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THE HEAT EXTRACTION IN A TECHNOLOGICAL PROCESS AS A NONLINEAR PROBLEM

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Abstract: The dissipative power in an electric arc to the REZ process assures the necessary quantity of heat in normal conditions. We offer the premises to a mathematical analysis of the thermal transfer in the electric arc. Using the thermal equilibrium equation, an initial time of the initiation of the electric arc, an integral transformation technique, we will formulate a boundary value problem of elliptic type. Some techniques of bifurcation theory can be applied to reveal that a ramification of solution is achieved from the first eigenvalue of the linear part and a continuation is given by means of implicit function theorems.

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1. Introduction

From the technological point of view, the implemented power in electric arc to a REZ process (160kVA order) on a pilot installation for ingots of half and low weight, assures the necessary quantity of heat for another melt of ingot-electrode in normal conditions of work (on standard installation parameters). With an energetic balance on the electric arc level developed between ingot-electrode and metallic bath, we create premises to thermodynamically property analysis of the arc. Difficulties in mathematical modeling of energetically equilibrium of the arc are risen up by the specification of power dissipation function through

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column and implicitly of heat extraction function on the electrode foot level.

We put the problem of perturbations analysis which appears on changing power steps. The calculation of gyved up heat could be done at electric current parameters supplied by transformer: 2000 A, 50-80 V (see [5], p. 45). In case of soliciting installation electric energy variations, we impose the preliminary calculus requirement of extract heat on the ingot-electrode foot level, for REZ process behavior characterization. We can specify the power dissipation function at the column arc level and of the heat extraction function on the electrode foot level. Taking account a linear perturbation, depending o a rate factor of power in changing the power step, we study the bifurcation of solution [4]. We will see how results of bifurcation theory assure the ramification of solution u of (3.1) problem from a solution u of an associated eigenvalue problem.

2. Some Technological Features of the Electric Arc and a Modeling Problem

We adopt for heat transmitted in metallic mass the thermal diffusion equation. In analyzing heat source context, we require heat quantity determination through an energetically result:

$$dQ = (U.I - \varphi)\,.dt, \tag{2.1}$$

where U is electrical tension, I electrical current intensity from electrode, φ a dissipation function of electrical energy in arc, dQ elementary variation of extraction heat. For determining the extraction heat dissipation function trough thermal conduction, it is necessary to determine the distribution of temperature in arc column. This is the equation of heat diffusion in liquid metal mass:

$$\frac{\partial T}{\partial t}(t, x) - a(x)\,\Delta T(t, x)\,, \tag{2.2}$$

T is thermodynamical temperature, a a thermal spreading. We accept the cylindrical form of electric arc developed between electrode and metallic bath. We consider $T{=}T(t, r, \omega, z)$, r, ω, z are the cylindrical coordinates of the system of reference. From grounds of symmetry, it is sufficient to have $T{=}T(t, r, z)$ and take the Laplacian in cylindrical coordinates:

$$\Delta. = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial .}{\partial r}\right) + \frac{\partial^2 .}{\partial z^2} \tag{2.3}$$

We have $\theta_0(r, z) = T(0, r, z)$ the temperature at the moment of the initiation of electric arc between ingot and the main board of crystal-maker, covered with electric-conducting slag. We impose as a boundary condition the proportionality of heat flux q with P, a distribution of power on the surface level of metallic bath:

$$q(r,t,0) = -\lambda \frac{\partial T}{\partial z}(t,r,0) = \begin{cases} \frac{P}{\pi b^2} r, \text{ for } r \le b\,, \\ 0, \text{ if } r > b\,, \end{cases} \tag{2.4}$$

where b is the thickness of electric arc. On this condition it is considered that power transmits itself like a step function on the contact surface electrode-melt steel. It is used the expression in cylindrical coordinates of diffusion equation (2.2), with expression (2.3) at Laplacian. We introduce the Laplacian transformer:

$$\Theta(r,z,p) := \int_0^\infty e^{-ip} T(t,r,z)\, dt \tag{2.5}$$

of T, thermodynamic temperature function, $p \in R^+$. The function Θ verifies the equation:

$$a\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Theta}{\partial r}(r,z,p)\right) + \frac{\partial^2 \Theta}{\partial z^2}(r,z,p)\right\} + p\Theta(r,z,p) = T_0(r,z)\,, \tag{2.6}$$

$$\text{for all } (r,z) \in \Omega,$$

where Ω is a 2D section in the arc column, with a boundary condition:

$$\Theta_0(r,p) = \begin{cases} -\frac{P}{\pi b^2 \lambda}\frac{1}{p^2} - \frac{1}{p}\theta_0(r,0)\,, \text{ if } r \le b\,, \\ -\frac{1}{p}\theta_0(r,0)\,, \text{ if } r > b\,. \end{cases} \,. \tag{2.7}$$

We consider that at the initial moment the temperature θ_0 is uniformly distributed on the contact surface electrode-slag. We introduce the variable:

$$u(r,z,p) = \Theta(r,z,p) - \Theta_0(r,p)\,, \tag{2.8}$$

and so, we obtain the boundary value problem:

$$a\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}(r,z,p)\right) + \frac{\partial^2 u}{\partial z^2}(r,z,p)\right\} + pu(r,z,p) = f(r,z,p) \text{ in } \Omega, \tag{2.9}$$

$$u = 0, \text{ for } z = 0,$$

where it is noted

$$f(r,z,p) = \begin{cases} \theta_0(r,z) + \frac{a}{p}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\Theta_0(r,0)\right) + \frac{P}{\pi b^2 \lambda}\frac{1}{p} + \theta_0(r,0)\,, \text{ for } r \le b\,, \\ \theta_0(r,z) + \frac{a}{p}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\Theta_0(r,0)\right) + \theta_0(r,0)\,, \text{ for } r > b\,. \end{cases} \,. \tag{2.10}$$

Because the function f has a discontinuity on the disk circumference:

$$\left\{(r,z) \in R^2 /\ 0 < r \le b,\ z = 0\right\}$$

and it is independent from u, it is accepted that $f \in L^2(\Omega)$ and we could see

(2.9) like a boundary value problem on a restricted domain Ω, associated with a quasi-linear operator.

3. Some Preliminaries Results

We consider a linear perturbation equation, generic writting $Lu - \lambda u = 0$, $\lambda \in R$ being a rate factor of power in changing the power step, for which the nonomogenious associated boundary value problem is:

$$Lu - \lambda u + f = 0,\ f(u) = ct.,\ \text{ in } \Omega,\ f \in L^2(\Omega)\ , \tag{3.1}$$
$$u = 0,\ \text{for } z = 0,\ \text{where } L. = \Delta. + p.$$

We will see that bifurcation of a branch of positive solution $u(\lambda)$ is achieved from the first eigenvalue of linear part of an application that defines the equation. The *Lyapunov-Schmitd* theory [4] is applicable for parametric representation of solution, defined near eigenvalue λ_1, and the solution continuation is achieved in implicit functions theorem meaning. The following considerations are for an eigenvalue problem, put in a more generally frame or a new point of view, for which we will obtain intrinsic results.

We consider the boundary value problem:

$$Lu - \lambda u + f(u) = 0,\ \text{in } \Omega,\ \ Bu = 0 \text{ on } \partial\Omega, \tag{3.2}$$

where $\lambda \in R$ and L is a second order operator, defined on a real functions space, B is a linear operator, defined on functions restricted to $\partial\Omega$, $f(.)$ is a convex function for $t > 0$, concave for $t < 0$, $f(0) = f'(0) = 0$, with $\lim\limits_{t\to-\infty} f'(t) = k_- > -\infty$, $\lim\limits_{t\to+\infty} f'(t) = k_+ < \infty$ (see Figure 1). For $f \in C^\alpha(R)$ with a finite rising rate $\lim\limits_{t\to\pm\infty} \left|f'(t)\right| < \lambda_1 - \delta$, $Lu = -\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}u\right) + au$, $a_{ij} \in C^{1+\alpha}(\Omega)$, $\alpha \in (0,1)$, $Bu = u$, it is accepted the operational form of equation (3.2):

$$F(u,\lambda) = K(\lambda u - f) = u, \tag{3.3}$$

where $K : C^\alpha(\Omega) \to C^{2+\alpha}(\Omega)$ is a linear, compact application, defined by the solution of boundary value problem:

$$Lu = h \text{ in } \Omega,\ h \in C^\alpha(\Omega)\,,\ Bu = 0 \text{ on } \partial\Omega. \tag{3.4}$$

The eigenvalue λ_1 of the operator L from (3.2) are characteristic value for application K from (3.3). It is considered the eigenvalue quasi-linear problem:

$$K(m.n) = \mu u,\ m \in C(\Omega)\,,\ m(x) > 0,\ \text{a.p.t. } x \in \Omega, \tag{3.5}$$

with $\mu_j(m) > 0$ eigenvalues which are accumulated to 0. We reconsider the

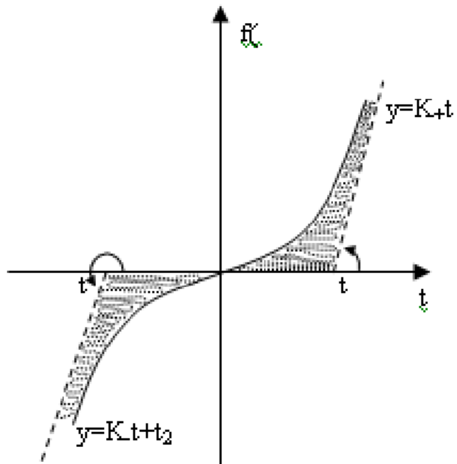

Figure 1: Example of function f

following result: eigenvalue of problem (3.5) are monotonous in comparison to function $m(x)$ on Ω. We will study the branches bifurcations of no vanish solutions from the vanish solution branche $\{(u, \lambda) \in C(\Omega) \times R / \ u = 0\}$; we see $(0, \lambda_1)$ as bifurcation point. For the formulation of the existence of positive solution branches, we will preliminary present some results. We introduce first:

Definition 1. Function $u \in C(\Omega)$, for which Fréchet derivative of application in u is a reversible application $F(., \lambda)$ in u, is a reversible application, it is named itself a notcritical point for problem (3.2). Otherwise, it is named itself critical point.

A regular element characterization introduced by F is made by

Lemma 2. *If u is a non zero solution of problem (3.2), $\lambda \geq \lambda_1$, which holds the same sign in Ω, then u is not a critical point for application $G(., \lambda)$: $C(\overline{\Omega}) \to C(\overline{\Omega})$, $G(u, \lambda) = u - F(u, \lambda)$.*

Proof. We suppose that u is a critical point for application $G(., \lambda)$ that is, to say there is $v \in C(\overline{\Omega})$, $v \neq 0$, so that $G'(u, \lambda) v = 0$. Then, equation $F'(u, \lambda) v = v$ has the solution $v \neq 0$, so the eigenvalue problem $K\left(\lambda - f'(u)\right)$ $v = \mu v$ has v as a eigenfunction, for a particular eigenvalue $\mu_k(\lambda - f(u)) = 1$, with $k > 1$, specified. On the other hand, u verifies $K(\lambda u - f(u)) = u$, but K

being linear, we have $K\left(\lambda-\frac{f(u)}{u}\right)u=u$, so that the eigenvalue problem:

$$K\left(\lambda-\frac{f(u)}{u}\right)v=\mu v, \tag{3.6}$$

has an eigenvalue $\mu_j\left(\lambda-\frac{f(u)}{u}\right)=1$, with u eigenfunction, positive or negative, which is the solution to equation $F(u,\lambda)=u$. We observe that function f satisfies $\frac{f(t)}{t}<f'(t)$, $(\forall)\,t\neq 0$. Indeed, the conditions $f(0)=f'(0)=0$, f convex, assure f as a local increasing function for $t>0$, so the chord slope $\frac{f(t)}{t}$ is inferior to graphic tangent: $f'(t)$. We have $\lambda-\frac{f(t)}{t}>\lambda-f'(t)=\delta>0$, from where, using monotony of eigenvalues, results $\mu_1\left(\lambda-f'(u)\right)\le\mu_1\left(\lambda-\frac{f(u)}{u}\right)$, that is to say $\mu_1\left(\lambda-\frac{f(u)}{u}\right)=1$. But, for (3.6), $1=\mu_1\left(\lambda-\frac{f(u)}{u}\right)>\mu_1\left(\lambda-f'(u)\right)>\mu_k\left(\lambda-f'(u)\right)$, is a contradiction (because $k>1$ and $\mu_k\to 0$). □

Lemma 3. *The equation $F(u,\lambda)=u$ has a unique positive solution, for $\lambda\in(\lambda_1,\lambda_1+k_+)$ and no solution for $\lambda>\lambda_1+k_+$, one negative solution for $\lambda\in(\lambda_1,\lambda_1+k_-)$ and no solution for $\lambda>\lambda_1+k_-$. More, $\lim\limits_{\lambda\to\lambda_1+k_+}\|u(\lambda)\|=+\infty$.*

Proof. We see that application $F(.,\lambda)=K(\lambda.-f(.))$ does not vary in the positive cone $P\subset C\left(\overline{\Omega}\right)$([1]) and it is an asymptotic linear application, having $F'_+(0,\lambda)v=K\left(\lambda v-f'(0)v\right)=\lambda Kv=\lambda\mu_1(1)v=\frac{\lambda}{\lambda_1}>v$, $(\forall)\,v\in P$, $F'_+(+\infty,\lambda)v=K\left(\lambda v-f'(+\infty)v\right)=(\lambda-k_+)Kv=\frac{\lambda-k_+}{\lambda_1}v>v$, $(\forall)\,v\in P$.

Having now $\lambda\in(\lambda_1,\lambda_1+k_+)$, then $\frac{\lambda}{\lambda_1}>1$, $\frac{\lambda-k_+}{\lambda_1}<1$ so $F'_+(0,\lambda)v-v>0$, $F'_+(+\infty,\lambda)v-v<0$, $(\forall)\,v\in P$, then the continuous application $F(u,.)-u$. has a unique positive zero: u.

We suppose $\lambda>\lambda_1+k_+$ and a unique positive solution u for $F(u,\lambda)=u$, for which $\mu_1\left(\lambda-\frac{f(u)}{u}\right)=1$ (Lemma 1). Because $\lambda-f'(u)>\lambda-k_+$, we have $1=\mu_1\left(\lambda-\frac{f(u)}{u}\right)>\mu_1\left(\lambda_1-f'(u)\right)>\mu_k(\lambda-k_+)>\mu_1(\lambda_1)=1$, a contradiction, so there is no solution for $F(u,\lambda)=u$, with $\lambda>\lambda_1+k_+$. □

4. The Main Result

The results from Lemmas 1 and 2 assure irreversibility of application $G(.,.)$, so it is a local isomorphism for $\lambda\le\lambda_1$. The following result shows us existence of two no vanish branches of solutions, making a bifurcation from solution $(0, \lambda_1)$ at the problem (3.3), parametrically appointed through:

$$\Gamma_+ = \left\{(u, \lambda) \in C\left(\overline{\Omega}\right) \times R/\ u = u_+(\lambda), \lambda_1 \leq \lambda \leq \lambda_1 + k_+\right\}, \qquad (4.1)$$
$$\Gamma_- = \left\{(u, \lambda) \in C\left(\overline{\Omega}\right) \times R/\ u = u_-(\lambda), \lambda_1 \leq \lambda \leq \lambda_1 + k_-\right\}.$$

Theorem 4. *Let $f \in C^1(R)$ be a function that satisfies the imposed conditions; then the multitude no vanish solutions of equation (3.3), making a bifurcation in point $(0, \lambda_1)$, form two continuous branches of solutions Γ_+, Γ_- so that $\lim_{\lambda \to \lambda_1 + k_+} \|u(\lambda)\| = +\infty$. Much more, equation (3.3) has only vanish solution for $\lambda \in (0, \lambda_1)$ and neither positive, nor negative solution for $\lambda > \lambda_2$.*

Proof. First we prove that $F(u, \lambda) = u$ has solution for $\lambda_1 < \lambda < \lambda_2$, with positive or negative necessity. If $u \neq 0$, then exist $k > 1$ so that $\mu_k\left(\lambda - \frac{f(u)}{u}\right) = 1$ (Lemma 1). Using continuation property of no vanish solution u, we have for $\lambda - \frac{f(u)}{u} < \lambda < \lambda_2$ the following inequalities: $\mu_2\left(\lambda - \frac{f(u)}{u}\right) < \mu_2(\lambda) = \frac{\lambda}{\lambda_2}$, and consequently $\mu_1\left(\lambda - \frac{f(u)}{u}\right) = 1$, so that the no vanish solution of equation $v = K\left(\lambda - \frac{f(u)}{u}\right) v$ is constant in sign. Because u is a solution of this equation, we have $u = K\left(\lambda - \frac{f(u)}{u}\right) u$ so u is positive or negative.

If $u = u_+(\lambda)$ is the positive solution for $F(u, \lambda) = u$, conforming Lemma 2, there exist for $\lambda_1 < \lambda < \lambda_1 + k_+$, the solution branch Γ_+. Analogous, a no vanish negative solution branch Γ_- exists. Since for$\lambda < \lambda_1$, the only solution is vanish, the solutions branches are bifurcated in the point $(0, \lambda_1)$. □

Corollary 5. *Having*

$$c_1 = \min\{\lambda_1 + k_+, \lambda_1 + k_-\} \quad \textit{and} \quad c_2 = \max\{\lambda_1 + k_+, \lambda_1 + k_-\},$$

then the boundary value problem (3.2) has two no vanish solutions branches for $\lambda \in (\lambda_1, \min\{c_1, c_2\})$, a no vanish solutions branch if $c_1 < \min\{c_2, \lambda_2\}$and $\lambda \in [c_1, \min\{c_2, \lambda_2\}]$, none of the no vanish solutions if $c_2 \leq \lambda_2$ and $\lambda \in [c_2, \lambda_2]$.

For many other considerations see [2], [3]. A numerical analysis of this solution and other technological considerations will be presented in a forthcoming paper. □

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