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KNOT INVARIANTS FOR CLOSED PERIODIC ORBITS IN A THREE SPECIES FOOD CHAIN DYNAMICAL SYSTEM

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Abstract: Closed periodic orbits in a given flow can be studied by using knot holders. A knot holder is a branched 2-dimensional manifold with semi-flow. In this work the arising knots and links for model involving a three-species food chain is investigated. It is shown that they are positive knots, that according to associated positive braid representatives. Also it is concluded that the number of prime knots in the derived knot holder is bounded.

AMS Subject Classification: 57M25
Key Words: food chain, knots, templates, periodic orbits

1. Introduction

A food chain is the flow of energy from one organism to the next. Organisms in a food chain are grouped into levels based on how many links they are removed from the primary producers. Levels may consist of either a single species or a group of species that are presumed to share both predators and prey. They usually start with a primary producer and end with a carnivore. Such mathematical models have long proven useful in describing how populations vary over time. Data about the various rates of growth, death, and interaction of species naturally lead to models involving differential equations. Ecological systems have all the elements to produce chaotic dynamics. Although chaos is

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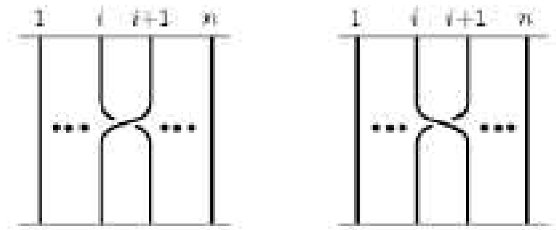
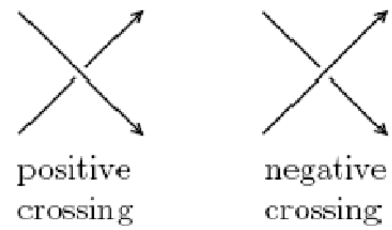


Figure 1:



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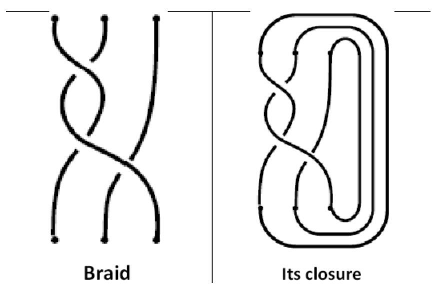

Figure 3:

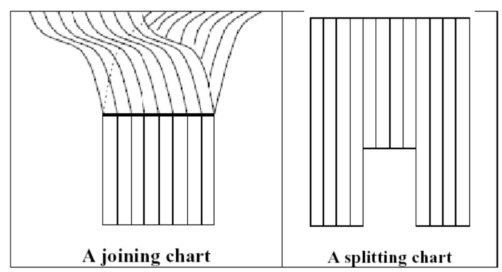

Figure 4:

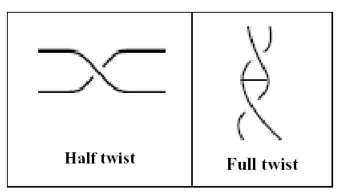

Figure 5:

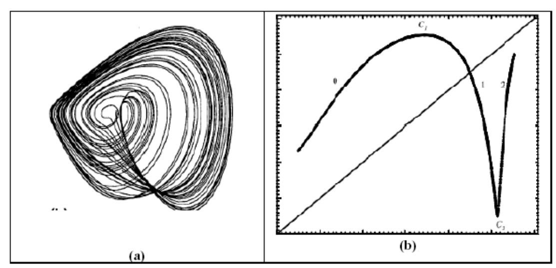

Figure 6:

commonly predicted by mathematical models, evidence for its existence in the natural world is scarce and inconclusive [6].

Any real ecological system contains many and many species, so it is very expensive and so hard to count them. One of the pioneer benefits of nonlinear dynamics is reconstruction of the phase portrait equivalent to the original species by applying the time evolution of a single species. In [8] a mathematical model that involving three-species which based on the couplings between a

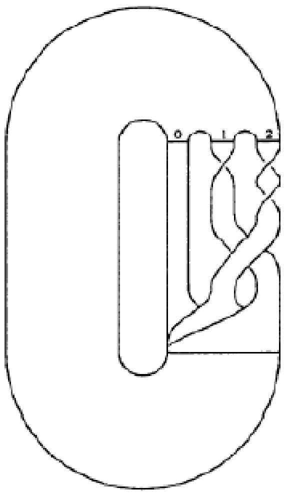

Figure 7:

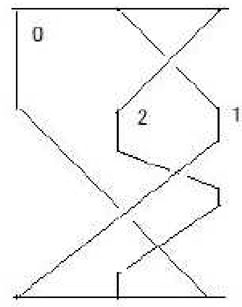

Figure 8:

Lotka-Volterra and a Leslie-Gover scheme is given by the equations:

$$\begin{aligned}
\dot{\widetilde{x}} &= a_1\,\widetilde{x} - b_1\,\widetilde{x}^{\,2} - \frac{\omega_0\,\widetilde{x}\,\widetilde{y}}{\widetilde{x}+d_0}, \\
\dot{\widetilde{y}} &= a_2\,\widetilde{y} + \frac{\omega_1\,\widetilde{x}\widetilde{y}}{\widetilde{x}+d_1} - \frac{\omega_2\,\widetilde{y}\widetilde{z}}{\widetilde{y}+d_2}, \quad \dot{\widetilde{z}} = c_0\,\widetilde{z}^{2} - \frac{\omega_3\,\widetilde{z}^{2}}{\widetilde{y}+d_3},
\end{aligned} \tag{1}$$

where $\widetilde{x}$ is the density of prey at the bottom of the food chain, $\widetilde{y}$ is the density of the specialist predator and $\widetilde{z}$ is the density of the generalist predator. While the parameters are as follows: a_1 is the rate of the self-growth for prey $\widetilde{x}$, a_2 measures the rate at which $\widetilde{y}$ will die out when there is no prey left, $\omega_i^{\prime}s$ are the maximum values attainable by each per capita rate, d_0, d_1 quantify the extent to which the environment provides protection to the prey $\widetilde{x}$, b_1 measures the strength of competition among prey individuals $\widetilde{x}$, d_2 is the value of $\widetilde{y}$ at which per capita removal rate of $\widetilde{y}$ becomes $\omega_2/2$, d_3 represents the residual loss in $\widetilde{z}$ population due to severe scarcity of its favorite food $\widetilde{y}$, c_0 describes the growth rate of the generalist predator $\widetilde{z}$ by sexual reproduction, the number of males and females being assumed to be equal. All parameters are positive values, where they reduced the number of parameters by the relations:

$$a = \frac{b_1\,d_0}{a_1}, \quad b = \frac{a_2}{a_1}, \quad c = \frac{\omega_1}{a_1}, \quad d = \frac{b_1\,d_1}{a_1}, \quad e = \frac{b_1\,d_2\,\omega_0}{a_1^2}, \quad f = \frac{c_0\,a_1^2}{b_1\,\omega_0\,\omega_2},$$

$$g=\frac{\omega_3}{\omega_2},\ h=\frac{b_1\,d_3\omega_0}{a_1^2},\ x=\frac{a_1}{b_1}\,\widetilde{x},\ y=\frac{a_1^2}{b_1\,\omega_0}\,\widetilde{y},\ z=\frac{a_1^3}{b_1\,\omega_0\,\omega_2}\,\widetilde{z},\ t=\frac{\widetilde{t}}{a_1}. \tag{2}$$

Then the system becomes

$$\begin{aligned}\dot{x} &= x\,(1-x)-\frac{x\,y}{x+a},\\ \dot{y} &= -b\,y+\frac{c\,x\,y}{x+d}-\frac{y\,z}{y+e}, \quad \dot{z}=f\,z^2-\frac{g\,z^2}{y+h}.\end{aligned} \tag{3}$$

2. Knots and Braids

A knot is an embedding of S^1 in R^3. Knots are classified up to ambient isotopy, a trivial knot is any knot equivalent to a circle. An interesting study of knots up to their regular projections in R^2, where a projection is regular if the set of self intersection consists of finite number of transverse double points. A knot is said to be prime if it can not be factored by a S^2 in R^3 to two non trivial knots, one inside S^2 and the other outside it, where the intersection of the knot with the sphere is just two points. The knot is composite if it does not prime, i.e. neither factor is the unknot. The braid group B_n has a group presentation

$$B_n = \left\{\sigma_i,\ \ i=1,2,\ldots,n-1 \;\middle|\; \begin{array}{ll}\sigma_i\,\sigma_{i+1}\,\sigma_i=\sigma_{i+1}\,\sigma_i\,\sigma_{i+1} & i=1,2,\ldots,n-2\\ \sigma_i\,\sigma_j=\sigma_j\,\sigma_i & |i-j|\geq 2\end{array}\right\}, \tag{4}$$

where the generators σ_i and their inverses σ_i^{-1} can be represented geometrically as illustrated in Figure 1. A braid is positive if all its crossings are positive according to the orientation in Figure 2. Any knot can be represented as a closed braid, [7], that is as in Figure 3.

3. Knot Holders

The existence and uniqueness of solutions of differential equations will open the door for the entrance and influence of knot theory, [10, 5]. That because periodic orbits are linked together, so it can be characterized using concepts from knot theory. The existence of a two-dimensional manifold, called knot holder, such that all periodic orbits can be placed on it while preserving their invariant linking properties, allows one then to describe concisely the global topological organization of the attractors [1, 2]. Knot holder is a compact

2-dimensional manifold with boundary and smooth expanding semi-flow built from a finite number of branch line charts and the semi-flow is tangent to the rest of the boundary. It is formed by attaching exit sets to entrance sets, such that the exit set consists of the middle portions of the splitting charts and that the entrance set be empty. It follows that the number of joining charts is equal to the number of splitting charts, as in Figure 4. The invariant set of a knot holder is the set of orbits of the semi-flow that never exit, [9]. Any knot holder can be braided, that it can be isotoped so that all the closed orbits are presented as braids, [4].

This manifold can be characterized by some algebraic invariants, such as linking numbers, torsion and layering. If we look to the knot holder as a braid template, that can be described by the braid word describing the crossing structure of the branches of the template, the framed braid which involves the twisting in each branch and the layering information which determines the order in which branches are glued at the branch line. A knot holder T is a braided holder if it can be embedded in $D^2 \times S^1$ in such a way that every knot on T is a closed braid.

Definition 1. The braid linking matrix of a knot holder with k-branches is a square symmetric $k \times k$ matrix $B = (b_{ij})$, where b_{ii} is the sum of half-twists in the i-th branch (the local torsion of the i-th branch), and b_{ij} is the sum of the crossings between the i-th and the j-th branches of the ribbon graph with standard insertion. The off-diagonal elements of B are twice the linking numbers lk_{ij} of the ribbon graph for the i-th and j-th branches.

Definition 2. The torsion matrix $T = (t_i)$ is a column matrix, where t_i is the number of half twists in branch i. The full and half twists are given in Figure 5. Looking at any branch as a ribbon, then the branch i is orientation preserving if t_i even number, and it is orientation reversing if t_i is odd.

Definition 3. The layering matrix $L = (L_{ij})$, is a matrix of entries ± 1, where for $i < j$, $L_{ij} = 1$, if branch i is stacked on branch j, and $L_{ij} = -1$, otherwise.

In [8] and by a numerical experiments, it is shown that the plane projections of the different embedding included by the three dynamical variables of the system (2) are in a faithful visual representation of the original attractors projected onto the different planes of the phase space, Figure 6a. Also the associated first-return maps are found to be constituted by three monotonic branches as in Figure 6b.

They also shown that the chaotic attractor generated by this system is characterized by the knot holder in Figure 7. That because the associated first return maps are found to be constituted by three monotonic branches, as in Fig 6b.

4. Classifying Knots Associated to our System

The topological invariants of the knot holder, Figure 7, associated to the given system are

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}. \tag{5}$$

The knot holder in Figure 7 can be described by a braid, Figure 8, where the numbers in graph give the number of half twists in each branch, where the branch is denoted by a string, and the pass of the branches together is represented by the braid word $\sigma_2^2\, \sigma_1\, \sigma_2\, B_3$, where three is the number of branches.

According to the torsion matrix of the knot holder, the branches number 0 and 2 are orientation preserving, while branch number 1 is an orientation reversing. So that the associated 3-ary tree with n-period (levels) can be written according to the following steps [10]:

1. In the first step, write the symbolic names for the branches from left to right as $0, 1, 2$.

2. In the second step, we write the symbolic names as $(0, 1, 2)$ for the orientation preserving and $(2, 1, 0)$ for orientation reversing.

3. In the third step, we do the same from the 1 st to the 2 nd steps, as in the following table:

period	preserve	reverse	preserve
1	0	1	2
2	$(0,1,2)$	$(2,1,0)$	$(0,1,2)$
3	$(0,1,2),(2,1,0),$ $(0,1,2)$	$(2,1,0),(0,1,2),$ $(2,1,0)$	$(0,1,2),(2,1,0),$ $(0,1,2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

(6)

Theorem 1. *A braid representative for all links of period n on the knot holder associated to the given three species system is $\beta = \Delta_{n,2}\, \Delta_{n,3}^2\, \gamma\, b \in B_{3n}$, where $b\, \varepsilon B_{3n}^+$, is a positive permutation braid [3], and*

$$\Delta_{n,2} = (\sigma_{2n-1}\sigma_{2n-2} \ldots \sigma_{n+1})(\sigma_{2n-1}\sigma_{2n-2} \ldots \sigma_{n+2}) \ldots (\sigma_{2n-1}\sigma_{2n-2})(\sigma_{2n-1}),$$

$$\gamma = (\sigma_{2n}\sigma_{2n-1}\ldots\sigma_{n+1})(\sigma_{2n+1}\sigma_{2n}\ldots\sigma_{n+2})\ldots(\sigma_{3n-1}\sigma_{3n-2}\ldots\sigma_{2n}),$$
$$\Delta^2_{n,3} = (\sigma_{2n+1}\sigma_{2n+2}\ldots\sigma_{3n-1})^{n-1}. \quad (7)$$

Proof. The proof will be a direct consequence from the knot holder in Figure 7, where the branches numerated one and two have a half twist in n-strands and a full twist in n-strands, which is equivalent to the braid words $\Delta_n = (\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1)(\sigma_{n-1}\sigma_{n-2}\ldots\sigma_2)\ldots(\sigma_{n-1}\sigma_{n-2})(\sigma_{n-1})$, and Δ^2_n, respectively. We are going to write them as $\Delta_{n,2}$ and $\Delta^2_{n,3}$, to indicate that the first in the second n-band of strands while the second is in the third n-band of strands in the third n-band of strands. Also the braid word $\sigma_2^2\sigma_1\sigma_2$ is the algebraic representation of an n-parallel band of strands cross over another n-parallel band of strands. Finally b is coming from the nature of ordering of the branches at the branch line, no self crossings in each band and arcs cross at most once. Hence we have a positive permutation braid, [3]. □

Corollary 1. *The number of prime factors of the knots derived from the system are bounded.*

Proof. Our knot holder has a braid word $\sigma_2^2\sigma_1\sigma_2\varepsilon B_3$, as in Figure 8, that which describe the crossing structure of three branches, Figure 7. Also the branches have torsion matrix $T = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$, which means that the branch number zero has no twist, while branches numerated one and two have positive half and positive full twists, respectively. This implies that the given template is positive. Hence the given knots are positive, at least all crossings have the same sign. Which implies that the associated knots are in bounded prime factors.

Examples. (a) Now let us see how to find a knot from a given knot holder and a symbolic name. To do that we must read down the k-ary tree recording the branch names through which we pass, where the ordering at the n-th level of the tree is the correct ordering for all the knots of period n on the template. Using the ordering at the bottom of the k-ary, we can draw the knots according to the associated symbolic names, see [4]. The ordering, up to period two from the above k-ary tree, is $00 \prec 01 \prec 02 \prec 12 \prec 11 \prec 10 \prec 20 \prec 21 \prec 22$. To draw the associated link, we collect the possible cyclic permutations, 00, 11, 22, 01 and 10, 02 and 20, 12 and 21. Then the derived link has 6 components.

(b) Consider the orbits 012, 011 on the knot holder in Figure 7 and from the table above. To construct the derived link, at first construct all cyclic permutations of the given orbits, which will be 012, 120, 201, 011, 110, 101. Then put them in the order $011 \prec 012 \prec 101 \prec 110 \prec 120 \prec 201$, but the branch numerated one has a reverse orientation. So the ordered sequence will be reordered $011 \prec 012 \prec 120 \prec 110 \prec 101 \prec 201$.

5. Conclusion

An algorithm to find positive braid representatives for closed periodic orbits arising from a very realistic three species model is given. It is shown that the associated knot holder is positive, by means of braid langue. Hence it is concluded that the resulting knots and links have a bounded number of prime factors. Also an algorithm to write down the symbolic name of a closed periodic orbit is described. In fact this can enables us to employ the knot invariants like linking numbers to classify the closed periodic orbits.

References

[1] J. Birman, R.F. Williams, Knotted periodic orbits in dynamical systems -II: Knot holders for fibered knots, *Cont. Math.*, **20** (1983), 1-60.

[2] J. Birman, R.F. Williams, Knotted periodic orbits in dynamical systems -I: Lorenz's, equations, *Topology*, **22**, No. 1 (1983), 47-82.

[3] E.A. Elrifai, H. Morton, Algorithms for positive braids, *Quart. J. Math. Oxford*, **45**, No. 2 (1994), 479-497.

[4] J. Franks, R. Williams, Entropy and knots, *Trans. Amer. Math. Soc.*, **291**, No. 1 (1985), 241-253.

[5] R. Gilmore, M. Lefranc, *The Topology of Chaos, Alice in Stretch and Squeezeland*, Wiley (2002).

[6] H.C.J. Godfrey, B.T. Grenfell, The continuing quest for chaos, *Trends Ecol. Evol.*, **8** (1993), 43-44.

[7] Akio Kawauchi, *A survey of knot theory*, Birkhäuser Verlag (1996).

[8] Christophe Letellier, Luis A. Aguirre, Jean Maquet, M.A. Aziz-Alaoui, Should all the species on a food chain be counted to investigate the global dynamics?, *Chaos, Solitons and Fractals*, **13** (2002), 1099-1113.

[9] Michael C. Sullivan, Knots on a positive template have a bounded number of prime factors, *Algebraic and Geometric Topology*, **5** (2005), 563-576.

[10] N.B. Tufillaro, T. Abbott, J. Reilly, *An Experimental Approach to Nonlinear Dynamics and Chaos*, Reading, Addison Wesley (1992).