International Journal of Pure and Applied Mathematics
Volume 49 No. 1 2008, 99-108

APPROXIMATION OF HARMONIC FUNCTIONS IN SOBOLEV SPACES

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Abstract: This paper is concerned with the regularity of harmonic functions which determines the efficiency of approximation in Sobolev space $W^{t,p}(\Omega)$. The result shows that the nonlinear approximation is always superior to linear approximation for harmonic functions on bounded Lipschitz domain.

AMS Subject Classification: 42C15, 94A11
Key Words: nonlinear approximation, wavelets, harmonic functions, Sobolev spaces

1. Introduction

This paper is concerned with the Besov regularity of harmonic function ($\triangle u = 0$). The particular scale of Besov spaces that we consider is of interest to us because it is connected to the rate of convergence of approximation in Sobolev space. In fact, Dahlke and DeVore had studied the similar problem in [5], but their work is restricted to the case of approximation in Lebesgue space. In this paper, we will follow the same line as in [5], but we do in more detailed Sobolev spaces.

Received: September 23, 2008

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We shall restrict our discussion to nonlinear wavelet approximation, then there are two questions: firstly what is the Besov regularity of the function which governs its approximate efficiency in Sobolev spaces; secondly does harmonic functions possess such regularity. Therefore, the first question is solved in Section 2. In Section 3, we will show that the nonlinear approximate efficiency is always superior to that of linear approximation for harmonic functions on bounded Lipschitz domain.

2. Approximation and Besov Spaces

Daubechies in [6] has constructed a univariate family D_m of compactly supported wavelets. When $m = 1$, D_1 is the Haar function. Larger values of m correspond to higher smoothness of the wavelet D_m and the wavelet D_m has m vanishing moments. We fix an arbitrary value of m and let $\phi = \phi_m$ be the univariate scaling function which generates the wavelet $\psi = D_m$. We define $\psi^0 =: \phi$ and $\psi^1 =: \psi$. Further, let E denote the nonzero vertices of the square $[0,1]^d$, Ψ denote the set of $2^d - 1$ functions

$$\psi^e(x_1, x_2, \dots x_d) =: \prod_{j=1}^{d} \psi^{e_j}(x_j), \quad e \in E\,,$$

and D denote the set of dyadic cubes in R^d. Each cube $I \in D$ is of the form $I = 2^{-j}k + 2^{-j}[0,1]^d$ with $k \in Z^d, j \in Z$. Then the functions

$$\eta_I =: \eta_{j,k} = 2^{\frac{jd}{2}} \eta(2^j \cdot -k), \quad k \in Z^d, \quad j \in Z, \quad \eta \in \Psi\,,$$

form an orthonormal basis for $L^2(R^d)$.

If $h \in R^d$, we denote by Ω_h the set of all $x \in \Omega$ such that the line $[x, x+h]$ is contained in Ω. The modulus of smoothness $\omega_r(F,t)_{L^p(\Omega)}$ of a function $F \in L^p(\Omega), 0 < p \le \infty$ is defined by

$$\omega_r(F,t)_{L^p(\Omega)} =: \sup_{|h| \le t} \|\triangle_h^r(F,\cdot)\|_{L^p(\Omega_{rh})}, \quad t > 0$$

with $\triangle_h^r$ the r-th difference of step h. For $\alpha > 0$ and $0 < q, p \le \infty$, the Besov space $B_q^\alpha(L_p(\Omega))$ is defined as the space of all functions F for which

$$|F|_{B_q^\alpha(L_p(\Omega))} =: \begin{cases} (\int_0^\infty [t^{-\alpha}\omega_r(F,t)_{L^p(\Omega)}]^q \frac{dt}{t})^{\frac{1}{q}}, & 0 < q < \infty, \\ \sup\limits_{t>0} t^{-\alpha}\omega_r(F,t)_{L^p(\Omega)}, & q = \infty, \end{cases}$$

is finite with $r =: [\alpha] + 1$. Adding $\|F\|_{L^p(\Omega)}$, we obtain a quasi-norm for $B_q^\alpha(L_p(\Omega))$. It is possible to characterize Besov spaces by wavelet decompositions. We restrict the wavelet expansion to those η_I with $|I| \le 1$ and denote the corresponding dyadic cubes by D^+. Let P_0 be the orthogonal projector which maps $L^2(R^d)$ onto S_0, where S_0 is expanded by the finite linear combinations of the integer shifts of the function $\phi(x_1)\ldots\phi(x_d)$. If $0 < p \le \infty$, we define $\eta_{I,p} =: |I|^{\frac{1}{2}-\frac{1}{p}}\eta_I$. Let p' be the conjugate index to p, that is, $\frac{1}{p}+\frac{1}{p'}=1$, we know the following characterization:

Lemma 2.1. (see [5]) *Let ϕ and ψ be in $C^r(R)$. If $0 < p \le \infty$ and $r > \alpha > d(\frac{1}{p}-1)$, then a function F is in the Besov space $B_p^\alpha(L_p(R^d))$ if and only if*

$$F = P_0(F) + \sum_{I\in D^+}\sum_{\eta\in\Psi} < F, \eta_{I,p'} > \eta_{I,p}$$

with $\|P_0(F)\|_{L^p(R^d)} + (\sum\limits_{I\in D^+}\sum\limits_{\eta\in\Psi}|I|^{-\frac{\alpha p}{d}}| < F, \eta_{I,p'} > |^p)^{\frac{1}{p}} < \infty$ *providing an equivalent quasi-norm for $B_p^\alpha(L_p(R^d))$.*

Proposition 2.1. *Let ϕ, ψ be in $C^{r+[t]+1}(R)$ and $t \ge 0$. If $1 < p \le \infty$, $r > \alpha > 0$ and $\frac{1}{\tau} = \frac{\alpha}{d}+\frac{1}{p}$, then a function F is in the Besov space $B_\tau^{\alpha+t}(L_\tau(R^d))$ if and only if*

$$F = P_0(F) + \sum_{I\in D^+}\sum_{\eta\in\Psi} < F, \eta_{I,p'} > \eta_{I,p}$$

with $\|P_0(F)\|_{L^\tau(R^d)} + (\sum\limits_{I\in D^+}\sum\limits_{\eta\in\Psi}|I|^{-\frac{t\tau}{d}}| < F, \eta_{I,p'} > |^\tau)^{\frac{1}{\tau}} < \infty$ *providing an equivalent quasi-norm for $B_\tau^{\alpha+t}(L_\tau(R^d))$.*

Proof. Using the fact that $\eta_{I,\tau'} = |I|^{\frac{1}{p'}-\frac{1}{\tau'}}\eta_{I,p'}$, a simple computation gives

$$|I|^{-\frac{\alpha\tau}{d}}| < F, \eta_{I,\tau'} > |^\tau = | < F, \eta_{I,p'} > |^\tau.$$

Then the desired result follows from Lemma 2.1.

In the following, we will use Sobolev spaces. We say $f \in W^{t,p}(R^d)$ for $t \ge 0$, if both f and $(-\triangle)^{\frac{t}{2}}f =: (|\xi|^t\widehat{f}(\xi))^\vee$ belong to $L^p(R^d)$. We let $\|f\|_{W^{t,p}(R^d)} =: \|f\|_{L^p(R^d)} + \|(-\triangle)^{\frac{t}{2}}f\|_{L^p(R^d)}$ and $|f|_{W^{t,p}(R^d)} =: \|(-\triangle)^{\frac{t}{2}}f\|_{L^p(R^d)}$. We define

$$\sigma_n(f)_{W^{t,p}(R^d)} =: \inf_{\sharp\Lambda\le n}|f - \sum_{(I,\eta)\in\Lambda} a_{I,\eta}\eta_I|_{W^{t,p}(R^d)}.$$

Lemma 2.2. (see [1]) *Let ϕ and ψ be in $C^r(R)$. If $1 < p < \infty$ and $r > \alpha > 0$, then*

$$F \in B_\tau^\alpha(L_\tau(R^d)), \quad \frac{1}{\tau} = \frac{\alpha}{d}+\frac{1}{p} \Longleftrightarrow \sum_{n=1}^\infty[n^{\frac{\alpha}{d}}\sigma_n(F)_{L^p(R^d)}]^\tau\frac{1}{n} < \infty.$$

Proposition 2.2. *Let ϕ, $\psi \in C^{r+[t]+1}(R)$ and $t \geq 0$. If $1 < p < \infty$ and $r > \alpha > 0$, then*

$$F \in B_\tau^{\alpha+t}(L_\tau(R^d)), \quad \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p} \Longleftrightarrow \sum_{n=1}^{\infty} [n^{\frac{\alpha}{d}} \sigma_n(F)_{W^{t,p}(R^d)}]^\tau \frac{1}{n} < \infty.$$

Proof. If $f = \sum_{I \in D} \sum_{\eta \in \Psi} \langle f, \eta_I \rangle \eta_I$, we let $I_{-t} f =: \sum_{I \in D} \sum_{\eta \in \Psi} |I|^{-\frac{t}{d}} \langle f, \eta_I \rangle \eta_I$. Then we first prove

$$F \in B_\tau^{\alpha+t}(L_\tau(R^d)) \Longleftrightarrow I_{-t} F \in B_\tau^{\alpha}(L_\tau(R^d)).$$

In fact, we know from Proposition 2.1 that

$$\begin{aligned} &F \in B_\tau^{\alpha+t}(L_\tau(R^d)) \\ &\qquad \Longleftrightarrow \|P_0(F)\|_{L^\tau(R^d)} + (\sum_{I \in D^+} \sum_{\eta \in \Psi} |I|^{-\frac{t\tau}{d}} | < F, \eta_{I,p'} > |^\tau)^{\frac{1}{\tau}} < \infty. \end{aligned}$$

Moreover, it is easy to verify that the right hand side equals to

$$\begin{aligned} &\|P_0(I_{-t}F)\|_{L^\tau(R^d)} + (\sum_{I \in D^+} \sum_{\eta \in \Psi} | < I_{-t}F, \eta_{I,p'} > |^\tau)^{\frac{1}{\tau}} \\ &\qquad = \|P_0(I_{-t}F)\|_{L^\tau(R^d)} + (\sum_{I \in D^+} \sum_{\eta \in \Psi} |I|^{-\frac{\alpha\tau}{d}} | < I_{-t}F, \eta_{I,\tau'} > |^\tau)^{\frac{1}{\tau}}, \end{aligned}$$

which means by Lemma 2.1 that $I_{-t}F \in B_\tau^{\alpha}(L_\tau(R^d))$. Furthermore, we can obtain from Lemma 2.2 that

$$\sum_{n=1}^{\infty} [n^{\frac{\alpha}{d}} \sigma_n(I_{-t}F)_{L^p(R^d)}]^\tau \frac{1}{n} < \infty.$$

Therefore, it is enough to prove $\sigma_n(I_{-t}F)_{L^p(R^d)} \sim \sigma_n(F)_{W^{t,p}(R^d)}$ in the following:

In fact, for $1 < p < \infty$, we know from [3] that $\|F\|_{L^p(R^d)} \sim \|S(F, \cdot)\|_{L^p(R^d)}$ with

$$S(F, x) =: (\sum_{I \in D} \sum_{\eta \in \Psi} |\langle F, \eta_I \rangle|^2 |I|^{-1} \chi_I(x))^{\frac{1}{2}}.$$

Moreover, for $t \geq 0$, we know from [3] that

$$|F|_{W^{t,p}(R^d)} \sim \|(\sum_{\eta \in \Psi} \sum_{I \in D} |\langle F, \eta_I \rangle|^2 |I|^{-1} |I|^{-\frac{2t}{d}} \chi_I(x))^{\frac{1}{2}}\|_{L^p(R^d)}.$$

Therefore, we can obtain the desired result from the following argument:

$$\sigma_n(F)_{W^{t,p}(R^d)} = \inf_{\sharp\Lambda \leq n} |F - \sum_{(I,\eta) \in \Lambda} a_{I,\eta} \eta_I|_{W^{t,p}(R^d)}$$

$$\sim \inf_{\sharp\Lambda\leq n} \|(\sum_{\eta\in\Psi}\sum_{I\in D} |\langle F,\eta_I\rangle - a_{I,\eta}\delta_\Lambda|^2 |I|^{-1}|I|^{-\frac{2t}{d}}\chi_I(x))^{\frac{1}{2}}\|_{L^p(R^d)}$$

$$\sim \inf_{\sharp\Lambda\leq n} \|(\sum_{\eta\in\Psi}\sum_{I\in D} |\langle I_{-t}F,\eta_I\rangle - |I|^{-\frac{t}{d}}a_{I,\eta}\delta_\Lambda|^2 |I|^{-1}\chi_I(x))^{\frac{1}{2}}\|_{L^p(R^d)}$$

$$= \inf_{\sharp\Lambda\leq n} \|I_{-t}F - \sum_{(I,\eta)\in\Lambda} |I|^{-\frac{t}{d}}a_{I,\eta}\eta_I\|_{L^p(R^d)} = \sigma_n(I_{-t}F)_{L^p(R^d)}. \quad \square$$

Standard finite element methods approximate the functions by elements from linear spaces S_j of piecewise polynomials on uniform grids. Moreover, the approximate efficiency in the norm of $W^{t,p}(\Omega)$, $1 < p < \infty$ is given by the following lemma:

Lemma 2.3. (see [4]) *If n_j is the dimension of S_j and $t \geq 0$, then*

$$\text{dist}\,(u, S_j)_{W^{t,p}(\Omega)} = O(n_j^{-\frac{\alpha}{d}}) \Longleftrightarrow u \in B_\infty^{\alpha+t}(L_p(\Omega)).$$

3. Regularity of Harmonic Functions

The following lemma is a simple transformation of Theorem 3.1 in [1].

Lemma 3.1. *Let $1 \leq p \leq \infty, \beta > 0$, $t \geq 0$ and $k > \beta$ be an integer. Then there is a constant $C > 0$ depending only on k, β and Ω such that whenever v is a harmonic function on Ω which is in $B_p^{\beta+t}(L_p(\Omega))$, we have*

$$\|\delta(x)^{k-\beta}|\nabla^{k+[t]+1}v(x)|\|_{L^p(\Omega)} \leq C\|v\|_{B_p^{\beta+t}(L_p(\Omega))}$$

with $\delta(x) =: dist(x, \partial\Omega)$, $\nabla^k v$ denotes the vector of all kth order derivatives of v and $|\nabla^k v|$ is its Euclidean length.

The following is the main result of this paper.

Theorem 3.1. *Let Ω be a bounded Lipschitz domain in R^d and $t \geq 0$. If v is a harmonic function on Ω which is in the Besov space $B_p^{\lambda+t}(L_p(\Omega))$ for some $1 < p < \infty$ and $\lambda > 0$, then*

$$v \in B_\tau^{\alpha+t}(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}, \quad 0 < \alpha < \frac{\lambda d}{d-1}.$$

Proof. We fix τ and α as in the statement of the theorem. Because Ω is a Lipschitz domain, we can extend v to all of R^d with the extension in $B_p^{\lambda+t}(L_p(R^d))$ (see [5]). We denote this extension also by v.

Let $\psi =: D_{m+[t]+1}$ be the Daubechies's wavelets with parameter m. We require that m is large enough that the function ϕ and ψ are in $C^{s+[t]+1}(R)$, $s =: [\frac{d\lambda}{d-1}]+1$. Since ψ and ϕ have compact support, there is a cube $Q \subset R^d$, centered at the origin, such that supp$\eta \subset Q$ for all $\eta \in \Psi$. By shifts and dilates, we obtain the cubes $Q(I) =: 2^{-j}k + 2^{-j}Q, I = 2^{-j}k + 2^{-j}[0,1]^d$ which contain *supp* η_I for all $\eta \in \Psi$. Let Λ denote the set of pairs $(I,\eta), I \in D^+, \eta \in \Psi$, for which $Q(I)\bigcap \Omega \neq \emptyset$. Since $v \in B_p^{\lambda+t}(L_p(R^d))$, we have

$$v = P_0 v + v_0, \quad v_0 = \sum_{(I,\eta)\in\Lambda} < v, \eta_I > \eta_I = \sum_{(I,\eta)\in\Lambda} < v, \eta_{I,p'} > \eta_{I,p},$$

due to Lemma 2.1. The function $P_0 v$ is in $L^\tau(R^d)$, because it is a finite linear combination of shifts of $\phi(x_1)\dots\phi(x_d)$. According to Proposition 2.1, it is enough to show

$$(\sum_{(I,\eta)\in\Lambda} |I|^{-\frac{t\tau}{d}}| < v, \eta_{I,p'} > |^\tau)^{\frac{1}{\tau}} < \infty \tag{3.1}$$

for completing the proof. For $I \in D^+$, let $\delta_I =: \inf_{x\in Q(I)} \delta(x)$. Let Λ_j denote the set of those pairs $(I,\eta) \in \Lambda$ with $|I| = 2^{-jd}$. For each $k = 0, 1, \dots,$ let $\Lambda_{j,k} \subset \Lambda_j$ be the set of those $(I,\eta) \in \Lambda_j$ such that $k2^{-j} \leq \delta_I < (k+1)2^{-j}$. From the Lipschitz character of Ω, it follows that $|\Lambda_{j,k}| \leq C2^{j(d-1)}$ for $j, k = 0, 1, \dots$. Since Ω is bounded, we have $\Lambda_{j,k} = \emptyset$ if $k \geq C2^j$. Let $\Lambda_j^0 = \Lambda_j \backslash \Lambda_{j,0}$, we now fix j with $0 \leq j < \infty$ and estimate the portion of the sum in (3.1) corresponding to $(I,\eta) \in \Lambda_j^0$. For each $(I,\eta) \in \Lambda_j^0$, $\delta_I \geq 2^{-j}$, so $Q(I)$ is contained strictly in Ω. In addition, according to the Bramble-Hilbert Lemma in numerical analysis, there is a polynomial P_I of total degree $< m + [t] + 1$ such that

$$\begin{aligned}\|v - P_I\|_{L^p(Q(I))} &\leq C|Q(I)|^{\frac{m+[t]+1}{d}} |v|_{W^{m+[t]+1}(L^p(Q(I)))} \\ &\leq C|I|^{\frac{m+[t]+1}{d}} |v|_{W^{m+[t]+1}(L^p(Q(I)))}.\end{aligned}$$

Recall that $\eta_{I,p'}$ is orthogonal to any polynomial of total degree $< m + [t] + 1$, hence

$$\begin{aligned}| < v, \eta_{I,p'} > | &= | < v - P_I, \eta_{I,p'} > | \leq \|v - P_I\|_{L^p(Q(I))} \|\eta_{I,p'}\|_{L^{p'}(R^d)} \\ &\leq C|I|^{\frac{m+[t]+1}{d}} |v|_{W^{m+[t]+1}(L^p(Q(I)))} \\ \leq C|I|^{\frac{m+[t]+1}{d}} \delta_I^{\lambda-m} (\int_{Q(I)} &|\delta(x)^{m-\lambda} |\nabla^{m+[t]+1} v||^p dx)^{\frac{1}{p}} = C|I|^{\frac{m+[t]+1}{d}} \delta_I^{\lambda-m} \mu_I,\end{aligned}$$

where μ_I is defined by the last equality and

$$|v|_{W^{m+[t]+1}(L^p(Q(I)))} =: \sum_{|\nu|=m+[t]+1} \|D^\nu v\|_{L^p(Q(I)}.$$

$$\sum_{(I,\eta)\in\Lambda_j^0} |I|^{-\frac{t\tau}{d}}| < v, \eta_{I,p'} > |^\tau \leq C \sum_{(I,\eta)\in\Lambda_j^0} 2^{jt\tau} 2^{-(m+t)j\tau} \delta_I^{(\lambda-m)\tau} \mu_I^\tau$$

$$= C \sum_{(I,\eta)\in\Lambda_j^0} 2^{-mj\tau} \delta_I^{(\lambda-m)\tau} \mu_I^\tau.$$

We use Holder's inequality with exponent $\frac{p}{\tau}$ and $\frac{p}{p-\tau}$ to show

$$\sum_{(I,\eta)\in\Lambda_j^0} |I|^{-\frac{t\tau}{d}} |< v, \eta_{I,p'} >|^\tau$$

$$\leq C(\sum_{(I,\eta)\in\Lambda_j^0} 2^{-\frac{mpj\tau}{p-\tau}} \delta_I^{\frac{(\lambda-m)p\tau}{p-\tau}})^{\frac{p-\tau}{p}} \cdot (\sum_{(I,\eta)\in\Lambda_j^0} \mu_I^p)^{\frac{\tau}{p}}.$$

Now, by Lemma 3.1, we obtain

$$(\sum_{(I,\eta)\in\Lambda_j^0} \mu_I^p)^{\frac{\tau}{p}} = (\sum_{(I,\eta)\in\Lambda_j^0} \int_{Q(I)} |\delta(x)^{m-\lambda} |\nabla^{m+[t]+1} v||^p dx)^{\frac{\tau}{p}}$$

$$\leq C(\int_\Omega |\delta(x)^{m-\lambda} |\nabla^{m+[t]+1} v||^p dx)^{\frac{\tau}{p}} \leq C\|v\|^\tau_{B_p^{\lambda+t}(L^p(\Omega))} \leq C.$$

Therefore, summing over the sets $\Lambda_{j,k}, k = 1, 2\dots$ gives

$$\sum_{(I,\eta)\in\Lambda_j^0} |I|^{-\frac{t\tau}{d}} |< v, \eta_{I,p'} >|^\tau \leq C(\sum_{k=1}^{C2^j} \sum_{(I,\eta)\in\Lambda_{j,k}} 2^{-\frac{mpj\tau}{p-\tau}} \delta_I^{\frac{(\lambda-m)p\tau}{p-\tau}})^{\frac{p-\tau}{p}}$$

$$\leq C(\sum_{k=1}^{C2^j} 2^{j(d-1)} 2^{-\frac{mpj\tau}{p-\tau}} (k2^{-j})^{\frac{(\lambda-m)p\tau}{p-\tau}})^{\frac{p-\tau}{p}} \leq C(2^{j(d-1-\frac{p\lambda\tau}{p-\tau})} \sum_{k=1}^{C2^j} k^{\frac{(\lambda-m)p\tau}{p-\tau}})^{\frac{p-\tau}{p}}.$$

We now choose m large enough that $(m-\lambda)\tau > 1 - \frac{\tau}{p}$ and obtain

$$\sum_{(I,\eta)\in\Lambda_j^0} |I|^{-\frac{t\tau}{d}} |< v, \eta_{I,p'} >|^\tau \leq C 2^{j(\frac{(d-1)(p-\tau)}{p} - \lambda\tau)}.$$

We now define $\Lambda^0 = \bigcup_{j=0}^{\infty} \Lambda_j^0$ and sum over all dyadic levels $j = 0, 1, \dots$ to find

$$\sum_{(I,\eta)\in\Lambda^0} |I|^{-\frac{t\tau}{d}} |< v, \eta_{I,p'} >|^\tau \leq C \sum_{j=0}^{\infty} 2^{j(\frac{(d-1)(p-\tau)}{p} - \lambda\tau)} \leq C$$

provided $\frac{(d-1)(p-\tau)}{p} - \lambda\tau < 0$, i.e. $\tau > \frac{p(d-1)}{p\lambda+d-1}$. This condition on τ is equivalent to the condition on α given in the theorem.

Finally, we need to estimate the sum of wavelet coefficients corresponding to the sets $\Lambda_{j,0}, j = 0, 1, \dots$. Holder's inequality and the fact that $|\Lambda_{j,0}| \leq C2^{j(d-1)}$ gives

$$\sum_{(I,\eta)\in\Lambda_{j,0}} |I|^{-\frac{t\tau}{d}}| < v, \eta_{I,p'} > |^{\tau}$$
$$\leq C2^{j(d-1)(1-\frac{\tau}{p})}(\sum_{(I,\eta)\in\Lambda_{j,0}} |I|^{-\frac{tp}{d}}| < v, \eta_{I,p'} > |^{p})^{\frac{\tau}{p}}$$
$$= C2^{j(d-1)(1-\frac{\tau}{p})}2^{-j\lambda\tau}(\sum_{(I,\eta)\in\Lambda_{j,0}} 2^{jp\lambda}2^{jtp}| < v, \eta_{I,p'} > |^{p})^{\frac{\tau}{p}}.$$

Hence, summing over all dyadic levels j and using Holder's inequality again, we find

$$\sum_{j=0}^{\infty}\sum_{(I,\eta)\in\Lambda_{j,0}} |I|^{-\frac{t\tau}{d}}| < v, \eta_{I,p'} > |^{\tau}$$
$$\leq C\sum_{j=0}^{\infty} 2^{j(d-1)(1-\frac{\tau}{p})}2^{-j\lambda\tau}\cdot(\sum_{(I,\eta)\in\Lambda_{j,0}} 2^{jp(\lambda+t)}| < v, \eta_{I,p'} > |^{p})^{\frac{\tau}{p}}$$
$$\leq C(\sum_{j\geq 0}\sum_{(I,\eta)\in\Lambda_{j,0}} 2^{jp(\lambda+t)}| < v, \eta_{I,p'} > |^{p})^{\frac{\tau}{p}}\cdot(\sum_{j\geq 0} 2^{(-\frac{p\lambda\tau j}{p-\tau}+(d-1)j)})^{\frac{p-\tau}{p}}.$$

From Lemma 2.1, we know that

$$(\sum_{j\geq 0}\sum_{(I,\eta)\in\Lambda_{j,0}} 2^{jp(\lambda+t)}| < v, \eta_{I,p'} > |^{p})^{\frac{1}{p}} \leq C\|v\|_{B_p^{\lambda+t}(L^p(R^d))} \leq C,$$

the second sum on the right hand side is finite if the exponent of 2^j is negative, that is $\tau > \frac{p(d-1)}{p\lambda+d-1}$, which is the same restriction we had on α in the theorem. We have completed the verification of (3.1) and have therefore proved the theorem. □

Remark 3.1. The condition $\sum\limits_{n=1}^{\infty}[n^{\frac{\alpha}{d}}\sigma_n(F)_{W^{t,p}(R^d)}]^{\tau}\frac{1}{n} < \infty$ is slightly stronger than $\sigma_n(F)_{W^{t,p}(R^d)} = O(n^{-\frac{\alpha}{d}})$. We first introduce weak sequence space to show this fact: we say sequence $v = \{v_\lambda\}_{\lambda\in\nabla} \in \ell_\tau^\omega(\nabla)$ for any $\tau > 0$ if $\sup\limits_{n\geq 1} n^{\frac{1}{\tau}}v_n^* < \infty$, where $\{v_n^*\}_{n\in N}$ is the decreasing rearrangement of the absolute value of v. It is easy to see that $\ell^\tau(\nabla) \subset \ell_\tau^\omega(\nabla) \subset \ell^{\tau+\varepsilon}(\nabla)$ for any $\tau > 0$ and $\varepsilon > 0$. Since $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$, we obtain

$$\sum_{n=1}^{\infty}[n^{\frac{\alpha}{d}}\sigma_n(F)_{W^{t,p}(R^d)}]^{\tau}\frac{1}{n} < \infty \Longleftrightarrow \sum_{n=1}^{\infty}[n^{-\frac{1}{p}}\sigma_n(F)_{W^{t,p}(R^d)}]^{\tau} < \infty,$$

which means that $\{n^{-\frac{1}{p}}\sigma_n(F)_{W^{t,p}(R^d)}\}_{n\in N} \in \ell^\tau(N)$, furthermore,

$$\{n^{-\frac{1}{p}}\sigma_n(F)_{W^{t,p}(R^d)}\}_{n\in N} \in \ell_\tau^\omega(N),$$

that is, $\sup_{n\geq 1} n^{\frac{1}{\tau}} n^{-\frac{1}{p}} \sigma_n(F)_{W^{t,p}(R^d)} = \sup_{n\geq 1} n^{\frac{\alpha}{d}} \sigma_n(F)_{W^{t,p}(R^d)} < +\infty$. Therefore, Proposition 2.2 shows that $B_\tau^{\alpha+t}(L_\tau(R^d))$, $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$ determines the nonlinear wavelet approximate efficiency $O(n^{-\frac{\alpha}{d}})$ in Sobolev space $W^{t,p}(\Omega)$. On the other hand, we know from [5] that increasing the secondary index q in Besov spaces gives a larger space, i.e. $B_{q_1}^\alpha(L_p(\Omega)) \subset B_{q_2}^\alpha(L_p(\Omega))$, $q_1 < q_2$. Therefore, Lemma 2.3 tells us that $B_p^{\alpha+t}(L_p(\Omega))$ determines the same linear approximate efficiency in Sobolev space $W^{t,p}(\Omega)$. Our main result shows that whenever an harmonic function on Lipschitz domain Ω is know to be in a Besov space $B_p^{\lambda+t}(L_p(\Omega))$, then it automatically has additional smoothness in Besov spaces $B_\tau^{\alpha+t}(L_\tau(R^d))$. Therefore, nonlinear approximation is always superior to linear approximation.

Acknowledgements

The project is supported by fund Z20710 of Guilin University of Electronic Technology.

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