\leq w_1\leq w_0$ on $[\rho(a),\sigma(b)]$.

Next we prove $v_1\leq w_1$ on $[\rho(a),\sigma(b)]$. To prove this, we show that v_1 is a lower solution of BVPs (1.1) and w_1 is an upper solution. In view of (3.1) and (A_3), we have

$$\begin{aligned} 0=v_1^{\Delta\nabla}&+h(t)F(t,v_1;v_0,w_0)\\ &\leq v_1^{\Delta\nabla}+h(t)(f(t,v_1)-(f_x(t,v_0)+\phi_x(t,v_0))(v_1-v_0)\\ &+(\phi(t,v_1)-\phi(t,v_0))+(f_x(t,v_0)+\phi_x(t,v_0))(v_1-v_0)\\ &\qquad -(\phi(t,v_1)-\phi(t,v_0)))=v_1^{\Delta\nabla}+h(t)f(t,v_1), \end{aligned}$$

$$\begin{aligned} 0=w_1^{\Delta\nabla}&+h(t)G(t,w_1;v_0,w_0)\\ &\geq w_1^{\Delta\nabla}+h(t)(f(t,w_1)+(f_x(t,w_1)+\phi_x(t,w_1))(w_0-w_1)\\ &-(\phi(t,w_0)-\phi(t,w_1))+(f_x(t,v_0)+\phi_x(t,v_0))(w_1-w_0)\\ &\qquad -(\phi(t,w_1)-\phi(t,w_0)))\geq w_1^{\Delta\nabla}+h(t)f(t,w_1). \end{aligned}$$

Hence, by Lemma 2.1 we have $v_1 \leq w_1$ on $[\rho(a), \sigma(b)]$. Consequently these results yield

$$v_0 \leq v_1 \leq w_1 \leq w_0 \quad \text{on} \quad [\rho(a), \sigma(b)].$$

Continuing this process by induction, we can obtain the sequences $\{v_n\}$ and $\{w_n\}$ such that

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq w_n \leq \cdots \leq w_1 \leq w_0 \quad \text{on} \quad [\rho(a), \sigma(b)],$$

where for each $n \in \mathbb{N}$, v_{n+1} and w_{n+1} satisfy BVPs

$$\begin{cases} x^{\Delta\nabla} + h(t)F(t, x; v_n, w_n) = 0, \quad t \in [a, b], \\ \alpha x(\rho(a)) - \beta x^{\Delta}(\rho(a)) = 0, \quad x(\sigma(b)) - \lambda x(\xi) = 0, \end{cases}$$

and

$$\begin{cases} x^{\Delta\nabla} + h(t)G(t, x; v_n, w_n) = 0, \quad t \in [a, b], \\ \alpha x(\rho(a)) - \beta x^{\Delta}(\rho(a)) = 0, \quad x(\sigma(b)) - \lambda x(\xi) = 0. \end{cases}$$

Moreover

$$\begin{aligned} w_{n+1}(t) = \frac{1}{D}\left(\frac{\lambda w_{n+1}(\xi)}{\sigma(b)}(\beta - \alpha\rho(a))\right)(\sigma(b) - t) + \frac{\lambda w_{n+1}(\xi)}{\sigma(b)}t \\ + \frac{1}{D}\int_{\rho(a)}^{\sigma(b)} G(t, s)h(s)G(s, w_{n+1}; v_n, w_n)\nabla s, \quad t \in [a, b]. \end{aligned}$$

Using standard argument, the sequences $\{v_n\}$ and $\{w_n\}$ converge uniformly to some function x, that is $\lim_{n\to\infty} v_n = \lim_{n\to\infty} w_n = x$. Note that $\lim_{n\to\infty} F(t, v_{n+1}; v_n, w_n) = f(t, x)$. Now it is straightforward to see that

$$\begin{aligned} x(t) = \frac{1}{D}\left(\frac{\lambda x(\xi)}{\sigma(b)}(\beta - \alpha\rho(a))\right)(\sigma(b) - t) + \frac{\lambda x(\xi)}{\sigma(b)}t \\ + \frac{1}{D}\int_{\rho(a)}^{\sigma(b)} G(t, s)a(s)f(s, x)\nabla s, \quad t \in [a, b]. \end{aligned}$$

Hence $x(t)$ is a unique solution of BVPs (1.1) as desired.

Set $p_{n+1} = x - v_{n+1}$ and $q_{n+1} = w_{n+1} - x$. Notice that $p_{n+1} \geq 0$ and $q_{n+1} \geq 0$. By the mean value theorem, there exists η, $v_n \leq \eta \leq w_n$, such that

$$\begin{aligned} q_{n+1} &= \frac{1}{D}\left(\frac{\lambda(q_{n+1}(\xi)}{\sigma(b)}(\beta - \alpha\rho(a))\right)(\sigma(b) - t) + \frac{\lambda(q_{n+1}(\xi)}{\sigma(b)}t \\ &\quad + \frac{1}{D}\int_{\rho(a)}^{\sigma(b)} G(t, s)h(s)(G(s, w_{n+1}; v_n, w_n) - f(s, x))\nabla s \\ &\leq \frac{1}{D}\left(\frac{\lambda q_{n+1}(\xi)}{\sigma(b)}(\beta - \alpha\rho(a))\right)(\sigma(b) - t) + \frac{\lambda q_{n+1}(\xi)}{\sigma(b)}t \end{aligned}$$

$$+\frac{1}{D}\int_{\rho(a)}^{\sigma(b)} G(t,s)h(s)(f_{xx}(t,\eta)+\phi_{xx}(t,\eta))q_n(p_n+q_n)\nabla s$$
$$\leq \frac{1}{D}\left(\frac{\lambda q_{n+1}(\xi)}{\sigma(b)}(\beta-\alpha\rho(a))\right)(\sigma(b)-t)+\frac{\lambda q_{n+1}(\xi)}{\sigma(b)}t$$
$$+\frac{1}{D}\int_{\rho(a)}^{\sigma(b)} G(t,s)h(s)(f_{xx}(t,\eta)+\phi_{xx}(t,\eta))\left(\frac{1}{2}p_n^2+\frac{3}{2}q_n^2\right)\nabla s.$$

Taking into account the following inequality

$$\frac{(\beta-\alpha\rho(a))(\sigma(b)-t)}{D\sigma(b)}+\frac{t}{\sigma(b)}=\frac{\sigma(b)(\beta-\alpha\rho(a)+\alpha t)}{D\sigma(b)}\leq 1,$$

we obtain

$$\|p_{n+1}\|<\lambda\|p_{n+1}\|+\frac{1}{D}MNL\left(\frac{1}{2}\|p_n\|^2+\frac{3}{2}\|q_n\|^2\right),$$

where

$$M=\max_{t\in[a,b]}|f_{xx}(t,x)+\phi_{xx}(t,x)|,\ N=\max_{t\in[a,b]}|h(t)| \text{ and } L=\max_{t\in[a,b]}\int_{\rho(a)}^{\sigma(b)} G(t,s)\nabla s.$$

Hence, we get

$$\|q_{n+1}\|\leq\frac{MNL}{D(1-\lambda)}\left(\frac{1}{2}\|p_n\|^2+\frac{3}{2}\|q_n\|^2\right).$$

Similarly we obtain

$$\|p_{n+1}\|\leq\frac{MNL}{D(1-\lambda)}\left(\frac{3}{2}\|p_n\|^2+\frac{1}{2}\|q_n\|^2\right).$$

Corollary 3.1. *Assume that:*

(B_1) (A_1) and (A_2) hold;

(B_2) There exists $\phi(t,x)$ satisfying $\phi_{xx}(t,x)\leq 0$, $f_{xx}(t,x)+\phi_{xx}(t,x)\leq 0$, where $\phi_x(t,x)$ and $\phi_{xx}(t,x)$ are continuous in x on $[v_0,w_0]$ and ld-continuous in t on $[\rho(a),\sigma(b)]$.

Then the conclusion of Theorem 3.1 remains valid.

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