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A STUDY OF HYPERSURFACES ON SPECIAL FINSLER SPACES

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Abstract: The study of special Finsler spaces has been introduced by M. Matsumoto [4]. The purpose of the present paper is to study hypersurfaces of special Finsler spaces like quasi-C-reducible, C-reducible, Semi-C-reducible, P2-like, P-reducible, S3-like, and C2-like. And also we proved some hypersurfaces are Riemannian under the condition that the vector C_i is tangential to hypersurface.

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1. Introduction

The study of spaces endowed with generalized metrics was initiated by P. Finsler in 1918. Since then many important results have been achieved with respect to both the differential geometry of Finsler space and its application to variational problems, theoretical physics and engineering. L. Berwald and E. Cartan made a great contribution in developing tensor calculus of Finsler spaces corresponding to that on Riemannian spaces.

The theory of hypersurfaces in general depends to a large extent on the

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study of the behavior of curves in them. The authors G.M. Brown, Moor, C. Shibata, M. Matsumoto, B.Y. Chen, C.S. Bagewadi, L.M. Abatangelo, Dragomir and S. Hojo have studied different properties of subspaces of Finsler, Kahler and Riemannian spaces.

The author G.M. Brown [1] has published a paper – A study on tensors which characterize hypersurfaces of a Finsler space. M. Kitayama [2]-Finsler hypersurfaces and metric transformations. U.P. Singh-Hypersurfaces of C-reducible Finsler spaces, The authors S.K. Narasimhamurthy and C.S. Bagewadi (see [5], [6], [7], [8]) have studied C-conformal properties of special Finsler spaces and its applications.

2. Preliminaries

Let $F^n = (M^n, L)$ be a Finsler space on a differentiable manifold M endowed with a fundamental function $L(x, y)$. By a Finsler space, we mean a triple $F^n = (M, D, L)$, where M denotes n-dimensional differentiable manifold, D is an open subset of a tangent vector bundle TM endowed with the differentiable structure induced by the differentiable structure of the manifold TM and $L : D \longrightarrow R$ is a differentiable mapping having the properties:

$$
\begin{aligned}
&i) && L(x,y) > 0, \quad \text{for } (x,y)\,\varepsilon\, D,\\
&ii) && L(x,\lambda y) = |\lambda| L(x,y), \quad \text{for any } (x,y)\,\varepsilon\, D \,and\, \lambda\,\varepsilon\, R,\\
& && \text{such that } (x,\lambda y)\,\varepsilon\, D,\\
&iii) && \text{The d-tensor field } g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \quad \text{for } (x,y)\,\varepsilon\, D,
\end{aligned}
$$

where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$, is non degenerate on D.

We have the following identities (see [1] and [9]):

$$
\begin{aligned}
&a) && g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \;\; g^{ij} = (g_{ij})^{-1}, \;\; \dot{\partial}_i = \frac{\partial}{\partial y^i},\\
&b) && C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}, \; C^k_{ij} = \frac{1}{2} g^{km}(\dot{\partial}_m g_{ij}), \; h_{ij} = g_{ij} - l_i l_j\\
&c) && \gamma^i_{jk} = \frac{1}{2} g^{ir}(\partial_j g_{rk} + \partial_k g_{rj} - \partial_r g_{jk}),\\
&d) && G^i = \frac{1}{2}\gamma^i_{jk} y^j y^k, \quad G^i_j = \dot{\partial}_j G^i,\\
& && G^i_{jk} = \dot{\partial}_k G^i_j, \; G^i_{jkl} = \dot{\partial}_l G^i_{jk},\\
&e) && F^i_{jk} = \frac{1}{2} g^{ir}(\delta_j g_{rk} + \delta_k g_{rj} - \delta_r g_{jk}),
\end{aligned}
\tag{2.1}
$$

$$f)\qquad P_{hijk} = u_{(hi)}\{C_{ijk/h} + C_{hjr}C^r_{ik/0}\},$$
$$g)\qquad S_{hijk} = u_{(jk)}\{C_{hkr}C^r_{ij}\},$$

where $\delta_j = \partial_j - G^r_j\dot{\partial}_r$, the index 0 means contraction by y^i and the notation $u_{(jk)}$ denotes the interchange of indices j, k and subtraction.

3. Hypersurface F^{n-1} of the Finsler Space F^n

Finsler hypersurface $F^{n-1} = (M^{n-1}, L(u,v))$ of a Finsler space $F^n = (M^n, L(x, y))$ $(n \geq 4)$ may be parametrically represented by the equation $x^i = x^i(u^\alpha)$, where Latin indices i, j, … take values $1, \ldots, n$ and Greek indices α, β, … take values $1, 2, \ldots, n-1$.

The fundamental metric tensor $g_{\alpha\beta}$ and Cartan's C-tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by [2], [9]:

$$a)\qquad g_{\alpha\beta}(u,v) = g_{ij}(x,\dot{x})B^i_\alpha B^j_\beta, \tag{3.1}$$
$$b)\qquad C_{\alpha\beta\gamma} = C_{ijk}B^i_\alpha B^j_\beta B^k_\gamma,$$

where the matrix of projection factor $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ is of range $n-1$. The following notation are also employed

$$B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \quad B^i_{o\beta} = v^\alpha B^i_{\alpha\beta}, \quad B^{ijk\ldots}_{\alpha\beta\gamma\ldots.} = B^i_\alpha B^j_\beta B^k_\gamma \ldots ..$$

If the supporting element y^i at a point u^α of M^{n-1} is assumed to be tangential to M^{n-1}, we may then write $y^i = B^i_\alpha(u)v^\alpha$, where v^α is thought of as the supporting element of M^{n-1} at a point u^α.

We use the following notation on Finsler hypersurface (see [8], [9]):

$$a)\qquad C^\alpha_{\beta\gamma} = B^\alpha_i C^i_{jk} B^{jk}_{\beta\gamma},$$
$$b)\qquad C_\alpha = B^i_\alpha C_i,$$
$$c)\qquad B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta, \tag{3.2}$$
$$d)\qquad g^{\alpha\beta} = g^{ij} B^{\alpha\beta}_{ij},$$
$$e)\qquad h_{\alpha\beta} = g_{\alpha\beta} - l_\alpha l_\beta, \quad \text{and} \quad h_{\alpha\beta} = h_{ij}B^{ij}_{\alpha\beta},$$
$$f)\qquad l_\alpha = B^i_\alpha l_i.$$

4. Hypersurface of the Special Finsler Spaces

Now we consider the special Finsler spaces like quasi-C-reducible, C-reducible, P2-like, P-reducible, S3-like, and C2-like. Then we prove all these special Finsler space are well-defined in Finsler hypersurface F^{n-1} under some conditions.

Definition 1. (see [5]) A Finsler space $F^n(n > 2)$ is called a quasi-C-reducible, if the torsion tensor C_{ijk} is written as

$$C_{ijk} = A_{ij}C_k + A_{jk}C_i + A_{ki}C_j, \tag{4.1}$$

where A_{ij} is a symmetric Finsler tensor field satisfying $A_{io} = A_{ij}y^j = 0$.

Contracting (4.1) by projection factor $B^{ijk}_{\alpha\beta\gamma}$, we obtain

$$\begin{aligned} C_{ijk}B^{ijk}_{\alpha\beta\gamma} &= (A_{ij}C_k + A_{jk}C_i + A_{ki}C_j)B^{ijk}_{\alpha\beta\gamma}, \\ C_{ijk}B^{ijk}_{\alpha\beta\gamma} &= A_{ij}B^{ij}_{\alpha\beta}C_kB^k_\gamma + A_{jk}B^{jk}_{\beta\gamma}C_iB^i_\alpha + A_{ki}B^{ki}_{\gamma\alpha}C_jB^j_\beta. \end{aligned}$$

Using equation (3.1) and (3.2) we obtain

$$C_{\alpha\beta\gamma} = A_{\alpha\beta}C_\gamma + A_{\beta\gamma}C_\alpha + A_{\gamma\alpha}C_\beta, \tag{4.2}$$

where we setting $A_{\alpha\beta} = A_{ij}B^{ij}_{\alpha\beta}$ is a symmetric Finsler tensor field on hypersurface F^{n-1}. Thus we have:

Theorem 1. *A hypersurface F^{n-1} of a quasi-C-reducible Finsler space F^n is quasi-C-reducible.*

Suppose we assume that $C_\alpha = 0$, that implies

$$C_iB^i_\alpha = 0, \tag{4.3}$$

it means that C_i is tangential to the hypersurface F^{n-1}, then from (4.2), we have $C_{\alpha\beta\gamma} = 0$, therefore by Deicke's theorem the quasi-C-reducible Finsler hypersurface is Riemannian, which proves the following:

Theorem 2. *A quasi-C-reducible Finsler hypersurface F^{n-1} is Riemannian, if the vector C_i is tangential to hypersurface F^{n-1}.*

Definition 2. (see [3]) A Finsler space $F^n(n > 2)$ with non-zero length C of the torsion vector C_i is said to be semi-C-reducible, if the torsion tensor C_{ijk} is of the form

$$C_{ijk} = p(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j)/(n + 1) + qC_iC_jC_k/C^2, \tag{4.4}$$

where $C^2 = g^{ij}C_iC_j = C_iC^i$ and $p + q = 1$.

Contracting (4.4) by $B^{ijk}_{\alpha\beta\gamma}$ and using (3.1) and (3.2(b), (e)), we obtain

$$C_{ijk}B^{ijk}_{\alpha\beta\gamma} = p(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j)B^{ijk}_{\alpha\beta\gamma}/(n+1) + q(C_iC_jC_k/C^2)B^{ijk}_{\alpha\beta\gamma}.$$

A direct calculation will give

$$C_{\alpha\beta\gamma} = p(h_{\alpha\beta}C_\gamma + h_{\beta\gamma}C_\alpha + h_{\gamma\alpha}C_\beta)/(n) + qC_\alpha C_\beta C_\gamma/\overline{C}^2, \tag{4.5}$$

where $\overline{C}^2 = C_\alpha C^\alpha$, $\quad C_\alpha = C_i B^i_\alpha$, $\quad C^\alpha = C^i B^\alpha_i$. Therefore we have:

Theorem 3. *A hypersurface F^{n-1} of a semi-C-reducible Finsler space F^n is semi-C-reducible.*

Again by using condition (4.3) in (4.5), we obtain the following:

Theorem 4. *A hypersurface F^{n-1} of semi-C-reducible Finsler space is Riemannian, if the torsion vector C_i is tangential to the hypersurface F^{n-1}.*

Definition 3. (see [6]) A Finsler space $F^n (n > 2)$ is said to be C-reducible, if it satisfies the equation

$$(n+1)C_{ijk} = h_{ij}C_k + h_{jk}C_i + h_{ki}C_j, \tag{4.6}$$

where $C_i = g^{jk}C_{ijk}$.

Contracting (4.6) by $B^{ijk}_{\alpha\beta\gamma}$ and using (3.1) and (3.2(b), (e)), we obtain

$$nC_{\alpha\beta\gamma} = h_{\alpha\beta}C_\gamma + h_{\beta\gamma}C_\alpha + h_{\alpha\gamma}C_\beta, \tag{4.7}$$

where $C_\alpha = C_i B^i_\alpha = g^{\beta\gamma}C_{\alpha\beta\gamma}$. Hence we have:

Theorem 5. *(see [10]) A hypersurface of a C-reducible Finsler space is a C-reducible.*

Using the condition (4.3) in (4.7), we state that the following result:

Theorem 6. *A hypersurface F^{n-1} of a C-reducible Finsler space is Riemannian, if the torsion vector C_i is tangential to hypersurface F^{n-1}.*

Definition 4. (see [7]) A Finsler space $F^n (n \geq 2)$ with $C^2 = C_iC^i \neq 0$ is called C2-like, if the torsion tensor C_{ijk} is satisfies the equation

$$C_{ijk} = C_iC_jC_k/C^2. \tag{4.8}$$

Contracting (4.8) by $B^{ijk}_{\alpha\beta\gamma}$ and using (3.1) and (3.2(b), (e)), we obtain

$$C_{\alpha\beta\gamma} = C_\alpha C_\beta C_\gamma/\overline{C}^2, \tag{4.9}$$

where $\overline{C}^2 = C_\alpha C^\alpha$. Now we consider the special case for $p = 0$ in equation (4.5), and by virtue of $p+q = 1$, we have $q = 1$ and thus we led to the following theorem.

Theorem 7. *A semi-C-reducible Finsler hypersurface F^{n-1} is C2-like Finsler hypersurface, if $p = 0$.*

Consequently, taking into account of condition (4.3) in (4.9), we state the following:

Theorem 8. *A C2-like Finsler hypersurface F^{n-1} is Riemannian, if the torsion vector C_i is tangential to hypersurface F^{n-1}.*

Definition 5. A Finsler space $F^n (n > 2)$ is P2-like, if it is characterized by

$$P_{hijk} = K_h C_{ijk} - K_i C_{hjk}, \tag{4.10}$$

where $K_h = K_h(x, y)$ is a covariant vector field.

Contracting (4.10) by $B^{hijk}_{\delta\alpha\beta\gamma}$ and using (3.1), we have

$$\begin{aligned} P_{hijk} B^{hijk}_{\delta\alpha\beta\gamma} &= (K_h C_{ijk} - K_i C_{hjk}) B^{hijk}_{\delta\alpha\beta\gamma}, \\ P_{\delta\alpha\beta\gamma} &= K_h B^h_\delta C_{ijk} B^{ijk}_{\alpha\beta\gamma} - K_i B^i_\alpha C_{hjk} B^{hjk}_{\delta\beta\gamma}, \\ P_{\delta\alpha\beta\gamma} &= K_\delta C_{\alpha\beta\gamma} - K_\alpha C_{\delta\beta\gamma}, \end{aligned}$$

where we set $K_\alpha = K_i B^i_\alpha$ is a covariant vector field on F^{n-1}. Thus we obtain:

Theorem 9. *A hypersurface of a P2-like Finsler space is P2-like.*

Definition 6. (see [7]) A Finsler space F^n is called a P-reducible, if the torsion tensor P_{ijk} is written as

$$P_{ijk} = (h_{ij} P_k + h_{jk} P_i + h_{ki} P_j)/(n + 1), \tag{4.11}$$

where $P_i = P^m_{im} = C_{i/0}$.

Contracting (4.11) by $B^{\alpha\beta\gamma}_{ijk}$ and using (3.2(e)), we obtain

$$\begin{aligned} P_{ijk} B^{ijk}_{\alpha\beta\gamma} &= (h_{ij} P_k + h_{jk} P_i + h_{ki} P_j) B^{ijk}_{\alpha\beta\gamma}/(n + 1), \\ P_{ijk} B^{ijk}_{\alpha\beta\gamma} &= (h_{\alpha\beta} P_\gamma + h_{\beta\gamma} P_\alpha + h_{\gamma\alpha} P_\beta)/n, \end{aligned}$$

where we set $P_i B^i_\alpha = P_\alpha = C_{\alpha/0}$. Hence we have the following result.

Theorem 10. *A hypersurface of a P-reducible Finsler space is P-reducible.*

Next we consider the curvature tensor of F^n

$$S_{hijk} = C_{hkr} C^r_{ij} - C_{hjr} C^r_{ik}.$$

Contracting above equation by $B^{hijk}_{\delta\alpha\beta\gamma}$ and using (2.1), we have

$$S_{hijk} B^{hijk}_{\delta\alpha\beta\gamma} = (C_{hkr} C^r_{ij} - C_{hjr} C^r_{ik}) B^{hijk}_{\delta\alpha\beta\gamma}$$

$$\begin{aligned} &= C_{hkr}C^r_{ij}B^{hijk}_{\delta\alpha\beta\gamma} - C_{hjr}C^r_{ik}B^{hijk}_{\delta\alpha\beta\gamma}, \\ S_{\delta\alpha\beta\gamma} &= C_{\delta\gamma\theta}C^\theta_{\alpha\beta} - C_{\delta\beta\theta}C^\theta_{\alpha\gamma}. \end{aligned}$$

Hence $S_{\delta\alpha\beta\gamma}$ is the curvature tensor of F^{n-1}.

Definition 7. (see [7]) A Finsler space $F^n (n > 3)$ is called S3-like, if the curvature tensor S_{hijk} is satisfies the equation

$$L^2 S_{hijk} = S(h_{hj}h_{ik} - h_{hk}h_{ij}), \tag{4.12}$$

where the scalar curvature $S = S_{hijk}g^{hj}g^{ik}$ is a function of position alone.

Contracting (4.12) by $B^{hijk}_{\delta\alpha\beta\gamma}$ and using (3.2(e)), we get

$$\begin{aligned} L^2 S_{hijk}B^{hijk}_{\delta\alpha\beta\gamma} &= S(h_{hj}h_{ik} - h_{hk}h_{ij})B^{hijk}_{\delta\alpha\beta\gamma}, \\ L^2 S_{\delta\alpha\beta\gamma} &= S(h_{\delta\beta}h_{\alpha\gamma} - h_{\delta\gamma}h_{\alpha\beta}), \end{aligned}$$

where the scalar curvature $S = S_{\delta\alpha\beta\gamma}g^{\delta\beta}g^{\alpha\gamma}$ and $g^{\alpha\beta} = g^{ij}B^{\alpha\beta}_{ij}$. Thus we state:

Theorem 11. *A hypersurface of a S3-like Finsler space is S3-like.*

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