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PERIODIC APPROXIMATIONS BASED ON SINC

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Abstract: In this paper we derive some novel formulas for interpolating functions that are periodic with period T on $\mathbb{R} = \{x : -\infty < x < \infty\}$. These formulas are all based on the Whittaker Cardinal series expansion. Let N be a positive integer. If the spacing h of this interpolatory expansion is defined by $h = T/(2\,N)$, then the infinite Cardinal series reduces to a Fourier interpolation polynomial, which is obtainable by interpolation with the Dirichlet kernel,

$$D_e(N,T,x) = \frac{\sin\{2\,N\,\pi\,x/T\}}{2\,N\,\tan\{\pi\,x/T\}}\,.$$

On the other hand, if the spacing h of this interpolatory expansion is defined by $h = T/(2\,N+1)$, then the infinite Cardinal series reduces to a Fourier interpolation polynomial, which is obtainable by interpolation with the Dirichlet

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kernel,

$$D_o(N,T,x) = \frac{\sin\{(2\,N+1)\,\pi\,x/T\}}{(2\,N+1)\,\sin\{\pi\,x/T\}}\,.$$

These results show that Fourier polynomials are a special case of Cardinal expansions.

Two standard families of approximations are thus obtainable, one, starting with Cardinal interpolation at the points $\{k\,h\,:\,k \in \mathbb{Z}\}$, and the other, starting with Cardinal interpolation at the points $\{(k+1/2)\,h\,:\,k \in \mathbb{Z}\}$. In this way the well known formulas of e.g., the trapezoidal rule over the real line, reduce to the trapezoidal rule over $[0,T]$, and similarly for the midordinate rule.

The coefficients of each type of expansion are point evaluations of functions to be approximated, i.e., we differ from Fourier polynomial approximations in that no computations are required for obtaining the Fourier approximations.

We then also derive some relations with polynomials in y via use of the transformation $y = \cos(2\,\pi\,x/T)$. It thus follows that algebraic polynomials are a special case of Fourier polynomials.

We give some comparative examples of approximations of smooth periodic functions and discontinuous functions via both our periodic basis as well as with corresponding polynomial approximations.

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1. Formula Derivations

We use two well known identities to derive our formulas, which are valid for all $z \in \mathbb{C}$:

$$\frac{\pi\,z}{\tan(\pi\,z)} = \lim_{N\to\infty} \sum_{k=-N}^{N} \frac{1}{z-k}\,, \quad \frac{\pi\,z}{\sin(\pi\,z)} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{z-k}\,. \tag{1}$$

We shall assume throughout the paper that f is periodic on the real line $\mathbb{R}$, with period $T > 0$, i.e., that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$. In addition, we shall assume that f takes on its mean value at all points on $\mathbb{R}$, which is especially important at points of discontinuity, i.e., that $\lim_{\delta\to 0}(f(x-\delta) - 2\,f(x) + f(x+\delta)) = 0$ for all $x \in \mathbb{R}$.

Let us define the Cardinal series,

$$F(a,h,x) = \sum_{k\in\mathbb{Z}} f(k\,h - a\,h)\,\frac{\sin\left\{\frac{\pi}{h}(x + a\,h - k\,h)\right\}}{\frac{\pi}{h}(x + a\,h - k\,h)}\,, \tag{2}$$

which approximates f on $\mathbb{R}$. We shall be primarily interested in two cases: $a = 0$ and $a = 1/2$.

Theorem. *Let f be periodic, with period T on $\mathbb{R}$.*

(i) If h is defined by $h = T/(2\,N)$, where N is a positive integer, then

$$F(a,h,x) = \sum_{k=0}^{2\,N-1} f(k\,h - ah)\,s(a,k,h,x)\,, \tag{3}$$

where

$$s(a,k,h,x) = \frac{\sin\left\{\frac{\pi}{h}(x + a\,h - k\,h)\right\}}{2\,N\;\tan\left\{\frac{\pi}{P}(x + a\,h - k\,h)\right\}}\,. \tag{4}$$

(ii) If h is defined by $h = T/(2\,N - 1)$, where N is a positive integer, then

$$F(b,h,x) = \sum_{k=0}^{2\,N-2} f(k\,h - b\,h)\,S(a,k,h,x) \tag{5}$$

where

$$S(a,k,h,x) = \frac{\sin\left\{\frac{\pi}{h}(x + a\,h - k\,h)\right\}}{(2\,N - 1)\;\sin\left\{\frac{\pi}{T}(x + a\,h - k\,h)\right\}}\,. \tag{6}$$

Proof. Part (i). Under the assumption that f has period T on $\mathbb{R}$, i.e., that $f(x + T) = f(x)$ for all $x \in \mathbb{R}$, and if a is an arbitrary real number, we have

$$\begin{aligned}
F(a,h,x) &= \sum_{k\in\mathbb{Z}} f(k\,h - a\,h)\,\frac{\sin\left\{\frac{\pi}{h}(x + a\,h - k\,h)\right\}}{\frac{\pi}{h}(x + b\,h - k\,h)} \\
&= \sum_{s\in\mathbb{Z}} \sum_{k=0}^{2\,N-1} f(k\,h - a\,h + 2\,s\,N\,h)\,\frac{\sin\left\{\frac{\pi}{h}(x + a\,h - k\,h - 2\,s\,N\,h)\right\}}{\frac{\pi}{h}(x + a\,h - k\,h - 2\,s\,N\,h)} \\
&= \sum_{k=-N}^{N-1} f(k\,h)\,s(a,k,h,x)\,,
\end{aligned} \tag{7}$$

and since $2\,N\,h = T$,

$$s(a,k,h,x) = \frac{h}{\pi}\,\sin\left\{\frac{\pi}{h}(x + a\,h - k\,h)\right\}\,\sum_{s=-\infty}^{\infty} \frac{1}{x + a\,h - k\,h - s\,T}\,. \tag{8}$$

Hence, using (1) (a), we get (4).

Part (ii). In this case, we have $(2\,N-1)\,h = T$, so that

$$\begin{aligned} F(a,h,x) &= \sum_{k\in\mathbb{Z}} f(k\,h - a\,h)\,\frac{\sin\left\{\frac{\pi}{h}(x+a\,h-k\,h)\right\}}{\frac{\pi}{h}(x+a\,h-k\,h)} \\ &= \sum_{s\in\mathbb{Z}} \sum_{k=0}^{2\,N-2} f(k\,h-a\,h+s\,T)\,\frac{\sin\left\{\frac{\pi}{h}(x+a\,h-k\,h-s\,T)\right\}}{\frac{\pi}{h}(x+a\,h-k\,h-s\,T)} \\ &= \sum_{k=0}^{2\,N-2} f(k\,h-a\,h)\,S(a,k,h,x)\,, \end{aligned} \tag{9}$$

where

$$S(a,k,h,x) = \frac{h}{\pi}\,\sin\left\{\frac{\pi}{h}(x+a\,h-k\,h)\right\}\sum_{s=-\infty}^{\infty}\frac{1}{x+a\,h-k\,h-s\,a}\,. \tag{10}$$

Applying (1) (b) to this equation yields (6). □

Remark. The following identities are easily verified, for any given integer k,

$$\begin{aligned} s(a,k,h,x) &= s(a,k+2\,N,h,x)\,, \\ S(a,k,h,x) &= S(a,k+2\,N-1,h,x)\,. \end{aligned} \tag{11}$$

When these identities are combined with the above theorem they readily yield the following eight formulas of interpolation over the interval $[0,T]$ of even and odd periodic functions of period T defined on the real line $\mathbb{R}$.

(i) If $T = 2\,N\,h$, $a = 0$, and if f is an even function defined on $\mathbb{R}$, with the above additional assumed properties, then

$$\begin{aligned} F(0,h,x) &= f(0)\,s(0,0,h,x) + f(Nh)\,s(0,N,h,x) \\ &\quad + \sum_{k=1}^{N-1} f(k\,h)\,\left\{s(0,k,h,x)+s(0,-k,h,x)\right\}\,. \end{aligned} \tag{12}$$

(ii) If $T = 2\,N\,h$, $a = 0$, and if f is an odd function defined on $\mathbb{R}$, with the above additional assumed properties, then

$$F(0,h,x) = \sum_{k=1}^{N-1} f(k\,h)\,\left(s(0,k,h,x)-s(0,-k,h,x)\right\}\,. \tag{13}$$

(iii) If $T = 2\,N\,h$, $a = 1/2$, and if f is an even function defined on $\mathbb{R}$, with the above additional assumed properties, then

$$F(1/2,h,x) = \sum_{k=1}^{N} f(k\,h-h/2)\,\left\{s(1/2,k,h,x)+s(1/2,-k,h,x)\right\}\,. \tag{14}$$

(iv) If $T = 2\,N\,h$, $a = 1/2$, and if f is an odd function defined on $\mathbb{R}$, with the above additional assumed properties, then

$$F(1/2,h,x) = \sum_{k=1}^{N} f(k\,h - h/2)\ \{s(1/2,k,h,x) - s(1/2,-k,h,x)\}\ . \qquad (15)$$

(v) If $T = (2\,N-1)\,h$, $a = 0$, and if f is an even function defined on $\mathbb{R}$, with the above additional assumed properties, then

$$\begin{aligned} F(0,h,x) &= F(0)\,S(0,0,h,x) \\ &\quad + \sum_{k=1}^{N-1} f(k\,h)\,(S(0,k,h,x) + S(0,-k,h,x))\,. \end{aligned} \qquad (16)$$

(vi) If $T = (2\,N-1)\,h$, $a = 0$, and if f is an odd function defined on $\mathbb{R}$, with the above additional assumed properties, then

$$F(0,h,x) = \sum_{k=1}^{N-1} f(k\,h)\ \{S(0,k,h,x) - S(0,-k,h,x)\}\ . \qquad (17)$$

(vii) If $T = (2\,N-1)\,h$, $a = 1/2$, and if f is an even function defined on $\mathbb{R}$, with the above additional assumed properties, then

$$\begin{aligned} &F(1/2,h,x) = f(T/2)\,S(1/2,N+1,h,x) \\ &\quad + \sum_{k=1}^{N-1} f(k\,h - h/2)\ \{S(1/2,k,h,x) + S(1/2,1-k,h,x)\}\ . \end{aligned} \qquad (18)$$

(viii) If $T = (2\,N-1)\,h$, $a = 1/2$, and if f is an odd function defined on $\mathbb{R}$, with the above additional assumed properties, then

$$F(1/2,h,x) = \sum_{k=1}^{N} f(k\,h - h/2)\ (S(1/2,k,h,x) - S(1/2,1-k,h,x))\ . \qquad (19)$$

2. Connections with Fourier Series and Numerical Integration

It is convenient here, to take $T = 2\,\pi$. Two types of Dirichlet kernels are used regularly to prove the convergence of Fourier series, namely

$$D_s(N,\theta) = \sum_{k=-N}^{N} e^{i\,k\,\theta} = \frac{\sin\{(N+1/2)\,\theta\}}{\sin(\theta/2)}, \qquad (20)$$

and also,

$$D_t(N,\theta) = \frac{1}{2}\, e^{-i\,N\,\theta)} + \sum_{k=-N+1}^{N-1} e^{i\,k\,\theta} + \frac{1}{2}\, e^{i\,N\,\theta} = \frac{\sin\{N\,\theta\}}{\tan(\theta/2)}\,. \tag{21}$$

Given F defined on $[-\pi\,,\pi]$, its Fourier series is given by

$$F(\theta) = \sum_{k\in\mathbb{Z}} c_k\, e^{i\,k\,\theta}\,, \tag{22}$$

with

$$c_k = \frac{1}{2\,\pi}\int_{-\pi}^{\pi} F(\theta)\, e^{-i\,k\,\theta}\, d\theta\,. \tag{23}$$

It thus follows that

$$\frac{1}{2\,\pi}\int_{-\pi}^{\pi} F(\theta')\, D_s(\theta-\theta')\, d\theta' = \sum_{k=-N}^{N} c_k\, e^{i\,k\,\theta}\,, \tag{24}$$

and also, that

$$\frac{1}{2\,\pi}\int_{-\pi}^{\pi} F(\theta')\, D_t(\theta-\theta')\, d\theta' = \frac{1}{2}c_{-N}\, e^{-i\,N\,\theta} + \sum_{k=-N+1}^{N-1} c_k\, e^{i\,k\,\theta} + \frac{1}{2}\, c_N\, e^{i\,N\,\theta}\,. \tag{25}$$

3. Trapezoidal and Midordinate Integration

We should also mention two popular methods, the trapezoidal and midordinate rule of numerical integration over an interval $[0,T]$, with spacing $h = T/N$. First, the *trapezoidal rule*, which is given by

$$\int_0^T f(x)\, dx \approx T_N(h,f)\,, \tag{26}$$

with

$$T_N(h,f) = h\left\{\frac{1}{2}f(0) + \sum_{k=1}^{N} f(k\,h) + \frac{1}{2}f(T)\right\}\,, \tag{27}$$

and the *midordinate rule*,

$$\int_0^T f(x)\, dx \approx M_N(h,f)\,, \tag{28}$$

with

$$M_N(h,f) = h\sum_{k=1}^{N} f((k-1/2)\,h)\,. \tag{29}$$

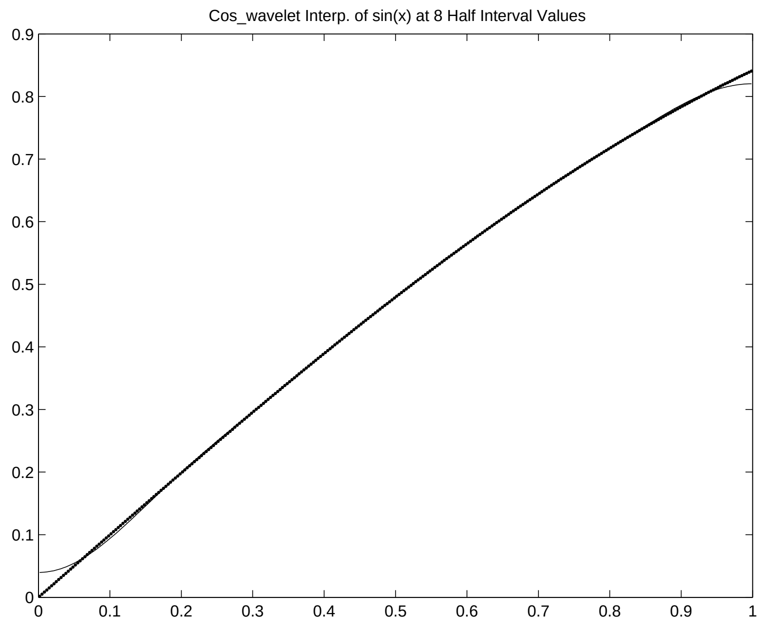

Figure 1: Cosine polynomial approximation of $\sin(x)$ on $[0, 1]$

Thus the formula in (6) is obtained by applying the trapezoidal rule (27)–(28) to the integral (25) involving the kernel D_t, while (2.11) can be derived by applying the midordinate rule to the integral (25). Indeed, Gabdulhaev [1] obtained (11) for the interval $[-\pi\,, \pi]$ in just this manner (see Stenger [2, Theorem 2.2.6]).

4. Error of Approximation

A bound on the error of approximation was obtained in the proof of Theorem 2.2.6 of Stenger [2]. However, a much more accurate bound on the error of approximation is possible via use of the approach of our paper. Namely, we can use the accurate bound of Sinc approximation derived in Chapter III of Stenger [2]. That is, if f is analytic and bounded by M on the strip $D_d = \{z \in \mathbb{C} : |\Im z| < d\}$, then $f(x) - f_h(x)| < (2\,M/d)\,\exp(-\pi\,d/h)$ for all $x \in \mathbb{R}$.

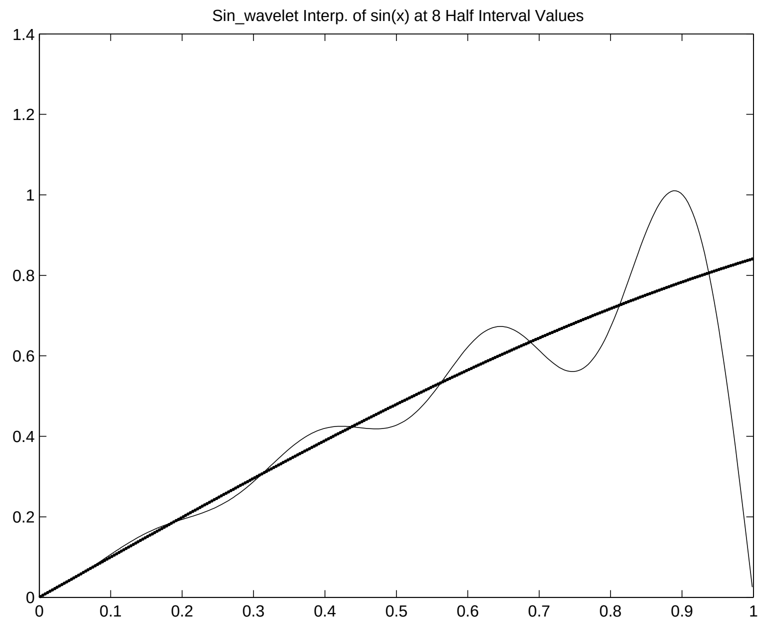

Figure 2: Sine polynomial approximation of $\sin(x)$ on $[0, 1]$

5. Connection with Polynomial Approximation

Suppose that we wish to approximate a given function $F(y)$ on a finite interval (c, d) of the real line $\mathbb{R}$. Then, setting $y = (1/2)(c + d) + (1/2(d - c)\,\cos(x)$, we get a new function, $G(x) = F(y) = F((1/2)(c + d) + (1/2)(d - c)\,\cos(x))$, which is a periodic function on all of $\mathbb{R}$, with period $2\,\pi$, and moreover, $G(x)$ is an even function of x. We can thus approximate G on the interval $[0, \pi]$ via use of either (7) or (13). Since G is representable via a rapidly convergent trigonometric cosine series expansion, as can be seen from either (24) or (25), and since $\cos(m\,x) = T_m(w)$, with $w = \cos(x)$, and with $T_M(w)$ denoting the Chebyshev polynomial, it follows that the corresponding function $F(y)$ is now approximated via a Chebyshev polynomial expansion in the variable

$$w = \frac{y - (1/2)(c + d)}{(1/2)(d - c)}\,. \tag{30}$$

Moreover, this approximation is a rapidly convergent function of N if F is analytic in an open region containing the interval $[c, d]$. However, we no longer

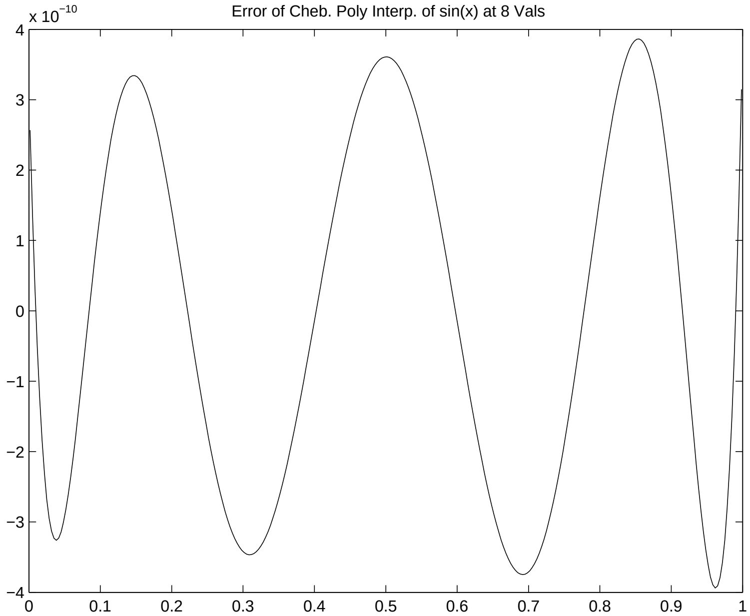

Figure 3: Error of deg. 7 Chebyshev polynomial approximation of $\sin(x)$ on $[0,1]$

have interpolation at a set of equi-spaced points, but rather at the points [1]

$$y_j = (1/2)(c+d) + (1/2)(d-c)\,\cos(x_j)\,, \tag{31}$$

where $x_j = \pi\, j/N$, $j = 0\,,\ 1\,,\ \ldots\,,\ N$ for the case of (12), and with $x_j = \pi(j-1/2)/N$, $j = 1\,,\ 2\,,\ \ldots\,,\ N$ for the case of (14).

6. Application of the Formulas

We now illustrate the application of the derived formulas. In all cases of our graphical illustrations, '—' refers to the exact graph, and '$\cdots$' refers to the approximation.

Examples. We illustrate the 8–point approximation of the function $\sin(x)$, in three ways:

1. The case of interpolation of $\sin(x)$ on $[0,1]$, where this function is defined

[1]End values may no longer be interpolations for the case of (13) or (16), since a Fourier series averages values at discontinuities.

by $\sin(|x|)$ on the periodic interval $[-1\,,1]$. We use the formula of Theorem. This is equivalent to a cosine polynomial approximation of $\sin(x)$ on $[0,1]$.

Notice that the function $f(x) = \sin(|x|)$ that is defined so on the periodic interval $[-1,1]$ has a continuous $\mathbf{Lip}^1$ extension to the real line $\mathbb{R}$, and so this cosine polynomial converges to f in the limit as the number of interpolation points approaches infinity.

2. The case of interpolation of $\sin(x)$ on $[0,1]$, where this function is defined by $\sin(x)$ on the periodic interval $[-1,1]$. We use the formula of Theorem. This is equivalent to a sine polynomial approximation of $\sin(x)$ on $[0,1]$.

Notice that the function $\sin(x)$ that is defined so on the periodic interval $[-1,1]$ is discontinuous at the points $\{x = 2\,k - 1\,; k \in \mathbb{Z}\}$, and we therefore see the effects of Gibb's phenomenon in the neighborhood of $x = 1$.

3. The case of interpolation of the function $g(x) = \sin(x)$ at the points $x = (1/2)(1 - \cos(u))$, with $u = (2\,k - 1)\,\pi/(2\,N)$, $k = 1\,,2\,,\ldots\,,N$, and then use of the formula of Theorem. This amounts to Chebyshev polynomial interpolation of g on $[0,1]$, which is, of course, very accurate. The error of this interpolation is given in Figure 3.

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