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ECONOMIC DEVELOPMENT AND SUSTAINABILITY IN A SOLOW MODEL WITH NATURAL CAPITAL AND LOGISTIC POPULATION CHANGE

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Abstract: This paper analyzes how a logistic labor growth rate affects the dynamics of the Solow model with the natural capital introduced as a factor of production. The conditions for long-run sustainability of the economy are investigated. For any given tax rate, the set of sustainable marginal propensity to consume is derived. The nature of non-trivial steady states of the economy is determined.

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1. Introduction

One of the most important models elaborated to explain economic growth is the neoclassical growth model originated with the work of Solow [8] and Swan [9], who independently proposed similar one-sector models. Solow developed an

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analysis of growth which is in several ways closely related to the one provided by the neoclassical model, while Swan gave a rather similar analysis, but in a fashion that was less mathematically explicit. Solow's purpose in developing the model was to deliberately ignore some important aspects of macroeconomics, such as short-run fluctuations in employment and savings rates, in order to develop a model that attempted to describe the long-run evolution of the economy. The Solow model is designed to show how growth in the capital stock, growth in the labor force, and advances in technology interact and how they affect a nation's total output. The key assumption of the Solow model is that capital is subject to diminishing returns. Given a fixed stock of labor, the impact on output of the last unit of capital accumulated will always be less than the one before. Diminishing returns implies that at some point the amount of new capital produced is only just enough to make up for the amount of existing capital lost due to depreciation. At this point, because of the assumptions of no technological progress or labor force growth, the economy ceases to grow. From simply being a tool for the analysis of the growth process, the Solow model has been generalized in several different directions (see, for example, Hall and Taylor [5]; Mankiw [6]; Romer [7]). However, as noticed by Dasgupta [3], there was no mention of environmental resources. The implicit assumption, clearly undesirable, was that natural resources are neither scarce now nor scarce in the future. An effort to address this omission by treating natural capital as an essential factor of production was done by Tran-Nam [10], who investigated an infinite-horizon aggregative closed economy in which production depends on physical capital, natural capital and labor. By modeling the natural capital stock as a renewable resource, in the sense that damages done to the environment production and consumption externalities are reversible, he showed the economy to be sustainable in the long-run only if, for any marginal propensity to consume (MPC), the rate of taxation to maintain the environment exceeds a critical level, or, equivalently, for any tax rate, the MPC is below a critical value. If human activities have a net zero or negative effect on the environment, then the economy is unsustainable in the long-run, i.e. physical and natural capital per worker will tend to zero as time grows indefinitely large. If human activities produce a net beneficial effect on the environment, then the economy will converge to a unique and stable steady state. A natural question to be asked in Tran-Nam's model is what the impact of changes in the population growth rate would be, i.e. to examine the consequences of relaxing the assumption of constant population growth rate. In general, population is assumed to growth at a constant rate. However, this is not a good approximation to reality. The main problem is that population grows exponentially, and so it tends to infinity as time goes to infinity, which is clearly unrealistic. What is often observed, instead, is that, as the population grows, some members interfere with each other in competition for some critical resource. That competition diminishes the growth rate, until the population ceases to grow. Therefore, it seems reasonable for a good population model to reproduce this behavior. The logistic growth model, as proposed by Verhulst [11], is just such a model, and this will be the population law that we assume in this paper. Regarding the natural capital stock, this is modeled as a renewable resource, so that there is no distinction between resource and environmental economists. Notice that resource economists, who are interested in population ecology, characterize complex systems by the population sizes of different species, while environmental economists, who are interested in systems ecology, summarize complex systems in terms of indices of quality of air, soil or water. Here, we combine both approaches by treating the environmental capital as a stock of measurable in some constant quality units. Since we focus on economic theory, all practical problems associated with measuring natural capital are assumed away. The change in the stock of natural capital depends on its autonomous evolution, production and consumption externalities, and environment maintenance programs. By modeling the natural capital stock as a renewable resource, we have that damages done to the environment production and consumption externalities are reversible, and can be corrected by collective maintenance actions. In this framework, we see that the long-run sustainability of the economy depends crucially on human activities and on the stock of the environment. In fact, there is long-run sustainability for the economy if human activities have a net zero effect on the environment and the stock of the environment grows or remains unchanged autonomously over time. Whether there is long-run unsustainability for the economy if human activities have a net zero or negative effect on the environment and the stock of the environment decays autonomously over time. In addition, for any given tax rate (or MPC), we derive the set of sustainable MPCs (or tax rates). Finally, by analyzing non-trivial steady states of a sustainable economy, we find that if human activities have a negative effect on the environment and the environment grows over time, then there is a unique steady state equilibrium, which is a saddle, whereas, if human activities have a positive effect on the environment and the environment decays over time, then there is still a unique steady state equilibrium, which is a node.

2. The Model

We consider a production function of Cobb-Douglas type in the Solow model (see Solow [8]), where the natural capital is introduced as a factor of production. The function at time t is, therefore, given by

$$Y_t = K_t^{\eta} E_t^{\theta} L_t^{1-\eta-\theta}, \quad \eta, \theta, \eta+\theta \in (0,1),$$

where Y_t, K_t, E_t, and L_t denote output, physical capital, natural capital, and labor, respectively. We assume that the technology has positive and diminishing marginal products with respect to each input, and it satisfies the Inada conditions. From national income accounting for a closed economy, we have that the total amount of final goods in the economy must be used for consumption C_t, for savings S_t, or spent to maintain or improve the environment. Thus,

$$Y_t = C_t + S_t + T_t,$$

where S_t and T_t stand for saving and tax, respectively. Individuals are assumed to save a constant fraction $\tau \in (0,1)$ of their tax revenue, i.e. $T_t = \tau Y_t$, and to consume a constant fraction of disposable income, i.e. $C_t = a(Y_t - T_t)$, with $a \in (0,1)$ the marginal propensity to consume (MPC). Note that we have $C_t = a(1-\tau)Y_t$. Physical capital is assumed to be a homogeneous good that depreciates at the constant rate $\delta > 0$, i.e., at each point in time, a constant fraction of the capital stock wears out and, hence, can no longer be used for production. The net increase in the stock of physical capital at a point in time equals gross investment I_t less depreciation, i.e.

$$\dot{K}_t = I_t - \delta K_t,$$

where a dot over a variable denotes differentiation with respect to time. Since the economy is closed, aggregate investment is equal to savings, whereas investments equal savings. Consequently, the capital accumulation equation takes the form

$$\dot{K}_t = (1-a)(1-\tau)Y_t - \delta K_t. \tag{1}$$

Based on the feature that the production function has homogeneity of degree one in K_t, E_t, and L_t, output (income) can be written in terms of capital per worker as $y_t = k_t^{\eta} e_t^{\theta}$, where $y_t = Y_t/L_t$ is output per worker, $k_t = K_t/L_t$ is capital per worker, and $e_t = E_t/L_t$ is natural capital per worker. In this way, equation (1) can be rewritten as

$$\frac{\dot{K}_t}{L_t} = (1-a)(1-\tau)y_t - \delta k_t. \tag{2}$$

The right-hand side of this equation contains per capita variable only, but the left-hand side does not. We can take the derivative of $k_t = K_t/L_t$ with respect to time to get

$$\dot{k}_t = \frac{\dot{K}_t}{L_t} - k_t \frac{\dot{L}_t}{L_t}. \tag{3}$$

If we substitute (2) into the expression (3), we can rearrange terms and obtain

$$\dot{k}_t = (1-a)(1-\tau)k_t^{\eta} e_t^{\theta} - [\delta + n(L_t)]\, k_t, \tag{4}$$

where $n(L_t) = \dot{L}_t/L_t$ denotes the labor growth rate. In classical mathematical models of economic growth, it is usually assumed that the labor force L_t has an independent growth equation as employed by Malthus, i.e. $\dot{L}_t = nL_t$, where n is a positive constant. Thus, population growth is modeled as an exponential growth model. However, Malthus's population model predicts population growth without bound, although it is obvious that the human population cannot grow at a constant rate indefinitely. To remove the prediction of unbounded population size in the very long-run, Verhulst [11] suggested to augment the exponential population growth model by a multiplicative factor $-bL_t$, where b is a positive constant such that $n - bL_0 > 0$. By normalizing the number of people at time 0 to 1, i.e. $L_0 = 1$, this means to consider

$$\frac{\dot{L}_t}{L_t} = n - bL_t, \tag{5}$$

with $n > b > 0$. Equation (5) is the population law assumed in this model, i.e. $n(L_t) = n - bL_t$. As a function of time, equation (5) is a Bernoulli differential equation whose solution is

$$L_t = \frac{ne^{nt}}{n - b + be^{nt}}\,, \tag{6}$$

where e^{nt} denotes the exponential function of nt. From (6), we derive that L_t is monotone increasing from 1 to $L_\infty = n/b$. Regarding the evolution over time of the environmental stock E_t, we assume it to be described by the differential equation

$$\dot{E}_t = \alpha E_t + \phi T_t - \beta Y_t - \gamma C_t,$$

where α, ϕ, β, and γ are some constants. Note that this equation can be rewritten as

$$\dot{E}_t = \alpha E_t + [(\phi + \gamma a)\tau - (\beta + \gamma a)]\, Y_t. \tag{7}$$

In case there is no human economic activity, E_t changes over time at the exponential rate α, with the parameter α positive, zero or negative according to whether the environment grows, remains unchanged or decays autonomously over time. In case there is human economic activity, we have depletion of β units of E_t for every unit of the final good produced (the production of the final good causes external damages to the environment), and also that each unit of the final good consumed depletes γ units of the environmental stock. Furthermore, environmental programs, funded by the entire tax revenue, generate ϕ units of the environmental stock per unit of the tax spent. The government runs a balanced budget at any instant of time, the taxation revenue is costlessly collected, and there are no government failures. In order to transform (7) into a differential equation in terms of capital stock per worker, we differentiate E_t/L_t with respect to time, and substitute this result into the expression (7). In this way, we arrive to $\dot{e}_t = [(\phi+\gamma a)\tau - (\beta+\gamma a)]k_t^\eta e_t^\theta - [n(L_t) - \alpha]e_t$. In conclusion, what we have found is that the economy of this modified Solow model, with logistic-type population growth law and natural capital as a factor of production, is described by the following dynamical system

$$\begin{cases} \dot{k}_t &= Ak_t^\eta e_t^\theta - (\delta + n - bL_t)\, k_t, \\ \dot{e}_t &= Bk_t^\eta e_t^\theta - (n - bL_t - \alpha)\, e_t, \\ \dot{L}_t &= L_t(n - bL_t), \end{cases} \tag{8}$$

where

$$A = (1-a)(1-\tau) > 0, \quad B = (\phi+\gamma a)\tau - (\beta+\gamma a).$$

Given $k_0 > 0$, $e_0 > 0$, this Cauchy problem has a unique solution (k_t, e_t, L_t) defined on $[0, \infty)$ (see Birkhoff and Rota [2]).

3. Long-Run Sustainability Conditions

An economy is said to be long-run sustainable if, at any finite time t, per capita consumption c_t equals or exceeds a given subsistence consumption level $\bar{c} > 0$. Written in terms of output per worker, this means

$$k_t^\eta e_t^\theta \geq \frac{\bar{c}}{a(1-\tau)},$$

for all t. This in turn requires that k_t and e_t must be both at least positive. Contrary to the neoclassical Solow model, where there is no mention of environmental resources, and the implicit assumption is that environmental stock is fixed and does not depend on human activities, now the natural capital is an essential input in the production process. Thus, the survival of the economy will depend not only on consumption, but also on the environment. Life can be sustained if human beings enjoy a sufficient amount of environment. If one thinks of the environment as a private good (no joint consumption), then long-run sustainability also requires that the environmental stock never falls below a minimum life-sustaining level $\bar{e} > 0$, i.e. $e_t \geq \bar{e}$, for all t. In general, there is no guarantee that the stock of physical capital as well as the stock of natural capital remain positive as time grows indefinitely large. The next results show how this depends crucially on the sign of B.

Proposition 1. *Let $B = 0$. For all t, the time path of natural capital is given by*

$$e_t = \frac{e_0 e^{\alpha t}\left(n - b + be^{nt}\right)}{ne^{nt}}. \tag{9}$$

Proof. For $B = 0$, the second equation of (8) writes $\dot{e}_t = -\left(n - bL_t - \alpha\right) e_t$. Using equations (5), (6), we can check that (9) is the unique solution of this separable differential equation. □

Corollary 2. *Let $B = 0$. The function e_t decreases monotonically to* 0 *(resp. $e_0 b/n$) if $\alpha < 0$ (resp. $\alpha = 0$), while it diverges towards infinity if $\alpha > 0$.*

Proof. Let $\alpha \leq 0$. Then $\dot{e}_t < 0$. The second part of the statement follows from (9) by taking t to infinity. □

Proposition 3. *Let $B = 0$. For all t, the time path of physical capital is described by*

$$k_t = \frac{(1 - ze^{nt})\left(k_0^{1-\eta} + (1-z)^{1-(\eta+\theta)} A e_0^{\theta} \int\limits_0^t e^{Mt}(1 - ze^{nt})^{-[1-(\eta+\theta)]} dt\right)^{\frac{1}{1-\eta}}}{(1-z)e^{(\delta+n)t}},$$

where

$$M = \alpha\theta + (1-\eta)\delta + [1 - (\eta+\theta)]n > 0, \quad z = \frac{b}{b-n}.$$

Proof. Replacing e_t, given by (9), in $\dot{k}_t = Ak_t^{\eta} e_t^{\theta} - (\delta + n - bL_t)k$, we obtain a differential equation of Bernoulli type in k_t. Substitution $u_t = k_t^{1-\eta}$, this becomes a linear differential equation in u_t, whose solution can be easily determined. The statement then follows expressing everything in terms of k_t. Notice that z in the statement of this proposition comes from having written $L_t = (1-z)e^{nt}/(1 - ze^{nt})$. □

The next result shows how the solution k_t can be expressed via Gauss hypergeometric functions.

Corollary 4. *Let ${}_2F_1$ denote the Gauss hypergeometric function (see*

Appendix). Then

$$k_t = \frac{1 - ze^{nt}}{(1-z)e^{(\delta+n)t}} \left\{ k_0^{1-\eta} + \frac{(1-z)^{1-(\eta+\theta)} A e_0^{\theta}}{M} \right.$$
$$\times \left[e^{Mt} \, {}_2F_1 \left(\frac{M}{n}, 1-(\eta+\theta), \frac{M}{n}+1; ze^{nt} \right) \right.$$
$$\left. \left. - \, {}_2F_1 \left(\frac{M}{n}, 1-(\eta+\theta), \frac{M}{n}+1; z \right) \right] \right\}^{\frac{1}{1-\eta}}.$$

Proof. By operating the change of variable $x = e^{nt}$, we get

$$\int_0^t e^{Mt}(1-ze^{nt})^{-[1-(\eta+\theta)]} dt = \frac{1}{n} \int_1^{e^{nt}} x^{\frac{M}{n}-1}(1-zx)^{-[1-(\eta+\theta)]} dx$$
$$= \frac{1}{n} \int_0^{e^{nt}} x^{\frac{M}{n}-1}(1-zx)^{-[1-(\eta+\theta)]} dx - \frac{1}{n} \int_0^{1} x^{\frac{M}{n}-1}(1-zx)^{-[1-(\eta+\theta)]} dx.$$

The purpose is now to rewrite the above two integrals in a different way. We have

$$\begin{aligned} \int_0^1 x^{\frac{M}{n}-1}(1-zx)^{-[1-(\eta+\theta)]} dx &= \int_0^1 x^{\frac{M}{n}-1} (1-x)^0 (1-zx)^{-[1-(\eta+\theta)]} dx, \\ &= \frac{n}{M} \, {}_2F_1 \left(\frac{M}{n}, 1-(\eta+\theta), \frac{M}{n}+1; z \right). \end{aligned}$$

Notice that we have used the integral representation of an hypergeometric function ${}_2F_1$ as well as the following two properties of the Γ-function: $\Gamma(1) = 1$, $\Gamma(v+1) = v\Gamma(v)$, for all $v > 0$. Next, operating the change of variable $r = e^{-nt}x$ yields

$$\int_0^{e^{nt}} x^{\frac{M}{n}-1} (1-zx)^{-[1-(\eta+\theta)]} dx = e^{Mt} \int_0^1 r^{\frac{M}{n}-1} \left(1 - ze^{nt} r\right)^{-[1-(\eta+\theta)]} dr,$$
$$= \frac{n}{M} e^{Mt} \, {}_2F_1 \left(\frac{M}{n}, 1-(\eta+\theta), \frac{M}{n}+1; ze^{nt} \right).$$

The statement now follows by substituting these expressions in the integral formula of Proposition 3. □

Remark 5. Let $c \geq 1$. Since $\int_0^c x^{\frac{M}{n}-1}(1-zx)^{-[1-(\eta+\theta)]} dx \sim \int_0^c x^{\frac{M}{n}-1} dx$, the fact that $M/n > 0$ implies the convergence of the integrals written in the proof of the previous corollary.

Corollary 6. *Let $B = 0$. In the long-run, k_t converges to 0 if $\alpha < 0$, and to $\left[Ae_0^\theta b^\theta/(1-\eta)\delta n^\theta\right]^{\frac{1}{1-\eta}}$ if $\alpha = 0$, while it diverges to $+\infty$ if $\alpha > 0$.*

Proof. From Proposition 3 we derive that

$$k_t^{1-\eta} = \frac{(1-ze^{nt})\left[k_0^{1-\eta} + (1-z)^{1-(\eta+\theta)}Ae_0^\theta \int\limits_0^t e^{Mt}(1-ze^{nt})^{-[1-(\eta+\theta)]}dt\right]}{(1-z)e^{(\delta+n)t}}.$$

Let $\alpha < 0$. Recalling that $1 \le L_t \le n/b$, as t grows toward infinity, we have

$$k_t^{1-\eta} \sim \frac{(1-z)^{1-(\eta+\theta)}Ae_0^\theta \int\limits_0^t e^{(\alpha\theta+\delta)t}dt}{e^{\delta t}} \sim e^{\alpha\theta t}.$$

Therefore, $k_t^{1-\eta}$ converges to zero, i.e. $k_t \to 0$. Let $\alpha \ge 0$. Then $\lim\limits_{t\to\infty} k_t^{1-\eta}$ leads to an indeterminate form since both the numerator and the denominator in the expression $k_t^{1-\eta}$ go to infinity as t grows to infinity. For the denominator this is immediate, while for the numerator this comes from the inequality

$$\int\limits_0^t e^{Mt}(1-ze^{nt})^{-[1-(\eta+\theta)]}dt \ge \int\limits_0^t e^{[\alpha\theta+(1-\eta)\delta]t}dt = \frac{e^{[\alpha\theta+(1-\eta)\delta]t}-1}{\alpha\theta+(1-\eta)\delta}.$$

An application of Hopital's rule gives

$$\lim_{t\to\infty} k_t^{1-\eta} = \frac{Ae_0^\theta}{(1-\eta)\delta}\left[\lim_{t\to\infty}\frac{e^{\alpha t}(n-b+be^{nt})}{ne^{nt}}\right]^\theta,$$

and so the statement considering the two cases $\alpha = 0$ and $\alpha > 0$. $\square$

Proposition 7. *Let $B < 0$. Set $e_\infty = \lim\limits_{t\to\infty} e_t$, and $k_\infty = \lim\limits_{t\to\infty} k_t$. Then $e_\infty = 0$ if $\alpha < 0$, $e_\infty < e_0 b/n$ if $\alpha = 0$, $k_\infty > 0$ no matter who is α. Let $B > 0$. Then nothing can be concluded on the long-run behavior of e_t and k_t.*

Proof. Let $B < 0$. The differential equations in (8) cannot be solved in terms of elementary functions, as done for the case $B = 0$. Thus, the study of the long-run behavior of the functions e_t, k_t is more complicated than the one done before. A common technique in this case is to compare the unknown solutions of the given equation with the known solutions of another, i.e. to use the so-called comparison theorems. We will use the following one (see Birkhoff and Rota [2]): "if $u_i(t)$, $i = 1, 2$, is the solution of the Cauchy problem $\dot{u} = \varphi_i(t, u)$, $u(0) = u_0$, and $\varphi_1(t, u) \le \varphi_2(t, u)$, for all (t, u), then $u_1(t) \le u_2(t)$, for all t". An application of this theorem gives the statement of our proposition.

For example, Proposition 1 and the fact that $\varphi_1 = Bk_t^\eta e_t^\theta - (n - bL_t - \alpha)e_t < -(n-bL_t-\alpha)e_t = \varphi_2$ imply the inequality $e_t < e_0 e^{\alpha t}(n-b+be^{nt})/ne^{nt}$, for all t. Thus, $e_\infty = 0$ if $\alpha < 0$, and $e_\infty < e_0 b/n$ if $\alpha = 0$. Similarly, to understand the long-run behavior of k_t, we need to study the differential equation $\dot{k}_t = Ak_t^\eta e_t^\theta - (\delta+n-bL_t)k_t$. From $\varphi_1 = -(\delta+n-bL_t)k_t < Ak_t^\eta e_t^\theta - (\delta+n-bL_t)k_t = \varphi_2$, we derive that $k_0(n-b+be^{nt})/ne^{(\delta+n)t} < k_t$, for all t. In particular, $k_\infty > 0$. □

We are now able to state the following result.

Theorem 8. *If human activities have a net zero or negative effect on the environment in every time period and the stock of the environment decays autonomously over time, i.e. if $B \leq 0$ and $\alpha < 0$, then the economy is unsustainable in the long-run. If human activities have a net zero effect on the environment and the stock of the environment grows or remains unchanged autonomously over time, i.e. $B = 0$ and $\alpha \geq 0$, then the economy is sustainable in the long-run.*

Remark 9. Tran-Nam [10] studied the model with $\dot{L}_t/L_t = n > \alpha$ and showed that the economy is always unsustainable in the long-run if $B \leq 0$, while a necessary condition for the economy to be long-run sustainable is that $B > 0$, i.e. if human activities produce a net beneficial effect on the environment for every time period.

4. Tax Rate and Sustainability

From Theorem 8 we derive that a necessary condition for the economy to be sustainable in the long-run is $B < 0$ ($\alpha \geq 0$), or $B > 0$ (α arbitrary), while a sufficient condition is $B = 0$ ($\alpha \geq 0$). The next three lemmas will be helpful in formulating these conditions in mathematical form.

Lemma 10. *For any given tax rate $\tau \in (0,1)$,*

$$B \gtreqless 0 \iff a \lesseqgtr \frac{\phi\tau - \beta}{(1-\tau)\gamma}.$$

Proof. Immediate from the definition of B. □

Remark 11. Similarly, for any given MPC $a \in (0,1)$,

$$B \gtreqless 0 \iff \tau \gtreqless \frac{a\gamma + \beta}{a\gamma + \phi}.$$

Recalling that $a \in (0,1)$, we compare the numbers $(\phi\tau - \beta)/(1-\tau)\gamma$ and a.

This is actually what the next lemma does.

Lemma 12. *i)* $\dfrac{\phi\tau-\beta}{(1-\tau)\gamma} \leq 0 \iff \tau \leq \dfrac{\beta}{\phi}$;

ii) $0 < \dfrac{\phi\tau-\beta}{(1-\tau)\gamma} < 1 \iff \dfrac{\beta}{\phi} < \tau < \dfrac{\beta+\gamma}{\phi+\gamma}$;

iii) $\dfrac{\phi\tau-\beta}{(1-\tau)\gamma} \geq 1 \iff \tau \geq \dfrac{\beta+\gamma}{\phi+\gamma}$.

Proof. An easy calculation. □

Lemma 13. *If $B \geq 0$, then $\phi > \beta$, whereas if $B < 0$ the relationship between ϕ and β is undetermined.*

Proof. Let $B = 0$. Then $\phi\tau - \beta = (1-\tau)a > 0$ implies $\phi\tau > \beta$, as well as $\phi\tau < \phi$ recalling that $0 < \tau < 1$. Thus, $\beta < \phi\tau < \phi$, i.e. $\phi > \beta$. Let $B > 0$. Then $\phi\tau - \beta > (1-\tau)a > 0$. The statement follows proceeding as done before. □

Proposition 14. *Let $\tau \in (0,1)$ be a given tax rate.*

1) Let $B > 0$ (α arbitrary):

1_1*) if $\tau \leq \beta/\phi$, the set of sustainable MPCs is empty*;

1_2) *if $\beta/\phi < \tau < (\beta+\gamma)/(\phi+\gamma)$, the set of sustainable MPCs is the interval* $(0, (\phi\tau-\beta)/(1-\tau)\gamma)$;

1_3) *if $\tau \geq (\beta+\gamma)/(\phi+\gamma)$, the set of sustainable MPCs is the interval* $(0,1)$.

2) *Let $B < 0$ ($\alpha \geq 0$), $\phi > \beta$:*

2_1) *if $\tau \leq \beta/\phi$, the set of sustainable MPCs is the interval* $(0,1)$;

2_2) *if $\beta/\phi < \tau < (\beta+\gamma)/(\phi+\gamma)$, the set of sustainable MPCs is the interval* $((\phi\tau-\beta)/(1-\tau)\gamma, 1)$;

2_3) *if $\tau \geq (\beta+\gamma)/(\phi+\gamma)$, the set of sustainable MPCs is empty.*

3) *Let $B < 0$ ($\alpha \geq 0$), $\phi \leq \beta$. Then the set of sustainable MPCs is the interval* $(0,1)$.

4) *Let $B = 0$ ($\alpha \geq 0$). For any given tax rate τ, the set of sustainable MPCs reduces to the point $a = (\phi\tau-\beta)/(1-\tau)\gamma$.*

Proof. The statement follows as an application of the previous lemmas. For example, let $B > 0$ and $\tau \leq \beta/\phi$. Supposing that there were a sustainable MPC $a \in (0,1)$, then this would imply $a < (\phi\tau-\beta)/(1-\tau)\gamma \leq 0$, i.e. the absurd $a < 0$. Consequently, the set of sustainable MPCs has to be empty. Let $B > 0$ and $\beta/\phi < \tau < (\beta+\gamma)/(\phi+\gamma)$. Since we have that $a < (\phi\tau-\beta)/(1-\tau)\gamma$, and $(\phi\tau - \beta)/(1-\tau)\gamma \in (0,1)$, then the statement follows immediately since $a \in (0,1)$. Let $B > 0$ and $\tau \geq (\beta+\gamma)/(\phi+\gamma)$. In this case we would have $a < (\phi\tau - \beta)/(1-\tau)\gamma < 1 < (\phi\tau - \beta)/(1-\tau)\gamma$. Thus, the set of sustainable MPCs is the interval $(0,1)$. Similarly for all the other remaining cases. □

Remark 15. Similarly, for any given MPC $a \in (0,1)$, the set of sustainable tax rates is the interval $((a\gamma+\beta)/(a\gamma+\phi), 1)$ if $B > 0$ (α arbitrary), the interval $(0, (a\gamma+\beta)/(a\gamma+\phi))$ if $B < 0$ ($\alpha \geq 0; \phi > \beta$), and $(0,1)$ if $B < 0$ ($\alpha \geq 0; \phi \leq \beta$). Let $B = 0$ ($\alpha \geq 0$). For any given MPC a, the set of sustainable tax rates reduces to a single element, $\tau = (a\gamma+\beta)/(a\gamma+\phi)$.

Remark 16. Some interesting things can be derived from the previous proposition. For example, if $B > 0$, we see that an increase in the tax rate in the relevant range widens the choice of sustainable MPCs, while a decrease in the tax rate narrows this choice. We also note that these results are intuitively clear. More resources are spent to repair the environment, then, keeping the economy sustainable, a larger fraction of the remaining output is available for consumption, while less resources are spent to repair the environment, then a smaller fraction of the remaining output is available for consumption.

5. Sustainable Steady State

A steady state of a sustainable economy is defined as a situation in which the growth rates of per capita physical capital, per capita natural capital, and labor growth rate are equal to zero. Let us denote the steady state equilibrium values of k_t, e_t, L_t, by k_*, e_*, L_*, respectively. Note that we will confine our analysis to interior steady states only, i.e. we will exclude the economically meaningless solutions such as $k_* = 0$, $e_* = 0$, or $L_* = 0$.

Proposition 17. *1) Let $B = 0$ ($\alpha > 0$), $B < 0$ ($\alpha = 0$), or $B > 0$ ($\alpha \geq 0$). Then the economy has no steady states.*

2) Let $B < 0$ ($\alpha > 0$), $B > 0$ ($\alpha < 0$). There is a unique steady state, which is given by

$$(k_*, e_*, L_*) = (\omega, -(\delta B/\alpha A)\omega, n/b)$$

where $\omega = \left[(A/\delta)^{1-\theta}(-B/\alpha)^{\theta}\right]^{1/[1-(\eta+\theta)]}$.

3) Let $B = 0$ ($\alpha = 0$). There are infinite steady states of the form

$$(k_*, e_*, L_*) = \left(\bar{k}, \left[(\delta/A)\bar{k}^{1-\eta}\right]^{1/\theta}, n/b\right), \text{ for all } \bar{k} > 0.$$

Proof. The steady states of the economy, reached when $\dot{k}_t = \dot{e}_t = \dot{L}_t = 0$, are obtained from the equations

$$Ak_t^{\eta-1}e_t^{\theta} = \delta, \quad Bk_t^{\eta}e_t^{\theta-1} = -\alpha, \quad L_t = n/b. \tag{10}$$

Let $B = 0$. It is then clear from (10) that there exists a steady state for each fixed $\bar{k} > 0$. Moreover, we deduce that there are no steady states if $\alpha \neq 0$. Let $B < 0$. If $\alpha = 0$, then (10) implies the absurd $Bk_t^{\eta}e_t^{\theta-1} = 0$. If $\alpha > 0$, then from (10) we get $e_t = -(\delta B/\alpha A)k_t$, and $k_t = \left[(A/\delta)^{1-\theta}(-B/\alpha)^{\theta}\right]^{1/[1-(\eta+\theta)]}$. Thus, the uniqueness of the steady state. Similarly the proof for the case $B > 0$. □

Remark 18. Let $B = 0$ ($\alpha = 0$). We know from Corollaries 2 and 6 that the economy converges to the point $\left(\left[Ae_0^{\theta}b^{\theta}/(1-\eta)\delta n^{\theta}\right]^{\frac{1}{1-\eta}}, e_0 b/n, b/n\right)$. Since there is no value $\bar{k} > 0$ such that $(k_\infty, e_\infty, L_\infty) = \left(\bar{k}, \left[(\delta/A)\bar{k}^{1-\eta}\right]^{1/\theta}, b/n\right)$, (otherwise, we would have the absurd $(1-\eta)\delta = \delta$), we can conclude that, in the long-run, the economy stabilizes to a point different from a steady state equilibrium.

Theorem 19. *Let $B = 0$ ($\alpha = 0$). Then every steady state equilibrium is unstable. Let $B < 0$ ($\alpha > 0$), then there is a unique steady state equilibrium which is a saddle with a two dimensional stable manifold. Let $B > 0$ ($\alpha < 0$). Then there is a unique steady state equilibrium which is a stable node.*

Proof. The local dynamic of the dynamical system (8) around a steady state equilibrium (k_*, e_*, L_*) is determined by the signs of the eigenvalues of the Jacobian matrix corresponding to its linearized system. This writes

$$\begin{bmatrix} \dot{k} \\ \dot{e} \\ \dot{L} \end{bmatrix} = J^* \begin{bmatrix} k - k_* \\ e - e_* \\ L - L_* \end{bmatrix}, \text{ where } J^* = \begin{bmatrix} J_{11}^* & J_{12}^* & J_{13}^* \\ J_{21}^* & J_{22}^* & J_{23}^* \\ J_{31}^* & J_{32}^* & J_{33}^* \end{bmatrix}.$$

J^* is the Jacobian matrix of the system (8) evaluated at (k_*, e_*, L_*). By definition, it is $J_{11}^* = (\partial\dot{k}/\partial k)\mid_{(k_*,e_*,L_*)}$, $J_{12}^* = (\partial\dot{k}/\partial e)\mid_{(k_*,e_*,L_*)}$, $J_{13}^* = (\partial\dot{k}/\partial L)\mid_{(k_*,e_*,L_*)}$, and so on for all the remaining matrix entries. After some calculations, we find

$$J^* = \begin{pmatrix} -(1-\eta)\delta & A\theta k_*^{\eta}e_*^{\theta-1} & bk_* \\ B\eta\delta/A & (1-\theta)\alpha & be_* \\ 0 & 0 & -n \end{pmatrix}.$$

This matrix has three eigenvalues, one of which is immediately seen to be given by the negative number $\lambda_1 = -n$. The other two, say λ_2, λ_3, can be determined by looking at the trace $\mathrm{Trace}(J^*)$ and at the determinant $\mathrm{Det}(J^*)$ of the matrix

J^*. We get

$$\begin{aligned} \mathrm{Det}\,(J^*) &= -n\left[-(1-\eta)(1-\theta)\delta\alpha - B\theta\eta\delta k_*^{\eta} e_*^{\theta-1}\right], \\ \mathrm{Trace}\,(J^*) &= J_{11}^* + J_{22}^* + J_{33}^* = -(1-\eta)\delta + (1-\theta)\alpha - n. \end{aligned}$$

Recalling that the determinant (resp. trace) of a matrix is also equal to the product (resp. sum) of its eigenvalues, we have

$$\begin{aligned} \lambda_2\lambda_3 &= -(1-\eta)(1-\theta)\delta\alpha - B\theta\eta\delta k_*^{\eta} e_*^{\theta-1}, \\ \lambda_2 + \lambda_3 &= -(1-\eta)\delta + (1-\theta)\alpha. \end{aligned} \tag{11}$$

Notice that both λ_2, λ_3 must be real numbers. Let $B = 0$ ($\alpha = 0$). Then (11) becomes $\lambda_2\lambda_3 = 0$, $\lambda_2 + \lambda_3 = -(1-\eta)\delta < 0$. Therefore, one eigenvalue is negative, the other is null. Let $B \neq 0$. Since (10) implies that $k_*^{\eta} e_*^{\theta-1} = -\alpha/B$, we may rewrite (11) as $\lambda_2\lambda_3 = -[1-(\theta+\eta)]\delta\alpha$, $\lambda_2+\lambda_3 = -(1-\eta)\delta+(1-\theta)\alpha$. If $B < 0$ ($\alpha > 0$), then $\lambda_2\lambda_3 < 0$. Thus, λ_2, λ_3 have opposite sign. Since the three eigenvalues of J^* have different signs, we can conclude that the system is saddle-path stable. Moreover, the stable manifold is a plane since two of the three eigenvalues are negative. If $B > 0$ ($\alpha < 0$), then $\lambda_2\lambda_3 > 0$ yields that λ_2, λ_3 must have the same sign, i.e. both negative being $\lambda_2+\lambda_3 < 0$. In conclusion, since all the eigenvalues of J^* are negative, the economy has a unique steady state, which is a stable node. □

Remark 20. In case of a constant population growth rate, Tran-Nam [10] showed that, if human activities produce a net beneficial effect on the environment, then the economy will converge to a unique and stable steady state.

6. Conclusion

In this paper, we have modified the Solow model by incorporating the natural capital as a factor of production and assuming the population to grow according to the logistic model. The natural capital stock is modeled as a renewable resource. Within this framework, we have investigated the long-run sustainability of the economy, and found that the economy is sustainable in the long-run if human activities have a net zero effect on the environment and the stock of the environment grows or remains unchanged autonomously over time, whereas, it is unsustainable if human activities have a net zero or negative effect on the environment and the stock of the environment decays autonomously over time. Furthermore, for any given tax rate (or MPC), we have determined the set of sustainable MPCs (or tax rates). Finally, we have analyzed the non-trivial steady states of a sustainable economy by treating the tax rate and MPC as exogenous to the economy. We have found that if if human activities have a negative effect on the environment and the environment grows over time, then there is a unique steady state equilibrium, which is a saddle. Whereas, if human activities have a positive effect on the environment and the environment decays over time, then there is still a unique steady state equilibrium, but it is a node.

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Appendix

Let recall some facts about Hypergeometric functions (see Abramowitz and Stegun [1]). The Gauss hypergeometric function ${}_2F_1(c_1, c_2, c_3; z)$, with complex arguments c_1, c_2, c_3, and z, is given by the series

$$ {}_2F_1(c_1, c_2, c_3; z) = \frac{\Gamma(c_3)}{\Gamma(c_1)\Gamma(c_2)} \sum_{m=1}^{\infty} \frac{\Gamma(c_1+m)\Gamma(c_2+m)}{\Gamma(c_3+m)} \frac{z^m}{m!}, $$

where $\Gamma(\cdot)$ is the special function Gamma. The above series is convergent for any c_1, c_2 and c_3 if $|z| < 1$, and for any c_1, c_2 and c_3 such that $Re(c_1 + c_2 - c_3) < 0$ if $|z| = 1$. Fortunately, there are many continuation formulas of the Gamma Hypergeometric function outside the unit circle. The most practical continuation formulas consist in the integral representations of the Gamma Hypergeometric function. We shall use the following formula

$$ {}_2F_1(c_1, c_2, c_3; z) = \frac{\Gamma(c_3)}{\Gamma(c_1)\Gamma(c_3-c_1)} \int_0^1 t^{c_1-1}(1-t)^{c_3-c_1-1}(1-zt)^{-c_2} dt, $$

where $Re(c_1) > 0$, $Re(c_3 - c_1) > 0$, commonly known as the Euler integral representation.