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ON THE ABSOLUTE RIESZ SUMMABILITY OF ORTHOGONAL SERIES

Abdulcabbar Sönmez
Department of Mathematics
Erciyes University
Kayseri, 38039, TURKEY
e-mail: sonmez@erciyes.edu.tr

Abstract: The purpose of this paper is to give some general theorems on the $|\bar{N}, p_n; \delta|_k$ summability of ortogonal series, which generalize two theorems due to Okuyama [6] related to summability of ortogonal series.

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1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty \quad (P_{-i} = p_{-i} = 0, i \geq 1).$$

The sequence-to-sequence transformation

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1}$$

defines the sequence (T_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [8]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$ or summable $|R, P_n, 1|_k$, $k \geq 1$, if (see [1])

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$$\sum_{n=1}^{\infty}(P_n/p_n)^{k-1} \mid \Delta T_{n-1} \mid^k < \infty.$$

The case $k = 1$ is reduced to the absolute Riesz summability $\mid R, P_n, 1 \mid$ and further, in the special case $p_n = 1/n + 1$, the summability $\mid R, P_n, 1 \mid$ is the same as the absolute logarithmic summability. Let $\{\phi_n(x)\}$ be an orthonormal system defined in the interval (a, b). For a function $f(x) \in L^2(a, b)$ such that

$$f(x) \approx \sum_{n=0}^{\infty} a_n \phi_n(x),$$

we denote by $E_n^{(2)}(f)$ the best approximation to $f(x)$ in the metric of L^2 by means of polynomials of $\phi_0(x), \ldots, \phi_{n-1}(x)$. It is well known that

$$E_n^{(2)}(f) = (\sum_{j=n}^{\infty} \mid a_j \mid^2)^{\frac{1}{2}}.$$

Let $P(x)$ be a strictly increasing function such that $P(n) = P_n$ for integer n and $P'(n) = p_n$. We denote the inverse function of $P(x)$ by $\Lambda(x)$ and put $v_n = [\Lambda(2^n)]$, where $[x]$ denotes the integral part of x. Then we put

$$C_n = (\sum_{j=v_n+1}^{v_{n+1}} \mid a_j \mid^2)^{\frac{1}{2}} \qquad (n = 0, 1, \ldots). \tag{2}$$

We write $\Delta\omega_n = \omega_n - \omega_{n-1}$ for any sequence $\{\omega_n\}$. It denotes a positive absolute constant that is not always the same.

2. Preliminary Results

Dealing with the absolute Riesz summability of orthogonal series, Moricz [3] proved the following generalization of a theorem due to Tandori [8].

Theorem A. *The orthogonal series*

$$\sum_{n=0}^{\infty} a_n \phi_n(x)$$

for every orthonornal system $\{\phi_n(x)\}$ is summable $\mid R, P_n, 1 \mid$ almost everywhere if and only if the series

$$\sum_{n=0}^{\infty} C_n \tag{3}$$

converges, where C_n is defined by (2).

Okuyama and Tsuchikure [4] proved the following equivalent theorem.

Theorem B. *The convergence of the series (3) is equivalent to convergence of the series*

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} (\sum_{j=1}^{\infty} P_{j-1}^2 \mid a_j \mid^2)^{\frac{1}{2}} .$$

Leindler [2] also proved a certain symmetrical analogue of Theorem B.

Theorem C. *The convergence of the series (3) is equivalent to the convergence of the series*

$$\sum_{n=0}^{\infty} p_n (\sum_{j=n}^{\infty} P_j^{-2} \mid a_j \mid^2)^{\frac{1}{2}} .$$

Theorem D. (see [6]) *Let $1 \leq k \leq 2$. Then the conditions*

i) $\sum_{n=0}^{\infty} C_n^k < \infty$;

ii) $\sum_{n=1}^{\infty} \frac{p_n}{P_n {P_{n-1}}^k} (\sum_{j=1}^{n} P_{j-1}^2 \mid a_j \mid^2)^{\frac{k}{2}} < \infty$;

iii) $\sum_{n=1}^{\infty} p_n (\sum_{j=n}^{\infty} P_j^{-2k} \mid a_j \mid^2)^{\frac{k}{2}} < \infty$

are mutually equivalent.

Theorem E. (see [6]) *Let $1 \leq k \leq 2$ and $\{p_n\}$ be a positive sequence. If the series*

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n {P_{n-1}}^k} (\sum_{j=1}^{n} P_{j-1}^2 \mid a_j \mid^2)^{\frac{k}{2}} < \infty,$$

then the orthogonal series

$$\sum_{n=0}^{\infty} a_n \phi_n(x) \tag{4}$$

is summable $\mid R, p_n, 1 \mid_k$ almost everywhere.

Now we will give a definition due to Bor [1].

The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n; \delta \mid_k$, $k \geq 1$ and $\delta \geq 0$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k + k - 1} \mid T_n - T_{n-1} \mid^k < \infty.$$

Here we generalize Theorems D and E as follows.

Theorem 1. *Let $1 \leq k \leq 2$, $\delta \geq 0$ and $1 - \delta k > 0$. Then the conditions:*

i) $$\sum_{n=0}^{\infty} C_n^k < \infty \,,$$

ii) $$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k} (\sum_{j=1}^{n} P_{j-1}^2 \mid a_j \mid^2)^{\frac{k}{2}} < \infty,$$

$$\sum_{n=v_p+1}^{v_{p+1}} (P_n/p_n)^{\delta k-1} (1/P_{n-1}) \geq (P_{v_{p+1}}/p_{v_{p+1}})^{\delta k} (\frac{1}{P_{v_p}} - \frac{1}{P_{v_{p+1}}}) \,,$$

iii) $$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k} p_n (\sum_{j=n}^{\infty} P_j^{-2k} \mid a_j \mid^2)^{\frac{k}{2}} < \infty$$

are mutually equivalent.

Theorem 2. *Let $1 \leq k \leq 2$, $\delta \geq 0$ and $\{p_n\}$ be a positive sequence. If the series*

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k} (\sum_{j=1}^{n} P_{j-1}^2 \mid a_j \mid^2)^{\frac{k}{2}} < \infty,$$

then the orthogonal series

$$\sum_{n=0}^{\infty} a_n \phi_n(x)$$

is summable $\mid \bar{N}, p_n; \delta \mid_k$ almost everywhere.

Proof of Theorem 1. First we prove that the condition (i) is equivalent to the condition (ii). To show that the condition (ii) implies the condition (i), we suppose that the number $m_0(n)$ is the integer such that $v_{m_0(n)} < n \leq v_{m_0(n)+1}$. Then we have

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k} (\sum_{j=1}^{n} P_{j-1}^2 \mid a_j \mid^2)^{\frac{k}{2}}$$

$$\geq \sum_{n=1}^{\infty} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k} (\sum_{m=0}^{m_0(n)-1} \sum_{j=v_m+1}^{v_{m+1}} P_{j-1}^2 \mid a_j \mid^2)^{\frac{k}{2}}$$

$$\geq \sum_{n=1}^{\infty} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k} (\sum_{m=0}^{m_0(n)-1} P_{v_{m_j}}^2 \sum_{j=v_m+1}^{v_{m+1}} \mid a_j \mid^2)^{\frac{k}{2}}$$

$$= \sum_{n=1}^{\infty} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k} (\sum_{m=0}^{m_0(n)-1} P_{v_m}^2 C_m^2)^{\frac{k}{2}}$$

$$\geq \sum_{p=1}^{\infty} \sum_{n=v_p+1}^{v_{p+1}} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k} (\sum_{m=0}^{m_0(n)-1} P_{v_m}^2 C_m^2)^{\frac{k}{2}}$$

$$\geq \sum_{p=1}^{\infty} \sum_{n=v_p+1}^{v_{p+1}} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k} (\sum_{m=0}^{p-1} P_{v_m}^2 C_m^2)^{\frac{k}{2}}$$

$$\geq \sum_{p=1}^{\infty} C_{p-1}^k \frac{P_{v_{p-1}}^k}{P_{v_{p+1}}^{k-1}} \sum_{n=v_p+1}^{v_{p+1}} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}}$$

$$\geq \sum_{p=1}^{\infty} C_{p-1}^k (\frac{P_{v_{p-1}}}{P_{v_{p+1}}})^k P_{v_{p+1}} (\frac{P_{v_{p+1}}}{p_{v_{p+1}}})^{\delta k} (\frac{1}{P_{v_p}} - \frac{1}{P_{v_{p+1}}})$$

$$= \sum_{p=1}^{\infty} C_{p-1}^k (\frac{P_{v_{p-1}}}{P_{v_{p+1}}})^k (\frac{P_{v_{p+1}}}{p_{v_{p+1}}})^{\delta k} (\frac{P_{v_{p+1}}}{P_{v_p}} - \frac{P_{v_{p+1}}}{P_{v_{p+1}}})$$

$$\geq \sum_{p=1}^{\infty} C_{p-1}^k (\frac{2^{p-1}}{2^{p+1}}) (\frac{2^{p+1}}{2^p})^{\delta k} (\frac{2^{p+1}}{2^p} - 1)$$

$$\geq \sum_{p=1}^{\infty} C_{p-1}^k (\frac{1}{4})^k (2)^{\delta k} = (2)^{k\delta-2k} \sum_{p=1}^{\infty} C_{p-1}^k = (2)^{k\delta-2k} \sum_{p=0}^{\infty} C_p^k$$

by virtue of the fact that

$$\frac{P_{v_{p+1}}}{P_{v_p}} - \frac{P_{v_{p+1}}}{P_{v_{p+1}}} \geq \frac{2^{p+1}}{2^p} - 1 = 1\,.$$

That the condition (i) implies the condition (ii) is similarly proved by the same method as that used by Moricz [3]. That the condition (i) is equivalent to the condition (iii) is also proved by the similar method that used by Leindler [2]. Thus the proof Theorem 1 is completed. □

Proof of Theorem 2. Let $T_n(x)$ be the n-th Riesz mean of the series (4). Then we have by (1)

$$\Delta T_n(x) = T_n(x) - T_{n-1}(x) = \frac{p_n}{P_n P_{n-1}} \sum_{j=1}^{n} P_{j-1} a_j \phi_j(x).$$

Using the Hölder inequality and the orthogonality

$$\int_a^b \mid \Delta T_n(x) \mid^k dx \leq A\{\int_a^b \mid \Delta T_n(x) \mid^2 dx\}^{k/2}$$

$$A(\frac{p_n}{P_n P_{n-1}})^k (\sum_{j=1}^{n} P_{j-1}^2 \mid a_j \mid^2)^{\frac{k}{2}}\,,$$

and then

$$\sum_{n=1}^{\infty}(P_n/p_n)^{\delta k+k-1}\int_a^b \mid \Delta T_n(x)\mid^k dx$$

$$\leq A\sum_{n=1}^{\infty}(P_n/p_n)^{\delta k+k-1}(\frac{p_n}{P_nP_{n-1}})^k(\sum_{j=1}^{n}P_{j-1}^2\mid a_j\mid^2)^{\frac{k}{2}}$$

$$= A\sum_{n=1}^{\infty}(P_n/p_n)^{\delta k-1}\frac{1}{P_{n-1}^k}(\sum_{j=1}^{n}P_{j-1}^2\mid a_j\mid^2)^{\frac{k}{2}},$$

which is convergent by the assumption and from Beppo-Lèvi Lemma we complete the proof.

In our theorems, if we take $\delta = 0$, then we establish the theorem due to Okuyama[6].

If we take $\delta = 0$ and k=1then we establish the theorem due to Moricz[3].

If we take $p_n = 1$ for all $n \in N$, $\delta = 0$ and k=1, then we establish the theorem due to Tandori [8].

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