International Journal of Pure and Applied Mathematics
Volume 49 No. 2 2008, 167-199

ON QUANTUM YANG-BAXTER COHERENT ALGEBRA SHEAVES

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Abstract: In this paper we begin by reviewing key concepts of classical and quantum Yang-Baxter algebras and then develop coherent sheaf theoretic structures to use in the definition and construction of quantum Yang-Baxter coherent algebra sheaves. The main results are structural theorems for quantum Yang-Baxter coherent algebra sheaves.

AMS Subject Classification: 81R50, 81R60
Key Words: classical Yang-Baxter algebras, quantum Yang-Baxter algebras, coherent algebra sheaves, quantum Yang-Baxter coherent algebra sheaves, quasi-triangular bi-algebras, quasi-triangular Hopf algebras, monoidal categories

1. Introduction and Background Material

The first aim of this introductory section is to provide a brief background material on Yang-Baxter algebras from both the classical and quantum points of view over an algebraically closed field $\mathbb{K}$ of characteristic zero, which will for the most part be the field of complex numbers $\mathbb{C}$. We cite here the papers of Yuri I. Manin [24], Louis H. Kauffman and David E. Radford [12, 13] for further details of the material reviewed in this section.

Received: September 12, 2008

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Definition 1.1. Let V be a $\mathbb{K}$-vector space and $R\colon V\otimes V\longrightarrow V\otimes V$ a $\mathbb{K}$-linear endomorphism. Suppose the $\mathbb{K}$-linear endomorphism:

$$R_{st}: V\otimes V\otimes V\longrightarrow V\otimes V\otimes V,$$

for $1\leq s<t\leq 3$ is defined by: $R_{12}:=R\otimes 1_V, R_{23}:=1_V\otimes R$ and $R_{13}:=(1_V\otimes\tau_{V,V})\circ(R\otimes 1_V)\circ(1_V\otimes\tau_{V,V})$, where the twist map: $\tau_{V,V}\colon V\otimes V\longrightarrow V\otimes V$ is given by $\tau_{V,V}(v\otimes w):=w\otimes v$, for all $v,w\in V$. The pair (V,R) is called a braided $\mathbb{K}$-vector space if and only if R satisfies the braid equation:

$$R_{12}R_{23}R_{12}=R_{23}R_{12}R_{23}\,. \tag{1.1}$$

It is well known from Yu.I. Manin [24] that the quantum Yang-Baxter equation $(QYBE)$ is given by

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}. \tag{1.2}$$

The $\mathbb{K}$-linear endomorphism $R\colon V\otimes V\longrightarrow V\otimes V$ satisfies (1.2) if and only if $R_\tau:=R\circ\tau_{V,V}$ satisfies equation (1.1). Equivalently R satisfies (1.1) if and only if ${}_\tau R:=\tau_{V,V}\circ R$ satisfies (1.2). This implies that solving the quantum Yang-Baxter equation is equivalent to solving the braid equation. It is also self evident that R satisfies the quantum Yang-Baxter equation or the braid equation if and only if ${}_\tau R_\tau:=\tau_{V,V}\circ R\circ\tau_{V,V}$ satisfies the quantum Yang-Baxter equation or the braid equation. If $R: V\otimes V\longrightarrow V\otimes V$ is an invertible $\mathbb{K}$-linear endomorphism then R satisfies (1.1) if and only if R^{-1} does and R satisfies (1.2) if and only if R^{-1} does. We now state the following lemma of D.E. Radford [30].

Lemma 1.2. *Let V be a $\mathbb{K}$-vector space and $R\in V\otimes V$. Define*

$$V_{[R]}:=\{(v^*\otimes 1_V)(R)+(1_V\otimes w^*)(R): v^*,w^*\in V^*\}. \tag{1.3}$$

Then:

(i) $R\in V_{[R]}\otimes V_{[R]}$ and $V_{[R]}$ is the smallest subspace U of V such that $R\in U\otimes U$.

(ii) If $\Phi\colon V\longrightarrow V$ is a $\mathbb{K}$-linear endomorphism of V such that $(\Phi\otimes\Phi)(R)=R$, then $\Phi(V_{[R]})=V_{[R]}$.

Proof. (i) is an immediate consequence of the definition of $V_{[R]}$ in (1.3).

(ii) Suppose $R\neq 0$ and $R=\sum_{j=1}^r v_j\otimes w_j$ for sufficiently small r. The sets $\{v_1,\cdots,v_r\},\{w_1,\cdots,w_r\}$ are linearly independent so that $V_{[R]}=Span\{v_1,\cdots,v_r,w_1,\cdots,w_r\}$. From the fact that $(\Phi\otimes\Phi)(R)=R$ we have that $\sum_{j=1}^r\Phi(v_j)\otimes\Phi(w_j)=\sum_{j=1}^r v_j\otimes w_j$, implies the sets $\{\Phi(v_1),\cdots,\Phi(v_r)\}$ and $\{\Phi(w_1),\cdots,\Phi(w_r)\}$ are also linearly independent. Thus $Span\{v_1,\cdots,v_r,\Phi(v_1),\cdots,\Phi(v_r)\}=Span\{w_1,\cdots,w_r,\Phi(w_1),\cdots,\Phi(w_r)\}$. It follows that $\Phi(V_{[R]})=V_{[R]}$. $\square$

Definition 1.3. Let $\mathcal{A}$ be an associative $\mathbb{K}$-algebra with unit, $dim_{\mathbb{K}}\mathcal{A} = r$ and $R\colon \mathcal{A}\otimes\mathcal{A} \longrightarrow \mathcal{A}\otimes\mathcal{A}$ a $\mathbb{K}$-linear endomorphism given as $R := \sum_{j=1}^{r} a_j \otimes b_j$. Suppose $R_{st}\colon \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \longrightarrow \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}$ for $1 \leq s < t \leq 3$, is a $\mathbb{K}$-linear endomorphism such that

$$R_{12} = R\otimes 1_{\mathcal{A}} = \sum_{j=1}^{r} a_j\otimes b_j\otimes 1_{\mathcal{A}}, R_{23} = 1_{\mathcal{A}}\otimes R = \sum_{j=1}^{r} 1_{\mathcal{A}}\otimes a_j\otimes b_j$$

and $R_{13} = (1_{\mathcal{A}}\otimes\tau_{\mathcal{A},\mathcal{A}})\circ(R\otimes 1_{\mathcal{A}})\circ(1_{\mathcal{A}}\otimes\tau_{\mathcal{A},\mathcal{A}}) = \sum_{j=1}^{r} a_j\otimes 1_{\mathcal{A}}\otimes b_j$.

From Yu.I. Manin [24], the quantum Yang-Baxter equation for R is:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{1.4}$$

or equivalently

$$\sum_{i,j,l=1}^{r} a_ia_j\otimes b_ja_l\otimes b_jb_l = \sum_{j,i,l=1}^{r} a_ja_i\otimes a_lb_i\otimes b_lb_j. \tag{1.5}$$

The pair $(\mathcal{A}, R)$ is called a Yang-Baxter algebra over $\mathbb{K}$ if the $\mathbb{K}$-linear endomorphism $R\colon \mathcal{A}\otimes\mathcal{A} \longrightarrow \mathcal{A}\otimes\mathcal{A}$ is invertible and satisfies the quantum Yang-Baxter equation (1.4). Next suppose $\mathcal{M}$ is a left $\mathcal{A}$-module and $R_{\mathcal{M}}\colon \mathcal{M}\otimes\mathcal{M} \longrightarrow \mathcal{M}\otimes\mathcal{M}$ an endomorphism of $\mathcal{M}\otimes\mathcal{M}$ defined by

$$R_{\mathcal{M}}(x\otimes y) = \sum_{j=1}^{r} a_j\cdot x\otimes b_j\cdot y, \tag{1.6}$$

for all $x, y \in \mathcal{M}$. Then it is easy to check that $R_{\mathcal{M}}$ is a solution of the quantum Yang-Baxter equation for all left $\mathcal{A}$-modules $\mathcal{M}$ if and only if

$$(R_{\mathcal{M}})_{12}(R_{\mathcal{M}})_{13}(R_{\mathcal{M}})_{23} = (R_{\mathcal{M}})_{23}(R_{\mathcal{M}})_{13}(R_{\mathcal{M}})_{12}. \tag{1.7}$$

If we set $R^{op} = \tau_{\mathcal{A},\mathcal{A}}\circ R = \sum_{j=1}^{r} b_j\otimes a_j$, then $(R^{op})_{\mathcal{M}} = {}_{\tau}(R_{\mathcal{M}})_{\tau} = \tau_{\mathcal{M},\mathcal{M}}\circ R_{\mathcal{M}}\circ\tau_{\mathcal{M},\mathcal{M}}$ for all $\mathcal{A}$-modules $\mathcal{M}$.

An example of Yang-Baxter algebra over $\mathbb{K}$ is given in the paper of L.H. Kauffman and D.E. Radford [13], which we describe below to illustrate the importance of this structure. Let V be an n-dimensional $\mathbb{K}$-vector space and $R\colon V\otimes V \longrightarrow V\otimes V$ a solution of the quantum Yang-Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{1.8}$$

Let the endomorphism $R \in End(V\otimes V) \cong End(V)\otimes End(V)$ and suppose $\epsilon = \{\epsilon_1, \cdots, \epsilon_n\}$ is a basis of V, $\eta = \{\epsilon_{ij}\}_{1\leq i,j\leq n}$ a basis of $V\otimes V$, where $\epsilon_{ij} = \epsilon_i\otimes\epsilon_j$, for $1 \leq i, j \leq n$, with the pair (i, j) ordered lexicographically. Then identify the pairs (i, j) with $\{1, 2, \cdots, n^2\}$ according to the lexicographical order and write $R \in End(V\otimes V)$ as $R(\epsilon_{k,l}) = \sum_{i,j=1}^{n} R_{k,l}^{i,j}\epsilon_{i,j}$, for $1 \leq k, l \leq n$ where $R_{k,l}^{i,j} \in \mathbb{K}$. The matrix of R with respect to the basis η is given by

$R_\eta = [R^{i,j}_{k,l}]_{1\leq i,j,k,l\leq n}$. Let $End(V) \cong M_n(\mathbb{K})$ be $\mathbb{K}$-algebra isomorphism with the $\mathbb{K}$-algebra of $n \times n$-matrices relative to the given basis over $\mathbb{K}$. Similarly $End(V \otimes V) \cong M_{n^2}(\mathbb{K})$ the $\mathbb{K}$-algebra isomorphism. If $A \in M_n(\mathbb{K})$, set matrix of $A = [A^i_j]_{1\leq i,j\leq n}$, then the composite algebra morphisms:

$$\mathcal{M}_n(K) \otimes \mathcal{M}_n(K) \cong End(V) \otimes End(V) \cong End(V \otimes V) \cong M_{n^2}(\mathbb{K}), \quad (1.9)$$

give for $A, B \in \mathcal{M}_n(\mathbb{K})$, the tensor product $A \otimes B$ considered as $n^2 \times n^2$-matrix such that

$$(A \otimes B)_{kl^{i,j}} := A^i_k B^j_l, \quad (1.10)$$

for all $1 \leq i, j, k, l \leq n$. Let $(\chi^i_j) \in \mathcal{M}_n(\mathbb{K})$ be defined by

$$(\chi^i_j)^k_l = \delta_{ik}\delta_{jk} \quad (1.11)$$

for all $1 \leq i, j, k, l \leq n$. The endomorphism $R : V \otimes V \longrightarrow V \otimes V$ can be given in matrix form as:

$$R_\eta = \sum_{i,j,k,l=1}^{n} R^{ij}_{kl} \chi^i_k \otimes \chi^j_l,$$

where $R^{ij}_{kl} \in \mathbb{K}$. Thus R satisfies the quantum Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

Equivalently $R_\eta = [R^{ij}_{kl}]_{1\leq i,j,k,l\leq n}$ satisfies

$$\sum_{p,q,y} R^{xy}_{pq} R^{ax}_{xr} R^{bc}_{yz} = \sum_{y,q,r} R^{yz}_{qr} R^{xc}_{yz} R^{ab}_{xy}.$$

It is clear that for $R : V \otimes V \longrightarrow V \otimes V$ an invertible endomorphism, the invertible matrices R_η also satisfy the quantum Yang-Baxter algebra. This shows that the pair $(\mathcal{M}_n(\mathbb{K}), R_\eta)$ is a Yang-Baxter algebra.

We can now define the tensor product of Yang-Baxter algebras. Suppose $(\mathcal{A}, R)$ and $(\tilde{\mathcal{A}}, \tilde{R})$ are two Yang-Baxter algebras over $\mathbb{K}$. The tensor product $(\mathcal{A} \otimes \tilde{\mathcal{A}}, \tilde{\tilde{R}})$ of $(\mathcal{A}, R)$ and $(\tilde{\mathcal{A}}, \tilde{R})$, where $\tilde{\tilde{R}} := (1_{\mathcal{A}} \otimes \tau_{\mathcal{A},\tilde{\mathcal{A}}} \otimes 1_{\tilde{\mathcal{A}}})(R \otimes \tilde{R})$ is a Yang-Baxter algebra over $\mathbb{K}$.

Definition 1.4. Let $(\mathcal{A}, R)$ and $(\mathcal{A}, \tilde{R})$ be Yang-Baxter algebras over $\mathbb{K}$. A morphism $\Psi \colon (\mathcal{A}, R) \longrightarrow (\tilde{\mathcal{A}}, \tilde{R})$ of Yang-Baxter algebras is a $\mathbb{K}$-algebra map $\Psi \colon \mathcal{A} \longrightarrow \tilde{\mathcal{A}}$ such that

$$\mathcal{A} \otimes \mathcal{A} \quad \xrightarrow{\Psi \otimes \Psi} \quad \tilde{\mathcal{A}} \otimes \tilde{\mathcal{A}} \colon R \longmapsto (\Psi \otimes \Psi)(R) = \tilde{R}.$$

Observe that the pair $(\mathbb{K}, 1 \otimes 1)$ is trivially a Yang-Baxter algebra over $\mathbb{K}$.

In particular, R^{op} satisfies (1.4) if and only if R satisfies (1.4). If $R \colon \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ is invertible, then $R_{\mathcal{M}} \colon \mathcal{M} \otimes \mathcal{M} \longrightarrow \mathcal{M} \otimes \mathcal{M}$ is also invertible, $(R^{-1})_{\mathcal{M}} = (R_{\mathcal{M}})^{-1}$ and satisfies (1.7). Recall that a Hopf algebra

$\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, S)$ is cocommotative if $\tau_{\mathcal{H},\mathcal{H}} \circ \Delta = \Delta$, where $\tau_{\mathcal{H},\mathcal{H}}\colon \mathcal{H}\otimes\mathcal{H} \longrightarrow \mathcal{H}\otimes\mathcal{H}$ is the twist map. We now consider a Hopf algebra which is cocommutative up to a conjugation by an element $R \in \mathcal{H}\otimes\mathcal{H}$.

Definition 1.5. A bialgebra $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon)$ or a Hopf algebra $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, S)$ is a quasi-triangular bialgebra or a quasi-triangular Hopf algebra if there exists an invertible element $R \in \mathcal{H}\otimes\mathcal{H}$, called the quasi-triangular structure, such that the pair $(\mathcal{H}, R)$ satisfies the following conditions:

(i) $(\Delta \otimes 1_{\mathcal{H}})R = R_{13}R_{23}$, $(1_{\mathcal{H}} \otimes \Delta)R = R_{13}R_{12}$ and (ii) $\tau_{\mathcal{H},\mathcal{H}} \circ \Delta(h) = R(\Delta(h)R^{-1}$, for all $h \in \mathcal{H}$ where

$$R = \sum_{(R)} R_{(1)} \otimes R_{(2)}, R_{st} := \sum 1_{\mathcal{H}} \otimes \cdots \otimes R_{(1)} \otimes 1_{\mathcal{H}} \otimes \cdots \otimes R_{(2)} \otimes \cdots \otimes 1_{\mathcal{H}},$$

the element of $\mathcal{H}\otimes\mathcal{H}\otimes\cdots\otimes\mathcal{H}$, which is R in the s-th and t-th positions and $\tau_{\mathcal{H},\mathcal{H}}$ the twist map.

We also observe that a quasi-triangular algebra $(\mathcal{H}, R)$ (respectively a quasi-triangular Hopf algebra $(\mathcal{H}, R)$) with $R \in \mathcal{H}\otimes\mathcal{H}$ given by $R = \sum_{i=1}^{r} h_i \otimes k_i$ and $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon)$ a bialgebra over $\mathbb{K}$ can be defined by requiring that the following conditions are satisfied:

(1) $\sum_{i=1}^{r} \Delta(h_i) \otimes k_i = \sum_{i,j=1}^{r} h_i \otimes h_j \otimes k_i k_j$,

(2) $\sum_{i=1}^{r} \epsilon(h_i) k_i = 1$,

(3) $\sum_{i=1}^{r} h_i \otimes \Delta^{cop}(k_i) = \sum_{i,j=1}^{r} h_i h_j \otimes k_i \otimes k_j$,

(4) $\sum_{i=1}^{r} h_i \epsilon(k_i) = 1$, and

(5) $(\Delta^{cop}(h))R = R(\Delta(h))$, for all $h \in \mathcal{H}$.

Thus (1) and (2) imply that the element $R \in \mathcal{H}\otimes\mathcal{H}$ is invertible. If $\mathcal{H}$ is a Hopf algebra $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, S)$ with antipode S, then in this case we obtain:

$$\sum_{i,j=1}^{r} S(h_i)h_j \otimes k_i k_j = \sum_{i=1}^{r} S(h_{i(1)})h_{i(2)} \otimes k_i = \sum_{i=1}^{r} \epsilon(h_i)1 \otimes k_i = 1 \otimes 1.$$

Thus $(S \otimes 1_{\mathcal{H}})(R)$ is a left inverse for R. Similarly $(1_{\mathcal{H}} \otimes S)(R)$ is a right inverse for R.

Proposition 1.6. (a) *Let the pair* $(\mathcal{H}, R)$ *be a quasi-triangular bialgebra. Then the quasi-triangular structure* $R \in \mathcal{H}\otimes\mathcal{H}$ *satisfies*

$$(\epsilon \otimes 1_{\mathcal{H}})(R) = 1 = (1_{\mathcal{H}} \otimes \epsilon)(R). \tag{1.12}$$

(b) *If* $(\mathcal{H}, R)$ *is a quasi-triangular Hopf algebra, then the quasi-triangular* *structure* $R \in \mathcal{H} \otimes \mathcal{H}$ *satisfies the conditions*:

$$(S \otimes 1_{\mathcal{H}})(R) = R^{-1}, \quad (1_{\mathcal{H}} \otimes S)(R^{-1}) = R\,, \tag{1.13}$$

and thus $(S \otimes S)(R) = R$.

Proof. (a) Apply ϵ to the left-hand side of Definition 1.5 (i) to get

$$(\epsilon \otimes 1_{\mathcal{H}} \otimes 1_{\mathcal{H}})(\Delta \otimes 1_{\mathcal{H}})(R) = R_{23} = (\epsilon \otimes 1_{\mathcal{H}} \otimes 1_{\mathcal{H}})R_{13}R_{23}.$$

This implies $(\epsilon \otimes 1_{\mathcal{H}})R = 1$ since R_{23} is invertible. Similarly for the right-hand side.

(b) Use part (a) and property of the antipode to get:

$$\sum_{(R)} (R_{(1)})S(R_{(1)})_{(2)} \otimes R_{(2)} = 1.$$

Definition 1.5 (i) implies that

$$\sum_{(R)} (R_{(1)})_{(1)}S(R_{(1)})_{(2)} \otimes R_{(2)} = 1 = (S \otimes 1_{\mathcal{H}})(R)R.$$

Similarly for the other side. Thus $(S \otimes 1_{\mathcal{H}})(R) = R^{-1}$. Since $\Delta\colon \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ is a $\mathbb{K}$-algebra homomorphism

$$(\Delta \otimes 1_{\mathcal{H}})S^{-1} = (R_{13}R_{23})^{-1} = R_{23}^{-1}R_{13}^{-1}$$

and so we deduce that $(1_{\mathcal{H}} \otimes S)(R^{-1}) = R$, which gives the result. □

Definition 1.7. Let $(\mathcal{H}, R)$ and $(\tilde{\mathcal{H}}, \tilde{R})$ be two quasi-triangular bialgebras over $\mathbb{K}$. A morphism $\Psi\colon (H, R) \longrightarrow (\tilde{H}, \tilde{R})$ of quasi-triangular bialgebras (respectively quasi-triangular Hopf algebras) over $\mathbb{K}$ is a bialgebra map (respectively a Hopf algebra map) $\Psi\colon H \longrightarrow \tilde{\mathcal{H}}$ such that $\tilde{R} := (\Psi \otimes \Psi)(R)$.

It is easy to show that the class of quasi-triangular bialgebras or quasi-triangular Hopf algebras over $\mathbb{K}$ together with their morphisms under composition form naturally a monoidal category. Observe that the bialgebra maps of Hopf algebras are Hopf algebra maps. The tensor product $(\mathcal{H} \otimes \tilde{H}, \tilde{\tilde{R}})$ of quasi-triangular bialgebras $(\mathcal{H}, R)$ and $(\tilde{\mathcal{H}}, R)$ over $\mathbb{K}$ with $\tilde{\tilde{R}}\colon = (1_{\mathcal{H}} \otimes \tau_{\mathcal{H},\tilde{\mathcal{H}}} \otimes 1_{\mathcal{H}})(R \otimes \tilde{R})$, where $\tau_{\mathcal{H},\tilde{\mathcal{H}}}\colon \mathcal{H} \otimes \tilde{\mathcal{H}} \longrightarrow \tilde{\mathcal{H}} \otimes \mathcal{H}$ is the twist map, is a quasi-triangular bialgebra over $\mathbb{K}$. The bialgebras $\mathcal{H}^{op}, \mathcal{H}^{cop}$ and $\mathcal{H}^{op\ cop}$ are also known to have quasi-triangular structure. Hence $(\mathcal{H}^{op}, R^{op}), (\mathcal{H}^{cop}, R^{op})$ and $(\mathcal{H}^{op\ cop}, R)$ are clearly quasi-triangular bialgebras. If $S\colon \mathcal{H} \longrightarrow \mathcal{H}$ is an antipode, then the fact that $S\colon \mathcal{H} \longrightarrow \mathcal{H}^{op\ cop}$ is a bialgebra map implies that $S\colon (\mathcal{H}, R) \longrightarrow (\mathcal{H}^{op\ cop}, R)$ is a morphism of quasi-triangular Hopf algebras. If $(\mathcal{H}, R)$ is finite-dimensional, then $S\colon (\mathcal{H}, R) \longrightarrow (\mathcal{H}^{op\ cop}, R)$ is a bijective map and hence an isomorphism of quasi-triangular Hopf algebras. In this case

$(\mathcal{H}^{op}, R^{op})$ and $(\mathcal{H}^{cop}, R^{op})$ are isomorphic quasi-triangular Hopf algebras. Let $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon)$ and $\tilde{\mathcal{H}} = (\tilde{\mathcal{H}}, \tilde{m}, \tilde{\eta}, \tilde{\Delta}, \tilde{\epsilon})$ be bialgebras. If $\pi\colon \mathcal{H} \longrightarrow \tilde{\mathcal{H}}$ is a surjective bialgebra map, then $(\tilde{\mathcal{H}}, (\pi \otimes \pi)(R))$ is a quasi-triangular bialgebra and $\pi\colon (\mathcal{H}, R) \longrightarrow (\tilde{\mathcal{H}}, (\pi \otimes \pi)(R))$ is a morphism. Therefore, the quotients of quasi-triangular bialgebras admit a quasi-triangular structure. Note however, that in general sub-bialgebra of quasi-triangular bialgebra does not necessarily admit a quasi-triangular structure. Now suppose $\tilde{\mathcal{H}}$ is a sub-bialgebra of $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon)$ such that $R \in \mathcal{H} \otimes \mathcal{H}$ implies automatically that $R \in \tilde{\mathcal{H}} \otimes \tilde{\mathcal{H}}$. Then the pair $(\tilde{\mathcal{H}}, R)$ is a quasi-triangular bialgebra which we call a sub-quasi-triangular bialgebra of $(\mathcal{H}, R)$. In the case $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, S)$ is a Hopf algebra and $\tilde{\mathcal{H}}$ a sub-Hopf algebra of $\mathcal{H}$, then the pair $(\tilde{\mathcal{H}}, R)$ is called a sub-quasi-triangular Hopf algebra of $(\mathcal{H}, R)$. Furthermore, if $(\mathcal{H}, R)$ is a finite-dimensional quasi-triangular Hopf algebra over $\mathbb{K}$ and $\mathcal{H}_{[R]}$ denotes the smallest sub-Hopf algebra $\tilde{\mathcal{H}}$ of $\mathcal{H}$ such that $R \in \tilde{\mathcal{H}} \otimes \tilde{\mathcal{H}}$, then the pair $(\mathcal{H}, R)$ is a minimal quasi-triangular Hopf algebra if $\mathcal{H} = \mathcal{H}_{[R]}$.

Definition 1.8. (a) A quasi-bialgebra over $\mathbb{K}$ is a bialgebra $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon)$ equipped with the coassociator $\Phi \in \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ satisfying the following conditions:

(i) $(1_{\mathcal{H}} \otimes \Delta)(\Delta(h)) = (\Phi^{-1}) \cdot (\Delta \otimes 1_{\mathcal{H}})(\Delta(h)) \cdot (\Phi)$, for all $h \in \mathcal{H}$,

(ii) $(\Delta \otimes 1_{\mathcal{H}} \otimes 1_{\mathcal{H}}) \cdot (\Phi) \cdot (1_{\mathcal{H}} \otimes 1_{\mathcal{H}} \otimes \Delta) \cdot (\Phi) = (\Phi \otimes 1)(1_{\mathcal{H}} \otimes \Delta \otimes 1_{\mathcal{H}}) \cdot (\Phi) \cdot (1 \otimes \Phi)$,

(iii) $(\epsilon \otimes 1_{\mathcal{H}}) \circ \Delta = 1_{\mathcal{H}} = (1_{\mathcal{H}} \otimes \epsilon) \circ \Delta$,

(iv) $(1_{\mathcal{H}} \otimes \epsilon \otimes 1_{\mathcal{H}}) \cdot (\Phi) = 1$.

Denote a quasi-bialgebra by $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, \Phi)$.

(b) A quasi-Hopf algebra is a quasi-bialgebra $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, \Phi)$ together with an anti-homomorphism $S\colon \mathcal{H} \longrightarrow \mathcal{H}$ called the antipode and elements $\alpha, \beta \in \mathcal{H}$ such that for all $h \in \mathcal{H}$ the following conditions hold:

(v) $\sum_{(h)} S(h_{(1)})\alpha h_{(2)} = \epsilon(h)\alpha, \quad \sum_{(h)} h_{(1)}\beta S(h_{(2)}) = \epsilon(h)\beta$,

(vi) $\sum_{(\varphi)} S(\varphi_{(1)})\alpha\varphi_{(2)}\beta S(\varphi_{(3)}) = 1, \quad \sum_{(\bar{\varphi})} \bar{\varphi}_{(1)}\beta S(\bar{\varphi}_{(2)})\alpha\bar{\varphi}_{(3)} = 1$, where $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}, \quad \Phi = \sum_{(\varphi)} \varphi_{(1)} \otimes \varphi_{(2)} \otimes \varphi_{(3)}, \quad \Phi^{-1} = \sum_{(\bar{\varphi})} \bar{\varphi}_{(1)} \otimes \bar{\varphi}_{(2)} \otimes \bar{\varphi}_{(3)}$.

Denote a quasi-Hopf algebra by $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, \Phi, \alpha, \beta)$.

Definition 1.9. Let $\tilde{\mathcal{H}} = (\tilde{\mathcal{H}}, \tilde{m}, \tilde{\eta}, \tilde{\Delta}, \tilde{\epsilon}, \tilde{\Phi})$ and $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, \Phi)$ be two quasi-bialgebras. An algebra homomorphism $\wp\colon \mathcal{H} \longrightarrow \tilde{\mathcal{H}}$ is a homomorphism of quasi-bialgebras if:

(i) $\tilde{\Delta} \circ \wp = (\wp \otimes \wp) \circ \Delta$, i.e. the diagram

$$\begin{array}{ccc} \mathcal{H} \otimes \mathcal{H} & \xrightarrow{\wp \otimes \wp} & \tilde{\mathcal{H}} \otimes \tilde{\mathcal{H}} \\ {\scriptstyle \Delta} \uparrow & & \uparrow {\scriptstyle \tilde{\Delta}} \\ \mathcal{H} & \xrightarrow{\wp} & \tilde{\mathcal{H}} \end{array}$$

commutes.

$ii)$ $\tilde{\epsilon} \circ \wp = \epsilon$, $\quad \tilde{\Phi} = \wp \otimes \wp$ and

$iii)$ $\quad \tilde{\Phi} = (\wp \otimes \wp \otimes \wp) \cdot \Phi$.

Similarly, for quasi-Hopf algebras $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, \Phi, \alpha, \beta)$. Note that a Hopf algebra is the same as a quasi-Hopf algebra if we set $\alpha = \beta = 1$ and $\Phi = 1 \otimes 1 \otimes 1$.

Definition 1.10. Let $\mathcal{A}$ be an associative $\mathbb{K}$-algebra with unit 1 and $R \in \mathcal{A} \otimes \mathcal{A}$ an invertible element. The $\mathbb{K}$-vector space dual $\mathcal{A}^*$ of $\mathcal{A}$ is naturally an associative algebra with unit 1. Define $\mathcal{A}_{[R]} := \{(a^* \otimes 1_{\mathcal{A}})(R) + (1_{\mathcal{A}} \otimes b^*)(R) : a^*, b^* \in \mathcal{A}^*\}$. We call $\mathcal{A}_{[R]}$ the subalgebra of the algebra $\mathcal{A}$ generated by $\mathcal{A}_{[R]} + \mathcal{A}_{[R^{-1}]}$.

(i) A Yang-Baxter algebra $(\mathcal{A}, R)$ over $\mathbb{K}$ is said to be a minimal Yang-Baxter algebra over $\mathbb{K}$ if $\mathcal{A} = \mathcal{A}_{[R]}$.

(ii) Let $(\mathcal{A}, R)$ be a Yang-Baxter algebra over $\mathbb{K}$. A Yang-Baxter subalgebra of $(\mathcal{A}, R)$ is a pair $(\mathcal{B}, R)$ such that $\mathcal{B}$ is a subalgebra of the algebra $\mathcal{A}$ and $R, R^{-1} \in \mathcal{B} \otimes \mathcal{B}$, thus $\mathcal{A}_{[R]} \subseteq \mathcal{B}$.

Remark 1.11. (i) Note that $(\mathcal{A}_{[R]}, R)$ is a unique minimal subalgebra of the Yang-Baxter algebra $(\mathcal{A}, R)$.

(ii) A Yang-Baxter algebra $(\mathcal{B}, \tilde{R})$ is a Yang-Baxter subalgebra of $(\mathcal{A}, R)$ if and only if $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ and the inclusion map $\imath_{\mathcal{B}} \colon (\mathcal{B}, \tilde{R}) \hookrightarrow (\mathcal{A}, R)$.

Let $\mathcal{J}$ be an ideal of the algebra $\mathcal{A}$. Then there is a unique Yang-Baxter algebra structure $(\mathcal{A} / \mathcal{J}, R)$ on the quotient $\mathcal{A} / \mathcal{J}$ such that the canonical projection map:$\pi_{\mathcal{A}} \colon \mathcal{A} \longrightarrow \mathcal{A} / \mathcal{J}$ induces a Yang-Baxter algebra morphism $\pi_{\mathcal{A}} \colon (\mathcal{A}, R) \longrightarrow (\mathcal{A} / \mathcal{J}, R)$.

Definition 1.12. (a) Let $\mathcal{A}$ be an associative $\mathbb{K}$-algebra with unit and

$$R = \sum_j a_j \otimes b_j \in \mathcal{A} \otimes \mathcal{A}.$$

Set $R_{12} = \sum_j a_j \otimes b_j \otimes 1_{\mathcal{A}}$, $\quad R_{13} = \sum_j a_j \otimes 1_{\mathcal{A}} \otimes b_j$ and $R_{23} = \sum_j 1_{\mathcal{A}} \otimes a_j \otimes b_j$. The pair $(\mathcal{A}, R)$ is called a Yang-Baxter algebra over $\mathbb{K}$ if and only if R is invertible and satisfies the Yang-Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{1.14}$$

or equivalently

$$\sum_{i,j,l} a_i a_j \otimes b_i b_l \otimes b_j b_l = \sum_{j,i,l} a_j a_i \otimes a_l b_i \otimes b_l b_j. \tag{1.15}$$

(b) Let $(\mathcal{A}, R)$ and $(\tilde{\mathcal{A}}, \tilde{R})$ be two Yang-Baxter algebras over $\mathbb{K}$. Then $(\mathcal{A} \otimes \tilde{\mathcal{A}}, \tilde{\tilde{R}})$ is a Yang-Baxter algebra over $\mathbb{K}$, called the tensor product of (A, R) and $(A, \tilde{R})$, where $\tilde{\tilde{R}}$ is defined by $\tilde{\tilde{R}} := (1_{\mathcal{A}} \otimes \tau_{\mathcal{A},\tilde{\mathcal{A}}} \otimes 1_{\tilde{\mathcal{A}}})(R \otimes \tilde{R})$.

(c) Quantum morphism $\Psi\colon (\mathcal{A}, R) \longrightarrow (\tilde{\mathcal{A}}, \tilde{R})$ of Yang-Baxter algebras over $\mathbb{K}$ is a $\mathbb{K}$-algebra map $\Psi : \mathcal{A} \longrightarrow \tilde{\mathcal{A}}$ which satisfies: $\tilde{R} = (\Psi \otimes \Psi)(R)$, that is :

$$R \in \mathcal{A} \otimes \mathcal{A} \xrightarrow{\Psi\otimes\Psi} \tilde{\mathcal{A}} \otimes \tilde{\mathcal{A}} \quad R \longmapsto (\Psi \otimes \Psi)(R) = \tilde{R}.$$

Furthermore, it can be shown that the category of Yang-Baxter algebras over $\mathbb{K}$ and their $\mathbb{K}$-algebra morphisms form a natural monoidal category.

Another interesting example of a Yang-Baxter algebra is constructed as follows:

Let V be a $\mathbb{K}$-vector space equipped with a Yang-Baxter operator

$$R\colon V \otimes V \longrightarrow V \otimes V : v \otimes w \longmapsto R(v \otimes w),$$

where $v, w \in V$. Let $T(V) = \bigoplus_{k\geq 0}^{\infty} T^k(V), \quad T^0(V) := \mathbb{K}$ with $T^k(V) := V^{\otimes k}$ the tensor algebra over $\mathbb{K}$. The ideal $< v \otimes w - R(v \otimes w) > \quad \subseteq T^2(V) := V \otimes V$ induces the quotient algebra $T(V) \diagup < v \otimes w - R(v \otimes w) >$ which is a quadratic algebra denoted by $Sym_R(V) := T(V) \diagup < v \otimes w - R(v \otimes w) >$, called an R-symmetric algebra. $Sym_R(V)$ is a quantum $\mathbb{K}$-algebra in the sense of Yu.I. Manin [24]. The associated matrix algebra is $Mat_R(V) := T(End(V)) \diagup < \psi \otimes \varphi - R(\psi \otimes \varphi >$. There is a natural inclusion map: $\imath_V : V \hookrightarrow Sym_R(V)$ and a unique R-structure on $Sym_R(V)$, which we denote again by

$$R : Sym_R(V) \otimes Sym_R(V) \longrightarrow Sym_R(V) \otimes Sym_R(V)$$

extending the Yang-Baxter operator $R\colon V \otimes V \longrightarrow V \otimes V$ on V. Thus the pair $(Sym_R(V), R)$ is a Yang-Baxter algebra relative to the R-structure

$$R\colon Sym_R(V) \otimes Sym_R(V) \longrightarrow Sym_R(V) \otimes Sym_R(V).$$

An R-morphism $\Psi\colon Sym_R(V) \longrightarrow Sym_{\tilde{R}}(\tilde{V})$, where

$$\tilde{R}\colon Sym_{\tilde{R}}(\tilde{V}) \otimes Sym_{\tilde{R}}(\tilde{V}) \longrightarrow Sym_{\tilde{R}}(\tilde{V}) \otimes Sym_{\tilde{R}}(\tilde{V})$$

is the Yang-Baxter operator on $Sym_{\tilde{R}}(\tilde{V})$ extending the Yang-Baxter operator

$\tilde{R}\colon \tilde{V}\otimes\tilde{V}\longrightarrow\tilde{V}\otimes\tilde{V}$ is such that the diagram

$$\begin{array}{ccc} Sym_R(V)\otimes Sym_R(V) & \xrightarrow{R} & Sym_R(V)\otimes Sym_R(V) \\ \big\downarrow{\scriptstyle \Psi\otimes\Psi} & & \big\downarrow{\scriptstyle \Psi\otimes\Psi} \\ Sym_{\tilde{R}}(\tilde{V})\otimes Sym_{\tilde{R}}(\tilde{V}) & \xrightarrow{\tilde{R}} & Sym_{\tilde{R}}(\tilde{V})\otimes Sym_{\tilde{R}}(\tilde{V}) \end{array}$$

commutes i.e. $\tilde{R}(\Psi\otimes\Psi)=(\Psi\otimes\Psi)R$.

Following Yu.I. Manin [23, 24, 25] a Yang-Baxter algebra $\mathcal{A}$ can be defined as an algebra object of a braided tensor category of $\mathbb{K}$-modules. Indeed if $\mathcal{C}$ is a braided tensor category, we say that $\mathcal{A}$ is an algebra in $\mathcal{C}$ if $\mathcal{A}\in Ob\mathcal{C}$ equipped with a multiplication map $m\in Hom_{\mathcal{C}}(\mathcal{A}\otimes\mathcal{A},\mathcal{A})$ satisfying:

(1) *Associativity Constraint*: $m(m\otimes 1_{\mathcal{A}})=m(1_{\mathcal{A}}\otimes m)$, that is the following diagram

$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} & \xrightarrow{m\otimes 1_{\mathcal{A}}} & \mathcal{A}\otimes\mathcal{A} \\ \big\downarrow{\scriptstyle 1_{\mathcal{A}}\otimes m} & & \big\downarrow{\scriptstyle m} \\ \mathcal{A}\otimes\mathcal{A} & \xrightarrow{m} & \mathcal{A} \end{array}$$

commutes and a unit map $\eta\in Hom_{\mathcal{C}}(\mathbb{K},\mathcal{A})$ satisfying

(2) *The Unit Conditions*: $m(\eta\otimes 1_{\mathcal{A}})={}_l\lambda,\quad m(1_{\mathcal{A}}\otimes\eta)=\lambda_r$, where ${}_l\lambda$ indicates left and λ_r right unit constraints with respect to the unit $\mathbb{I}$ of the braided tensor category $\mathcal{C}$. The following triangular diagrams

$$\begin{array}{ccc} \mathbb{K}\otimes\mathcal{A} & \xrightarrow{\eta\otimes 1_{\mathcal{A}}} & \mathcal{A}\otimes\mathcal{A} \\ {}_l\lambda\searrow & & \swarrow m \\ & \mathcal{A} & \end{array}$$

and

$$\begin{array}{ccc} \mathcal{A}\otimes\mathbb{K} & \xrightarrow{1_{\mathcal{A}}\otimes\eta} & \mathcal{A}\otimes\mathcal{A} \\ \lambda_r\searrow & & \swarrow m \\ & \mathcal{A} & \end{array}$$

commute. It is then easily proved that the associative $\mathbb{K}$-algebra $\mathcal{A}$ with unit, in the braided tensor category $\mathcal{C}$ is indeed a Yang-Baxter algebra with $R_{\mathcal{A},\mathcal{A}}=R\colon\mathcal{A}\otimes\mathcal{A}\longrightarrow\mathcal{A}\otimes\mathcal{A}$. The Yang-Baxter algebra $\mathcal{A}$ is then commutative if $m=mR_{\mathcal{A},\mathcal{A}}$. Note that even if $\mathcal{A}$ is not commutative in the above sense, $mR_{\mathcal{A},\mathcal{A}}:\mathcal{A}\otimes\mathcal{A}\longrightarrow\mathcal{A}$ is an associative product on $\mathcal{A}$. We can also associate to any Yang-Baxter algebra $\mathcal{A}$ in the braided tensor category $\mathcal{C}$ a Yang-Baxter algebra $\mathcal{A}^{op}$ the opposite algebra of $\mathcal{A}$ with multiplication map $m^{op}=mR_{\mathcal{A},\mathcal{A}}$, that is the diagram

$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{A} & \xrightarrow{R_{\mathcal{A},\mathcal{A}}} & \mathcal{A}\otimes\mathcal{A} \\ m^{op}\searrow & & \swarrow m \\ & \mathcal{A} & \end{array}$$

commutes, such that for every $a, b \in \mathcal{A}$, $m^{op}(a\otimes b) = mR(a\otimes b) = m(b\otimes a) \colon = ba$.

Definition 1.13. Let $(\mathcal{C}, \Delta, \epsilon)$ be a coalgebra. A Yang-Baxter coalgebra over $\mathbb{K}$ is a pair $(\mathcal{C}, \beta)$, with $\beta \colon \mathcal{C}\times\mathcal{C} \longrightarrow \mathbb{K}$ an invertible bilinear form such that

$$\beta(c_{(1)}, d_{(1)})\beta(c_{(2)}, e_{(1)})\beta(d_{(2)}, e_{(2)}) = \beta(c_{(2)}, d_{(2)})\beta(c_{(1)}, e_{(2)})\beta(d_{(1)}, e_{(1)}),$$

for all $c, d, \in \mathcal{C}$.

Let $(\mathcal{C}, \beta)$ and $(\tilde{\mathcal{C}}, \tilde{\beta})$ be Yang-Baxter coalgebras over $\mathbb{K}$. The tensor product $(\mathcal{C}\otimes\tilde{\mathcal{C}}, \tilde{\tilde{\beta}})$ is a Yang-Baxter coalgebra over $\mathbb{K}$, where

$$\tilde{\tilde{\beta}}(c\otimes\tilde{c}, d\otimes\tilde{d}) = \beta(c, d)\tilde{\beta}(\tilde{c}, \tilde{d}),$$

for $c, d \in \mathcal{C}$ and $\tilde{c}, \tilde{d} \in \tilde{\mathcal{C}}$.

A morphism $\Phi : (\mathcal{C}, \beta) \longrightarrow (\tilde{\mathcal{C}}, \tilde{\beta})$ of Yang-Baxter coalgebras over $\mathbb{K}$ is a coalgebra map $\Phi \colon \mathcal{C} \longrightarrow \tilde{\mathcal{C}}$ such that: $\beta(c, d) = \tilde{\beta}(\Phi(c), \Phi(d))$, for all $c, d \in \mathcal{C}$.

Note that $(\mathbb{K}, \beta)$ with the bilinear form $\beta \colon \mathbb{K}\times\mathbb{K} \longrightarrow \mathbb{K}$ such that $\beta(1, 1) = 1$ is trivially a Yang-Baxter coalgebra over $\mathbb{K}$. The category of all Yang-Baxter coalgebras over $\mathbb{K}$ with their $\mathbb{K}$-algebra morphisms naturally form a monoidal category. It is clear that if $(\mathcal{C}, \beta)$ is a Yang-Baxter coalgebra over $\mathbb{K}$, then $(\mathcal{C}^{cop}, \beta)$ and $(\mathcal{C}, \beta^{op})$ are also Yang-Baxter coalgebras over $\mathbb{K}$, where $\beta^{op}(c, d) := \beta(d, c)$, for all $c, d \in \mathcal{C}$. We also observe that Yang-Baxter algebras and Yang-Baxter coalgebras over $\mathbb{K}$ are dual notions. For if $(\mathcal{A}^0, R)$ is a Yang-Baxter algebra over $\mathbb{K}$, then $(\mathcal{A}^0, \beta_R)$ is a Yang-Baxter coalgebra over $\mathbb{K}$ with $\beta_R(a^0, b^0) = (a^0\otimes b^0)(R)$, for all $a^0, b^0 \in \mathcal{A}$, where $\beta_R : \mathcal{A}^0\times\mathcal{A}^0 \longrightarrow \mathbb{K}$ is a bilinear form of finite type and $\mathcal{A}^0 = \{\alpha \in \mathcal{A}^*, m^*(\alpha) \in \mathcal{A}^*\otimes\mathcal{A}^*\}$ is a subset of $\mathcal{A}^*$ consisting of all $f \in \mathcal{A}^*$ such that there exists an element $\Delta_f := \sum_{j=1}^r f_j\otimes g_j \in \mathcal{A}^*\otimes\mathcal{A}^*$ given by $f(ab) = \Delta_f(a\otimes b) = \sum_{j=1}^r f_j(a)g_j(b)$, for all $a, b \in \mathcal{A}$. $\mathcal{A}^0$ is a coalgebra over $\mathbb{K}$ called the dual coalgebra of $\mathcal{A}$. The coproduct of $\mathcal{A}^0$ is given by $\Delta(f) := \Delta_f : \mathcal{A}^0 \longrightarrow \mathcal{A}^0\otimes\mathcal{A}^0$, and the counit of $\mathcal{A}^0$ is $\epsilon : \mathcal{A}^0 \longrightarrow \mathbb{K}\colon f \longmapsto \epsilon(f) = f(1)$. If $(\mathcal{C}, \Delta, \epsilon)$ is a coalgebra over $\mathbb{K}$ and $\beta \colon \mathcal{C}\times\mathcal{C} \longrightarrow \mathbb{K}$ a bilinear form of finite type, then $(\mathcal{C}, \beta)$ is a Yang-Baxter coalgebra if and only if $(\mathcal{C}^*, R_\beta)$ is a Yang-Baxter algebra over $\mathbb{K}$. Now consider a coideal $\mathcal{J}$ of $\mathcal{C}$ so that the bilinear form $\beta \colon \mathcal{C}\times\mathcal{C} \longrightarrow \mathbb{K}$

satisfies $\beta(\mathcal{J},\mathcal{C}) = (0) = \beta(\mathcal{C},\mathcal{J})$. Then we obtain a unique Yang-Baxter coalgebra structure $(\mathcal{C}\diagup\mathcal{J},\tilde{\beta})$ on the quotient $\mathcal{C}\diagup\mathcal{J}$ so that the natural projection map $\pi_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \mathcal{C}\diagup\mathcal{J}$ induces a morphism $\pi_{\mathcal{C}}\colon (\mathcal{C},\beta) \longrightarrow (\mathcal{C}\diagup\mathcal{J},\tilde{\beta})$ of Yang-Baxter coalgebras. Further, suppose $(\mathcal{C},\Delta,\epsilon)$ is a coalgebra over $\mathbb{K}$ with a bilinear form $\beta\colon \mathcal{C}\times\mathcal{C} \longrightarrow \mathbb{K}$. $\mathcal{M}$ is a right $\mathcal{C}$-module if the endomorphism $\rho_{\mathcal{M}}\colon \mathcal{M}\otimes\mathcal{M} \longrightarrow \mathcal{M}\otimes\mathcal{M}$ is defined by $\rho_{\mathcal{M}}(x\otimes y) = \sum(x_{(1)}\otimes y_{(2)}\ \beta(x_{(2)},y_{(1)})$, for all $x,y\in\mathcal{M}$. Thus $(\mathcal{C},\beta)$ is a Yang-Baxter coalgebra over $\mathbb{K}$ if and only if $\rho_{\mathcal{M}}$ is an invertible solution of the Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ for all right $\mathcal{C}$- modules $\mathcal{M}$.

This paper is structured as follows. In Section 2 sheaves with $\mathbb{K}$-algebra structures are defined and quantum Yang-Baxter coherent $\mathbb{K}$-algebra sheaves are constructed. In Section 3 we formulate and prove the main structural theorems for the quantum Yang-Baxter coherent $\mathbb{K}$-algebra sheaves.

2. Presheaves and Sheaves with $\mathbb{K}$-Algebraic Structures

Throughout this paper we will take (M,g) to be a fixed m-dimensional complete, non-compact paracompact connected Riemannian manifold with the smooth structure given by the smooth atlas $\mathbb{A} := \{(U_\alpha,\varphi_\alpha)\}_{\alpha\in\mathfrak{A}}$ such that the family $\{U_\alpha\}_{\alpha\in\mathfrak{A}}$ is a locally finite open cover of M, that is, $M = \cup_{\alpha\in\mathfrak{A}}U_\alpha$. We consider the indexing set $\mathfrak{A}$ of the open cover $\{\cup_\alpha\}_{\alpha\in\mathfrak{A}}$ of M as a directed set with an order relation $\prec$ which satisfies the conditions:

(i) $\alpha\prec\alpha$, for all $\alpha\in\mathfrak{A}$,

(ii) if $\alpha\prec\beta$ and $\beta\prec\gamma$, then $\alpha\prec\gamma$,

(iii) for any $\alpha,\beta\in\mathfrak{A}$, there exists $\gamma\in\mathfrak{A}$ such that $\alpha\prec\gamma$ and $\beta\prec\gamma$.

It may happen that $\alpha\neq\beta$ but $\alpha\prec\beta$ and $\beta\prec\alpha$ simultaneously. The family $\{\cup_\alpha\}_{\alpha\in\mathfrak{A}}$ of open subsets of M indexed by the directed set $\mathfrak{A}$ will be called a directed system if, for any pair (α,β) with $\alpha\prec\beta$, there exists an inclusion morphism $\imath_{\alpha\beta} : U_\alpha \hookrightarrow U_\beta$ such that $\imath_{\alpha\alpha}\colon U_\alpha\hookrightarrow U_\alpha$, i.e. $\imath_{\alpha\alpha} = I_{U_\alpha}$ and $\imath_{\alpha\beta}\circ\imath_{\beta\gamma} = \imath_{\alpha\gamma}$, if $\alpha\prec\beta\prec\gamma$ and we have the inclusion composite morphism:

$$\imath_{\alpha\gamma} : U_\alpha \ \hookrightarrow\ U_\beta \hookrightarrow U_\gamma.$$

We let $Op(M)$ be the category with objects the members of the open cover $\{U_\alpha\}_{\alpha\in\mathfrak{A}}$ of M and morphisms inclusion maps $\imath_{\alpha\beta} : U_\alpha\hookrightarrow U_\beta$ if and only if $U_\alpha\subset U_\beta$ and $\alpha\prec\beta$ for all $\alpha,\beta\in\mathfrak{A}$. Let $\mathit{AssocAlg}_{\mathbb{K}}$ denote the monoidal category of associative unital $\mathbb{K}$-algebras. We now give the definition of presheaves and sheaves of associative unital $\mathbb{K}$-algebras and their $\mathbb{K}$-algebra morphisms over

(M, g).

Definition 2.1. A sheaf of rings on (M, g) is a sheaf of Abelian groups, denoted $(\mathcal{R}, \pi, M)$ such that:

(1) $\pi\colon \mathcal{R} \longrightarrow M$ is a surjective morphism and for each point $p \in M$ the stalk $\mathcal{R}_p := \pi^{-1}(p)$ of $\mathcal{R}$ is a ring.

(2) The multiplication:

$$\mu\colon \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}\colon (r, s) \longmapsto \mu(r, s) = r \cdot s \in \mathcal{R}_p \subseteq \mathcal{R}$$

is smooth such that $\pi(r) = \pi(s) = p \in M$. The sheaf of rings $(\mathcal{R}, \pi, M)$ is called a sheaf of commutative rings on M if for every $p \in M$, the stalk $\mathcal{R}_p$ is a commutative ring. The sheaf of rings $(\mathcal{R}, \pi, M)$ is said to have an identity element if the stalks $\mathcal{R}_p$, for each $p \in M$ as rings have identity elements 1_p, such that the corresponding sections to $\pi\colon \mathcal{R} \longrightarrow M$

$$1\colon M \longrightarrow \mathcal{R}\colon p \longmapsto 1_{(p)}\colon = 1_p \in \mathcal{R}_p$$

are smooth.

A morphism $\Phi\colon (\mathcal{R}, \pi, M) \longrightarrow (\tilde{\mathcal{R}}, \tilde{\pi}, M)$ of sheaves of rings, is a morphism of sheaves $\Phi\colon \mathcal{R} \longrightarrow \tilde{\mathcal{R}}$ such that the morphisms of stalks: $\Phi_p\colon \mathcal{R}_p \longrightarrow \tilde{\mathcal{R}}$ for each $p \in M$ are ring morphisms.

Definition 2.2. A sheaf of associative unital $\mathbb{K}$-algebras over (M, g) is a triple $\mathcal{A} := (\mathcal{A}, \rho, M)$ such that:

(1) $\mathcal{A}$ is a sheaf of rings on M, with multiplication: $m_{\mathcal{A}}\colon \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ and $\mathcal{A}$, is formally smooth [17] if the $\mathcal{A}$-bimodule $\Omega^1(\mathcal{A}) := ker\{m_{\mathcal{A}}\colon \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}\}$ is projective.

(2) $\rho\colon \mathcal{A} \longrightarrow M$ is a surjective morphism and for each $p \in M$, the stalk $\mathcal{A}_p := \rho^{-1}(p)$ of $\mathcal{A}$ is an associative unital $\mathbb{K}$-algebra with the unit element $1_p \in \mathcal{A}_p$ so that the corresponding section: $M \longrightarrow \mathcal{A}\colon p \longmapsto 1_{(p)} = 1_p \in \mathcal{A}$ is smooth and

(3) The scalar multiplication in $\mathcal{A}$: $\bullet\colon \mathbb{K} \times \mathcal{A} \longrightarrow \mathcal{A}\colon (\lambda, a) \longmapsto \lambda \cdot a \in \mathcal{A}_p \subset \mathcal{A}$ is smooth, where $\mathbb{K}$ is endowed with the discrete topology. Denote the monoidal category of associative unital $\mathbb{K}$-algebra sheaves by $\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}$-$\mathcal{S}h_M$.

Definition 2.3. A (graded) algebra $A \in Ob(\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}})$ is called (graded) right (left) coherent, if the following equivalent conditions hold:

(i) Every (homogeneous) finitely generated right-sided (left-sided) ideal in A is finitely presented, that is, A is (graded) coherent as a right (left) module over itself;

(ii) Every finitely presented (graded) right (left) A-module is (graded) coherent. Note that all the finitely presented (graded) right (left) A-modules form an abelian category. We denote, this monoidal category of associative unital $\mathbb{K}$-coherent algebras by $\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}$-algebras and the associated monoidal category of associative unital $\mathbb{K}$- coherent algebra sheaves by $\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}$-$\mathcal{S}h_M$.

In our noncommutative setting the coherent algebra sheaves we use should be thought of in the sense of category of coherent algebra sheaves given by the notion of a bundle of localizations formulated in the following definition.

Definition 2.4. (see [32]) A bundle of localizations $((M,g),C,\rho)$, consists of the fixed m-dimensional non-compact complete paracompact connected Riemannian manifold (M,g) endowed with the category $Op(M)$ with objects the locally finite open cover $\{U_\alpha\}_{\alpha\in\mathfrak{A}}$ of M and for each $U_\alpha\subset M$ a category $C(U_\alpha)$ of monoidal associative unital $\mathbb{K}$-coherent algebras such that for each inclusion morphism $\imath_{\alpha\beta}\colon U_\alpha\hookrightarrow U_\beta$ corresponds an exact right adjoint localization morphism $C(\imath_{\alpha\beta})\colon C(U_\beta)\longrightarrow C(U_\alpha)$. For any pair of inclusions $\imath_{\alpha\beta}\colon U_\alpha\hookrightarrow U_\beta$ and $\imath_{\beta\lambda}\colon U_\beta\hookrightarrow U_\lambda$, the function ρ assigns a functor isomorphism

$$\rho_{\imath_{\beta\lambda},\imath_{\alpha\beta}}\colon C(\imath_{\alpha\beta})\circ C(\imath_{\beta\lambda})\longrightarrow C(\imath_{\beta\lambda}\imath_{\alpha\beta}),$$

such that for any three inclusion morphisms $\imath_{\alpha\beta}\colon U_\alpha\hookrightarrow U_\beta$, $\imath_{\beta\lambda}\colon U_\beta\hookrightarrow U_\lambda$ and $\imath_{\lambda\gamma}\colon U_\lambda\hookrightarrow U_\gamma$ the composite map exists and the following diagram

$$\begin{array}{ccc} C(\imath_{\alpha\beta})\circ C(\imath_{\beta\lambda})\circ C(\imath_{\lambda\gamma}) & \xrightarrow{\rho_{\imath_{\beta\lambda},\imath_{\alpha\beta}}C(\imath_{\lambda\gamma})} & C(\imath_{\beta\lambda}\imath_{\alpha\beta})\circ C(\imath_{\beta\gamma}\imath_{\alpha\beta}) \\ {\scriptstyle C(\imath_{\alpha\beta})\rho_{\imath_{\lambda\gamma},\imath_{\beta\lambda}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \rho_{\imath_{\lambda\gamma},\imath_{\beta\lambda}\imath_{\alpha\beta}}} \\ C(\imath_{\alpha\beta})\circ C(\imath_{\lambda\gamma}\imath_{\beta\lambda}) & \xrightarrow{\rho_{\imath_{\lambda\gamma}\imath_{\beta\lambda},\imath_{\alpha\beta}}} & C(\imath_{\lambda\gamma}\imath_{\beta\lambda}\imath_{\alpha\beta}) \end{array}$$

commutes and $C(id)=Id$, $\rho_{id,\imath_{\alpha\beta}}=\rho_{\imath_{\alpha\beta},id}=id$.

Definition 2.5. A presheaf $\mathcal{F}$ in a bundle of localizations $\mathcal{M}=((M,g),C,\rho)$ is a function which assigns to an open set $U_\alpha\in\{U_\alpha\}_{\alpha\in\mathfrak{A}}$ an object $\mathcal{F}(U_\alpha)$ of the monoidal category $\mathcal{C}(U_\alpha)$ of associative unital $\mathbb{K}$-coherent algebras and to any inclusion morphism $\imath_{\alpha\beta}\colon U_\alpha\hookrightarrow U_\beta$ a morphism $\mathcal{F}(\imath_{\alpha\beta})\colon\mathcal{F}(U_\alpha)\longrightarrow\mathcal{F}(U_\beta)$ satisfying the compatibilty conditions: For any pair of inclusion morphisms $\imath_{\alpha\beta}\colon U_\alpha\hookrightarrow U_\beta,\imath_{\beta\lambda}\colon U_\beta\hookrightarrow U_\lambda$, the following diagram is commutative:

$$\begin{array}{ccc} C(\imath_{\alpha\beta})\circ C(\imath_{\beta\lambda}) & \xrightarrow{C(\imath_{\alpha\beta})\mathcal{F}(\imath_{\beta\lambda}))} & C(\imath_\alpha\mathcal{F}(U_\lambda) \\ {\scriptstyle \rho_{\imath_{\beta\lambda},\imath_{\alpha\beta}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{F}(\imath_{\alpha\beta})} \\ C(\imath_{\beta\lambda}\imath_{\alpha\beta}) & \xrightarrow{\mathcal{F}(\imath_{\beta\lambda}\imath_{\alpha\beta})} & \mathcal{F}(U_\alpha) \end{array}$$

Morphisms from presheaf $\mathcal{F}$ to presheaf $\mathcal{G}$ are functions η that assign to any open subset $U_\alpha \in \{U_\alpha\}_{\alpha\in\mathfrak{A}}$ amorphism $\eta(U_\alpha)\colon \mathcal{F}(U_\alpha \longrightarrow \mathcal{G}(U_\alpha)$ such that for any inclusion morphism $\imath_{\alpha\beta}\colon U_\alpha \hookrightarrow U_\beta$ we obtain

$$\mathcal{G}(\imath_{\alpha\beta}) \circ C(\imath_{\alpha\beta})(\eta(U_\beta)) = \eta(U_\alpha) \circ \mathcal{F}(\imath_{\alpha\beta}).$$

The composition of any two such morphisms η_1 and η_2 is defined by $\eta_2\circ\eta_1(U_\alpha) = \eta_2(U_\alpha) \circ \eta_1(U_\alpha)$. We denote the category of presheaves on $\mathcal{M}$ by $\mathcal{P}rsh_{\mathcal{M}}$ and call a presheaf $\mathcal{F}$ quasi-coherent, if the morphism $\mathcal{F}(\imath_{\alpha\beta})\colon \mathcal{F}(U_\alpha) \longrightarrow \mathcal{F}(U_\beta)$ is an isomorphism. Let $\mathcal{QC}oh\mathcal{M}$ be the subcategory of $\mathcal{P}rsh_{\mathcal{M}}$ generated by quasi-coherent presheaves. We easily obtain from this the Serre category of coherent algebra sheaves.

Definition 2.6. Let $\mathcal{A} = (\mathcal{A}, \rho, M)$ be an associative unital sheaf of $\mathbb{K}$-algebras on (M, g). A sheaf of $\mathcal{A}$-modules on (M, g) is a sheaf $\mathcal{M} = (\mathcal{M}, \pi, M)$ such that the following conditions are satisfied:

(1) $\mathcal{M}$ is a sheaf of abelian groups,

(2) The morphism $\pi\colon \mathcal{M} \longrightarrow M$ is surjective so that for each $p \in M$, the stalk $\mathcal{M}_p = \pi^{-1}(p)$ of $\mathcal{M}$ is a left $\mathcal{A}_p$-module and

(3) The module multiplication is given by:

$$\mu\colon \mathcal{A} \times_M \mathcal{M} \longrightarrow \mathcal{M}\colon (a, z) \longmapsto a \cdot z \in \mathcal{M}_p \subseteq \mathcal{M}$$

for each $p \in M$. The fiber product is defined by:

$$\mathcal{A} \times_M \mathcal{M} := \{(a, z) \in \mathcal{A} \times \mathcal{M}\colon \rho(a) = \pi(z)\}$$

such that the diagram:

$$\begin{array}{ccc} \mathcal{A} \times_M \mathcal{M} & \xrightarrow{proj_{\mathcal{M}}} & \mathcal{M} \\ {\scriptstyle proj_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{A} & \xrightarrow[\rho]{} & M \end{array}$$

is commutative.

Definition 2.7. Let

$$\mathcal{A} = (\mathcal{A}(U_\alpha), \rho_{U_\beta U_\alpha})_{\alpha,\beta\in\mathfrak{A}},$$

with $\imath_{\alpha\beta}\colon U_\alpha \hookrightarrow U_\beta$ if $U_\alpha \subseteq U_\beta$ in $Op(M)$ be a presheaf of associative unital $\mathbb{K}$-algebras and $\mathcal{M} = (\mathcal{M}(U_\alpha), \pi_{U_\beta U_\alpha})_{\alpha,\beta\in\mathfrak{A}}$, a presheaf of abelian groups on (M, g), such that:

(1) $\mathcal{M}(U_\alpha)$ is a left $\mathcal{A}(U_\alpha)$-module for every $U_\alpha \in Ob(Op(M))$ and

(2) if for every $U_\alpha, U_\beta \in Ob(Op(M))$ with $\imath_{\alpha\beta}\colon U_\alpha \hookrightarrow U_\beta$ if $U_\alpha \subset U_\beta$ in

$Op(M)$, we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{A}(U_\alpha) \times_M \mathcal{M}(U_\alpha) & \xleftarrow{\rho_{U_\beta U_\alpha} \times \ \pi_{U_\beta U_\alpha}} & \mathcal{A}(U_\beta) \times_M \mathcal{M}(U_\beta) \\ \bullet \Big\downarrow & & \Big\downarrow \bullet \\ \mathcal{M}(U_\alpha) & \xleftarrow{\pi_{U_\beta U_\alpha}} & \mathcal{M}(U_\beta) \end{array}$$

such that:

$$\pi_{U_\beta U_\alpha}(a \cdot s) = \rho_{U_\beta U_\alpha}(a) \cdot \pi_{U_\beta U_\alpha}(s),$$

for all $a \in \mathcal{A}(U_\beta)$ and $s \in \mathcal{M}(U_\beta)$, then $\mathcal{M}$ is called a presheaf of $\mathcal{A}$-modules on M.

Given any $\mathcal{A}(U_\alpha)$-module $\mathcal{M}$, the stalk of $\mathcal{M}$ at a point $p \in M$, defined by; $\mathcal{M}_p := \lim_{U_\alpha \ni p} \mathcal{M}(U_\alpha)$ is a left $\mathcal{A}_p$-module, with $\mathcal{A}_p := \lim_{U_\alpha \ni p} \mathcal{A}(U_\alpha)$ the stalk of $\mathcal{A}$ at $p \in M$.

We next formally, following A. Mallios [22] deduce that:

If $\mathcal{A} = (\mathcal{A}(U_\alpha), \rho_{U_\beta U_\alpha})_{\alpha,\beta \in \mathfrak{A}}$ with $\imath_{\alpha\beta}\colon U_\alpha \hookrightarrow U_\beta$, if $U_\alpha \subseteq U_\beta$ in $Op(M)$ is a presheaf of associative unital $\mathbb{K}$-algebras on M and $\mathcal{M} = (\mathcal{M}(U_\alpha), \pi_{U_\beta U_\alpha})_{\alpha,\beta \in \mathfrak{A}}$, a presheaf of $\mathcal{A}(U_\alpha)$-modules on M, then the corresponding sheafification functors:

$$\mathcal{S}hf\colon \mathcal{PS}h_M \longrightarrow \mathcal{S}h_M\colon \mathcal{A} \longmapsto \mathcal{S}hf(\mathcal{A}) := \mathcal{A}^{\blacklozenge} \equiv \mathcal{A},$$

$$\mathcal{S}hf\colon \mathcal{PS}h_M \longrightarrow \mathcal{S}h_M\colon \mathcal{M} \longmapsto \mathcal{S}hf(\mathcal{M}) := \mathcal{M}^{\blacklozenge} \equiv \mathcal{M}$$

of $\mathcal{A}$ and $\mathcal{M}$ on M respectively give rise to sheaves of associative unital $\mathbb{K}$-algebras $\mathcal{A}$ and Abelian group $\mathcal{M}$ so that $\mathcal{M}$ is an $\mathcal{A}$-module on M.

Definition 2.8. Let $\mathcal{A} = (\mathcal{A}, \rho, M)$ be a sheaf of associative unital $\mathbb{K}$-algebras on (M, g), $\mathcal{M} = (\mathcal{M}, \pi_{\mathcal{M}}, M)$ and $\mathcal{N} = (\mathcal{N}, \pi_{\mathcal{N}}, M)$ two left $\mathcal{A}$-modules on M. A morphism $\Phi\colon \mathcal{M} \longrightarrow \mathcal{N}$ is a morphism of $\mathcal{A}$-modules or an $\mathcal{A}$-morphism, if Φ is a morphism of sheaves such that for every point $p \in M$ the corresponding morphism of stalks: $\Phi_p := \Phi\restriction_{\mathcal{M}_p}\colon \mathcal{M}_p \longrightarrow \mathcal{N}_p$ is a morphism of left $\mathcal{A}_p$-modules.

Denote the category of left $\mathcal{A}$-modules on M by $\mathcal{A}$-Mod_M. If $\mathcal{A}$-morphism $\Phi\colon \mathcal{M} \longrightarrow \mathcal{N}$ is given, we define its kernel by $Ker\Phi := \{z \in \mathcal{M}\colon \Phi(z) = 0\}$ such that $\Phi(z) = \Phi_p(z) = 0 \equiv O_p \in \mathcal{N}_p$, and $z \in Ker\Phi$. If we denote the zeros of $\mathcal{N}$ by $O_M^{\mathcal{N}}$, then $Ker\Phi = \Phi^{-1}(O_M^{\mathcal{N}}) \subseteq \mathcal{M}$ is a subsheaf of $\mathcal{M}$. Hence $Ker\Phi \equiv (Ker\Phi, \pi_{\mathcal{M}\restriction_{Ker\Phi}}, M)$ is a sub-$\mathcal{A}$-module of $\mathcal{M}$ on M. The stalk of $Ker\Phi$ is given by $(Ker\Phi)_p = Ker(\Phi_p) \subseteq \mathcal{M}_p$ for every $p \in M$, i.e. an $\mathcal{A}_p$-submodule of $\mathcal{M}_p$. The image $Im\Phi := \Phi(\mathcal{M}) \subset \mathcal{N}$ is a subsheaf of $\mathcal{N}$ on M

and $Im\Phi = (Im\Phi, \pi_{\mathcal{N}}\restriction_{Im\Phi}, M)$ is a sub-$\mathcal{A}$-module of $\mathcal{N}$ with stalks given by $(Im\Phi)_p := Im(\Phi_p) \subseteq \mathcal{N}_p$, for every $p \in M$.

Definition 2.9. Let $\mathcal{A} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$ be an associative unital $\mathbb{K}$-algebra sheaf, and $\mathcal{M}$ an $\mathcal{A}$-module on M. Then we call $\mathcal{M}$ a free module of rank $l \in \mathbb{N}$, whenever there exists a $\mathbb{K}$-algebra sheaf isomorphism

$$\Psi\colon \mathcal{M} \xrightarrow{\simeq} \mathcal{A}^l,$$

equivalently for every $p \in M$, there is at the level of stalks a $\mathbb{K}$-algebra sheaf isomorphism

$$\Psi_p\colon \mathcal{M}_p \xrightarrow{\simeq} (\mathcal{A}^l)_p \cong (\mathcal{A}_p)^l.$$

If $\mathcal{M} \equiv (\mathcal{M}, \pi, M)$ is a given $\mathcal{A}$-module on M then $\mathcal{M}$ is said to be a locally free $\mathcal{A}$-module of finite rank on M, if for every $p \in M$, there exists $l \in \mathbb{N}$ and an open neighborhood $U_\alpha \in Ob(Op(M))$ of $p \in M$ such that we have the $\mathcal{A}\restriction_{U_\alpha}$-isomorphism: $\mathcal{M}\restriction_{U_\alpha} \cong \mathcal{A}^l$ of $\mathcal{A}(U_\alpha)$-modules. If $\mathcal{M}$ is a locally free $\mathcal{A}$-module of rank l on M write $l = rk_{\mathcal{A}}(\mathcal{M})$.

Definition 2.10. Suppose $\mathcal{A} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$ and $\mathcal{M},\mathcal{N}$ $\mathcal{A}$-modules on (M, g). For any $U_\alpha \in Ob(Op(M))$ define $Hom_{\mathcal{A}\restriction_{U_\alpha}}(\mathcal{M}\restriction_{U_\alpha}, \mathcal{N}\restriction_{U_\alpha})$ as the set of $\mathcal{A}\restriction_{U_\alpha}$-morphisms of $\mathcal{A}\restriction_{U_\alpha}$-modules. That is, for any $\sigma \in \mathcal{A}(U_\alpha)$ and $\Phi = (\Phi_{U_\beta}) \in Hom_{\mathcal{A}\restriction_{U_\alpha}}(\mathcal{M}\restriction_{U_\alpha}, \mathcal{N}\restriction_{U_\alpha})$ with the inclusion morphism $\imath_{\beta\alpha}\colon U_\beta \hookrightarrow U_\alpha$ if $U_\beta \subset U_\alpha$ in $Op(M)$ we have $(\sigma\cdot\Phi)_{U_\beta}(s)(p) := \sigma(p)\cdot\Phi_{U_\beta}(s)(p)$, where $s \in (\mathcal{M}\restriction_{U_\alpha})(U_\beta) = \mathcal{M}(U_\beta)$, and $p \in U_\beta$. We can also express this as

$$(\sigma\cdot\Phi)_{U_\beta} := (\sigma\restriction_{U_\beta})\cdot\Phi_{U_\beta} \equiv \sigma\cdot\Phi_{U_\beta},$$

for all $U_\beta \subseteq U_\alpha$ *in* $Op(M)$. Thus

$$Hom_{\mathcal{A}\restriction_{U_\alpha}}(\mathcal{M}\restriction_{U_\alpha}, \mathcal{N}\restriction_{U_\alpha}) \subseteq \Pi_{U_\alpha\supseteq U_\beta} Hom_{\mathcal{A}\restriction_{U_\beta}}(\mathcal{M}\restriction_{U_\beta}, \mathcal{N}\restriction_{U_\beta}).$$

It is immediate that for any $\mathcal{A}$-modules $\mathcal{M},\mathcal{N}$ the functor $\mathcal{H}om_{\mathcal{A}}(\mathcal{M},\mathcal{N})(U_\alpha) := Hom_{\mathcal{A}\restriction_{U_\alpha}}(\mathcal{M}\restriction_{U_\alpha}, \mathcal{N}\restriction_{U_\alpha})$ for all $U_\alpha \in Ob(Op(M))$ defines a presheaf of $\mathcal{A}$-modules with $\mathcal{A}(U_\alpha)$-isomorphism. Sheafification then gives the sheaf $Hom_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ of $\mathcal{A}$-modules on (M, g) such that $rk(Hom_{\mathcal{A}}(\mathcal{M}, \mathcal{N})) = (rk\mathcal{M})(rk\mathcal{N})$. Given any $\mathcal{A}$-module $\mathcal{M}$ $End_{\mathcal{A}}\mathcal{M} \equiv End\mathcal{M} := Hom_{\mathcal{A}}(\mathcal{M}, \mathcal{M})$ with composition as multiplication is an $\mathcal{A}$-algebra sheaf on M. If $\mathcal{L}$ is a sheaf of rank one then there is a sheaf algebra isomorphism $End\mathcal{L} \cong \mathcal{A}$.

Definition 2.11. A sequence

$$- - - \longrightarrow \mathcal{M}_{j-1} \xrightarrow{\Phi_{j-1}} \mathcal{M}_j \xrightarrow{\Phi_j} \mathcal{M}_{j+1} \longrightarrow - - - \tag{2.1}$$

of sheaves of $\mathcal{A}$-modules on (M, g) and $\mathcal{A}$-morphisms, called $\mathcal{A}$-sequence over M, is said to be exact at $\mathcal{M}_j$ if $Ker\Phi_j = Im\Phi_{j-1}$, for each $j \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We have an exact sequence of sheaves of $\mathcal{A}$-modules on M if (2.1) is exact at any $\mathcal{M}_j$. An exact sequence of sheaves of $\mathcal{A}$-modules of the form:

$$0 \longrightarrow \mathcal{M} \xrightarrow{\ \Psi\ } \mathcal{N} \xrightarrow{\ \Phi\ } \mathcal{P} \longrightarrow 0, \tag{2.2}$$

where 0 stands for the zero sheaf of $\mathcal{A}$-module, is called a short exact sequence of $\mathcal{A}$-module sheaves on M.

Note that given any morphism $\Phi\colon \mathcal{M} \longrightarrow \mathcal{N}$ of sheaves of $\mathcal{A}$-modules on M, there is associated a short exact sequence of sheaves of $\mathcal{A}$-modules

$$0 \longrightarrow Ker\Phi \xrightarrow{\ \imath\ } \mathcal{M} \xrightarrow{\ \Phi\ } Im\Phi \longrightarrow 0 \tag{2.3}$$

where $\imath$ is the restriction to the $Ker\Phi \subseteq \mathcal{M}$ of the identity $\mathcal{A}$-endomorphism of $\mathcal{M}$ which we still denote by $\imath \equiv 1_{\mathcal{M}}\colon \mathcal{M} \longrightarrow \mathcal{M}\colon z \longmapsto \imath(z) := z$.

Definition 2.12. Let $\mathcal{M} = (\mathcal{M}(U_\beta), \pi^{\mathcal{M}}_{U_\beta U_\alpha})_{\alpha,\beta\in\mathfrak{A}}$ and

$$\mathcal{N} = (\mathcal{N}(U_\beta), \pi^{\mathcal{N}}_{U_\beta U_\alpha})_{\alpha,\beta\in\mathfrak{A}}$$

with $\imath_{\alpha\beta}\colon U_\alpha \hookrightarrow U_\beta$, if $U_\alpha \subset U_\beta$ in $Op(M)$, be two $\mathcal{A}$-presheaves, i.e. presheaves of $\mathcal{A}$-modules on M with $\mathcal{A} = (\mathcal{A}(U_\beta), \rho_{U_\beta U_\alpha})_{\alpha,\beta\in\mathfrak{A}}$, $\imath_{\alpha\beta}\colon U_\alpha \hookrightarrow U_\beta$, if $U_\alpha \subset U_\beta$ in $Op(M)$ associative unital $\mathbb{K}$-algebras. A morphism $\Phi\colon \mathcal{M} \longrightarrow \mathcal{N}$ of $\mathcal{A}$-presheaves is a morphism of $\mathcal{A}$-presheaves or $\mathcal{A}$-morphism if $\Phi_{U_\beta}\colon \mathcal{M}(U_\beta) \longrightarrow \mathcal{N}(U_\beta)$, for each $U_\beta \in Op(M), \beta \in \mathfrak{A}$ is an $\mathcal{A}(U_\beta)$-morphism of $\mathcal{A}(U_\beta)$-modules such that the diagram:

$$\begin{array}{ccc} \mathcal{M}(U_\beta) & \xrightarrow{\ \Phi_{U_\beta}\ } & \mathcal{N}(U_\beta) \\ {\scriptstyle \rho^{\mathcal{M}}_{U_\beta U_\alpha}}\downarrow & & \downarrow{\scriptstyle \rho^{\mathcal{N}}_{U_\beta U_\alpha}} \\ \mathcal{M}(U_\alpha) & \xrightarrow{\ \Phi_{U_\alpha}\ } & \mathcal{N}(U_\alpha) \end{array}$$

is commutative.

Denote the category of $\mathcal{A}$-presheaves on (M,g) and $\mathcal{A}$-morphisms by $\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}$-$\mathcal{PS}h_M$. The above $\mathcal{A}$-morphisms of $\mathcal{A}$-presheaves become $\mathcal{A}$-morphisms of $\mathcal{A}$-sheaves by applying the sheafification functor. If $\Phi\colon \mathcal{M} \longrightarrow \mathcal{N}$ is any $\mathcal{A}$-morphism then the sheafification functor $\mathcal{S}hf : \mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}$-$\mathcal{PS}h_M \longrightarrow \mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}$-$\mathcal{S}h_M$ gives a sheaf morphism:

$$\Phi \equiv \Phi^{\blacklozenge}\colon \mathcal{M} \equiv \mathcal{M}^{\blacklozenge} = \mathcal{S}hf(\mathcal{M}) \longrightarrow \mathcal{S}hf(\mathcal{N}) = \mathcal{N}^{\blacklozenge} \equiv \mathcal{N}$$

such that the stalk morphisms $\Phi_p \equiv \Phi^{\blacklozenge}_p = \lim_{U_\alpha \ni p} \Phi(U_\alpha) : \mathcal{M}_P \longrightarrow \mathcal{N}_p$ for each $p \in M$ are $\mathcal{A}$-morphisms, where $\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}$-$\mathcal{S}h_M$ is the category of $\mathcal{A}$-sheaves and $\mathcal{A}$-morphisms. Furthermore, we obtain from the $\mathcal{A}$-presheaves $Ker\Phi \subseteq \mathcal{M}$ and $Im\Phi \subseteq \mathcal{N}$ the sub-$\mathcal{A}$-presheaves

$$Ker\Phi = (Ker\Phi_{U_\alpha}, \pi^{\mathcal{M}}_{U_\beta U_\alpha})_{\alpha,\beta\in\mathfrak{A}}$$

and $Im\Phi = (Im\Phi_{U_\alpha}, \pi^{\mathcal{N}}_{U_\beta U_\alpha})$ with $\imath_{\alpha,\beta}\colon U_\alpha \hookrightarrow U_\beta$ if $U_\alpha \subset U_\beta$ in $Op(M)$. The realization of these under the sheafification functors are sub-$\mathcal{A}$-sheaves. Finally, if $\mathcal{M} = (\mathcal{M}, \pi_{\mathcal{M}}, M)$ and $\mathcal{N} = (\mathcal{N}, \pi_{\mathcal{N}}, M)$ are $\mathcal{A}$-modules on M such that $\mathcal{M}$ is a sub-$\mathcal{A}$-module of $\mathcal{N}$ so that for every $p \in M, \mathcal{M}_p$ is a sub-$\mathcal{A}$-module of $\mathcal{N}_p$, then define the set $\mathcal{N} \diagup \mathcal{M} := \bigoplus_{p \in M} \mathcal{N}_p \diagup \mathcal{M}_p$, together with the natural projection map $\delta : \mathcal{N} \diagup \mathcal{M} \longrightarrow M$ such that $\delta(\mathcal{N}_p \diagup \mathcal{M}_p) := \{p\} = p$, for each $p \in M$. Thus by definition $(\mathcal{N} \diagup \mathcal{M})_p = \delta^{-1}(p) = \mathcal{N}_p \diagup \mathcal{M}_p$, for every $p \in M$. We can now make $(\mathcal{N} \diagup \mathcal{M},\ \delta,\ M)$ into $\mathcal{A}$-module on M called the quotient $\mathcal{A}$-module of $\mathcal{N}$ by $\mathcal{M}$.

Finally we construct a *Matrix algebra sheaf* as follows: Let

$$\mathcal{A} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}} - \mathcal{S}h_M)$$

and $n \in \mathbb{N}$. For any $U_\alpha \in Ob(Op(M))$, consider the assignment $U_\alpha \longmapsto M_n(\mathcal{A}(U_\alpha))$ giving rise to the the presheaf of full matrix $\mathbb{K}$-algebras $M_n(\mathcal{A}(U_\alpha))$ on M consisting of $n \times n$ matrices with entries in the associative unital commutative $\mathbb{K}$-algebra $\mathcal{A}(U_\alpha) \equiv \Gamma(U_\alpha, \mathcal{A})$. Note that if $\Gamma(\mathcal{A}) \equiv (\mathcal{A}(U_\alpha), \rho_{U_\beta U_\alpha})$ with the inclusion morphism: $\imath_{\alpha\beta} \colon U_\alpha \hookrightarrow U_\beta$ if and only if $U_\alpha \subset U_\beta$ *in* $Op(M)$, then we obtain the presheaf $(M_n(\mathcal{A}(U_\beta)), M_n(\rho_{U_\beta U_\alpha}))$ with the restriction $\mathbb{K}$-algebra morphisms: $M_n(\rho_{U_\beta U_\alpha}) \colon M_n(\mathcal{A}(U_\beta)) \longrightarrow M_n(\mathcal{A}(U_\alpha))$. The restriction morphism of the presheaf is given by $M_n(\rho_{U_\beta U_\alpha}) \colon = \underbrace{\rho_{U_\beta U_\alpha} \times \cdots \times \rho_{U_\beta U_\alpha}}_{n^2-times}$, i.e. for any section $[\sigma_{ij}^{U_\beta}] \in M_n(\mathcal{A}(U_\beta))$, such that $\sigma_{ij}^{U_\beta} \in \mathcal{A}(U_\beta) \equiv \Gamma(U_\beta, \mathcal{A})$, with $1 \leq i, j \leq n$ we have: $M_n(\rho_{U_\beta U_\alpha})([\sigma_{ij}^{U_\beta}]) := (\rho_{U_\beta U_\alpha}([\sigma_{ij}^{U_\beta}]) \equiv [\sigma_{ij}^{U_\beta}] \restriction_{U_\alpha}) \in M_n(\mathcal{A}(U_\alpha))$. Next denote by $M_n(\mathcal{A})$ the sheaf of $\mathbb{K}$-algebras on M generated by the presheaf $(M_n(\mathcal{A}(U_\beta)), M_n(\rho_{U_\beta U_\alpha}))$ called the sheaf of full matrix $\mathbb{K}$-algebras on M of order n. This sheaf of unital $\mathbb{K}$-algebras on M is in general non-commutative unless $n = 1$. Similarly if $n, m \in \mathbb{N}$, for any $U_\alpha \in Ob(Op(M))$ the correspondence $U_\beta \longmapsto M_{nm}(\mathcal{A}(U_\beta))$ is the set of all $n \times m$ matrices with entries in $\mathcal{A}(U_\beta)$ an associative unital commutative $\mathbb{K}$-algebra. Then the presheaf $(M_{nm}(\mathcal{A}(U_\beta)), M_{nm}(\rho_{U_\beta U_\alpha}))$ for any inclusion morphism $\imath_{\alpha\beta} \colon U_\alpha \hookrightarrow U_\beta$ whenever $U_\alpha \subset U_\beta \in Op(M)$ generates sheaf of full matrix $\mathbb{K}$-vector spaces $M_{nm}(\mathcal{A})$ on M. Thus we have an isomorphism $M_{nm}(\mathcal{A}(U_\beta)) \cong M_{nm}(\mathcal{A})(U_\beta)$ of $\mathbb{K}$-vector spaces for every $U_\beta \in Op(M)$. We also define the general linear group sheaf of order $n \in \mathbb{N}$ $\mathcal{GL}(n, \mathcal{A}) \colon = M_n(\mathcal{A})^\bullet$, where for any $\mathcal{A} \in Ob(Assoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$ we denote by $\mathcal{A}^\bullet$ the sheaf on M generated by the presheaf $U_\alpha \longmapsto \mathcal{A}(U_\alpha)$, for all $U_\alpha \in Op(M)$ as the group of units, i.e. invertible elements of the associative unital $\mathbb{K}$-algebra $\mathcal{A}(U_\alpha)$. $\mathcal{GL}(n, \mathcal{A})$ is a sheaf of groups on M not necessarily abelian, unless $n = 1$. The general linear group sheaf $\mathcal{GL}(n, \mathcal{A})$ on M is generated by the presheaf: $U_\alpha \longmapsto GL(n, \mathcal{A})(U_\alpha)$ for any $U_\alpha \in Op(M)$.

3. Monoidal Category of Sheaves of Unital $\mathbb{K}$-Algebras, Yang-Baxter $\mathbb{K}$-Algebras and Coalgebras, Quantum $\mathbb{K}$-Algebras and Coalgebras

In this section we use the various associative unital $\mathbb{K}$-algebras constructed in Section 2, using systematically the language of sheaves of associative unital $\mathbb{K}$-algebras on (M, g). We pay particular attention to the categories of sheaves of Hopf $\mathbb{K}$-algebras(quantum groups), which are the simplest examples of non-commutative geometry or quantum geometry with connections to braided categories of quasi-triangular Hopf $\mathbb{K}$-algebras (quasi-quantum groups) and their dual counter part the coquasi-triangular Hopf $\mathbb{K}$-algebras (coquasi-quantum groups), Yang-Baxter $\mathbb{K}$-algebras and $\mathbb{K}$-coalgebras, quantum $\mathbb{K}$-algebras and $\mathbb{K}$-coalgebras on (M, g).

Theorem 3.1. *Let $\mathcal{A} \in Ob(\mathcal{C}ohAssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$ and $R \in \mathcal{A}\otimes\mathcal{A}$ generate $\mathcal{A}$ if the only subsheaf $\mathcal{N}$ of $\mathcal{A}$ such that $R \in \mathcal{N} \otimes \mathcal{N}$ is $\mathcal{A}$ itself, that is, $\mathcal{N} = \mathcal{A}_{[R]} = \mathcal{A}$. Then there exists at most one $\mathbb{K}$-algebra sheaf isomorphism $S\colon \mathcal{A} \longrightarrow \mathcal{A}^{op}$ such that $(\mathcal{A}, R, S)$ is a quantum $\mathbb{K}$-algebra sheaf.*

Proof. Suppose that $\tilde{S}\colon \mathcal{A} \longrightarrow \mathcal{A}^{op}$ is another $\mathbb{K}$-algebra sheaf isomorphism such that $(\mathcal{A}, R, S)$ and $(\mathcal{A}, R, \tilde{S})$ are two quantum $\mathbb{K}$-algebra sheaf structures on $\mathcal{A}$. The fact that $R \in \mathcal{A} \otimes \mathcal{A}$ is invertible means that $R \neq 0$. Suppose $l \in \mathbb{N}$ is as small as possible such that for any $U_\alpha \in Ob(Op(M))$ we have $R_{U_\alpha} = \sum_{j=1}^{l} a_j \otimes b_j$ for $a_j, b_j \in \mathcal{A}(U_\alpha), U_\alpha \in Ob(Op(M))$ and $\{a_j\}_{1\le j\le l}$, $\{b_j\}_{1\le j\le l}$ form linearly independent sets of sections of $\mathcal{A}(U_\alpha)$. Then the definition of quantum $\mathbb{K}$-algebra sheaf implies that $\sum_{j=1}^{l} S_{U_\alpha}(a_j) \otimes b_j = R_{U_\alpha}^{-1} = \sum_{j=1}^{l} \tilde{S}_{U_\alpha}(a_j) \otimes b_j$, with

$$S_{U_\alpha}, \tilde{S}_{U_\alpha}\colon \mathcal{A}(U_\alpha) \longrightarrow \mathcal{A}(U_\alpha)^{op},$$

and

$$R_{U_\alpha}\colon \mathcal{A}(U_\alpha) \otimes \mathcal{A}(U_\alpha) \longrightarrow \mathcal{A}(U_\alpha) \otimes \mathcal{A}(U_\alpha)$$

which in turn implies that $S_{U_\alpha}(a_j) = \tilde{S}_{U_\alpha}(a_j)$, for all $1 \le j \le l$, by the assumption of linear independence of the sets of sections $\{a_j\}_{1\le j\le l}$ and $\{b_j\}_{1\le j\le l}$.

$$\sum_{j=1}^{l} S_{U_\alpha}(a_j) \otimes S_{U_\alpha}(b_j) = R_{U_\alpha} = \sum_{j=1}^{l} \tilde{S}_{U_\alpha}(a_j) \otimes \tilde{S}_{U_\alpha}(b_j) = \sum_{j=1}^{l} S_{U_\alpha}(a_j) \otimes \tilde{S}_{U_\alpha}(b_j)$$

and the linear independence of sets of sections $\{S_{U_\alpha}(a_j)\}_{1\le j\le l}$ as subsets of the $\mathbb{K}$-algebra sheaf $\mathcal{A}$ such that $S_{U_\alpha}(b_j) = \tilde{S}_{U_\alpha}(b_j)$, for all $1 \le j \le l$. Thus S_{U_α} and $\tilde{S}_{U_\alpha}$ agree on the generators of $\mathcal{A}(U_\alpha)$ which implies that $S_{U_\alpha} = \tilde{S}_{U_\alpha}$. But $S_{U_p} \cong S_p \cong \tilde{S}_p \cong \tilde{S}_p(U_\alpha)$ for any sheaf, hence by sheafification $S = \tilde{S}$. □

Definition 3.2. Let $\mathcal{A} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$ with $R \in \mathcal{A} \otimes \mathcal{A}$ invertible and $S\colon \mathcal{A} \longrightarrow \mathcal{A}^{op}$ a $\mathbb{K}$-algebra sheaf isomorphism such that $(\mathcal{A}, R, S)$ is a quantum $\mathbb{K}$-algebra sheaf on M. We define a twist quantum $\mathbb{K}$-algebra sheaf on M as the quadruple $(\mathcal{A}, R, S, B)$, where $(\mathcal{A}, R, S)$ is a quantum $\mathbb{K}$-algebra sheaf on M and $B \in \mathcal{A}$ an invertible element such that: $S(B) = B^{-1}$ and $S^2(X) = BXB^{-1}$ for all $X \in \mathcal{A}$.

We can now formulate a result of L.H. Kauffman and David E. Radford [12, 13] in our sheaf theoretic setting.

Theorem 3.3. *Let $M_n(\mathcal{A}), n \in \mathbb{N}$ be the associative unital $\mathbb{K}$-algebra sheaf on the complete Riemannian manifold (M, g) generated by the complete presheaf of full matrix $\mathbb{K}$-algebras $(M_n(\mathcal{A}(U_\alpha)), M_n(\rho_{U_\alpha U_\beta}))$ on M with restriction $\mathbb{K}$-algebra presheaf morphism*

$$M_n(\rho_{U_\alpha U_\beta}) : M_n(\mathcal{A}(U_\alpha)) \longrightarrow M_n(\mathcal{A}(U_\beta))$$

given by

$$M_n(\rho_{U_\alpha U_\beta})(\alpha_{ij}^{U_\alpha}) = \rho_{U_\alpha U_\beta})(\alpha_{ij}^{U_\alpha}) \equiv (\alpha_{ij}^{U_\alpha})\restriction_{U_\beta} \in M_n(\mathcal{A}(U_\beta))$$

for any section matrix $[a_{ij}^{U_\alpha}] \in M_n(\mathcal{A}(U_\alpha)) \equiv \Gamma(U_\alpha, \mathcal{A})$, for $1 \leq i, j \leq n$ and the inclusion map $\imath_{\beta\alpha}\colon U_\beta \hookrightarrow U_\alpha$ if $U_\beta \subseteq U_\alpha$ in $Op(M)$. Further, suppose that $\Phi \in Aut(M_n(\mathcal{A}))$ is of the form: $\Phi(X) = AXA^{-1}$ for all $X \in M_n(\mathcal{A})$ with $A \in M_n(\mathcal{A})$ invertible, X^τ the transpose of X and $(M_n(\mathcal{A}), R, S)$ a quantum $\mathbb{K}$-algebra sheaf on M. Then:

(1) There exists an invertible element $B \in M_n(\mathcal{A})$ such that $S(Y) = BY^\tau B^{-1}$ for any $Y \in M_n(\mathcal{A})$.

(2) $(M_n(\mathcal{A}), R, S, B(B^\tau)^{-1})$ is a twist quantum $\mathbb{K}$-algebra sheaf.

Proof. To prove (1), first note it is immediate that for all $X \in M_n(\mathcal{A}), \Phi \in Aut(M_n(\mathcal{A}))$ can be defined by $\Phi(X) := S(X^\tau)$ and hence $\Phi(X) = BXB^{-1}$ for some invertible element $B \in M_n(\mathcal{A})$. Because $S(X) = \Phi(X^\tau)$, the result follows.

The proof of (2) follows from definitions by computation. $\square$

Theorem 3.4. *Let $(\mathcal{A}, R, S)$ be a quantum $\mathbb{K}$-algebra sheaf and $\Phi \in Aut(\mathcal{A})$ a $\mathbb{K}$-algebra sheaf automorphism such that $R_\Phi := (\Phi \otimes \Phi) \circ R$ and $S_\Phi := \Phi S \Phi^{-1}$ make the following two diagrams commutative:*

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} & \xrightarrow{R} & \mathcal{A} \otimes \mathcal{A} \\ R_\Phi \searrow & & \downarrow \Phi \otimes \Phi \\ & \mathcal{A} \otimes \mathcal{A} & \end{array}$$

and

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{S} & \mathcal{A}^{op} \\ \Phi^{-1}\uparrow & & \downarrow\Phi \\ \mathcal{A} & \xrightarrow[S_\Phi]{} & \mathcal{A}^{op} \end{array}$$

(1) If $\Phi\colon \mathcal{A} \longrightarrow \mathcal{A}$ is a $\mathbb{K}$-algebra sheaf isomorphism, then $(\mathcal{A}, R_\Phi, S_\Phi)$ is a quantum $\mathbb{K}$-algebra sheaf and $\Phi\colon (\mathcal{A}, R, S) \longrightarrow (\mathcal{A}, R_\Phi, S_\Phi)$ is an isomorphism of quantum $\mathbb{K}$-algebra sheaves.

(2) If $\Phi : \mathcal{A} \longrightarrow \mathcal{A}^{op}$ is an isomorphism of $\mathbb{K}$-algebra sheaves, then $(\mathcal{A}, R_\Phi, S_\Phi)$ is a quantum $\mathbb{K}$-algebra sheaf and $\Psi\colon (\mathcal{A}, R, S) \longrightarrow (\mathcal{A}, R_\Phi, S_\Phi)$ is an isomorphism of quantum $\mathbb{K}$-algebra sheaves with Ψ given by the $\mathbb{K}$-algebra sheaf composite morphism $\Psi = \Phi \circ S$.

Proof. To prove part (1), since $\mathcal{A}$ is a $\mathbb{K}$-algebra sheaf, we can consider sections of $\mathcal{A}$ on $U_\alpha \in Ob(Op(M))$ such that $\mathcal{A}(U_\alpha) \equiv \Gamma(U_\alpha, \mathcal{A})$ and for all inclusion maps $\imath_{\beta\alpha} : U_\beta \hookrightarrow U_\alpha$ if $U_\beta \subset U_\alpha$ in $Op(M)$ we have $\mathbb{K}$-algebra presheaves $(\mathcal{A}(U_\alpha), \rho_{U_\alpha U_\beta})$ with the restriction presheaf morphisms: $\rho_{U_\alpha U_\beta}\colon \mathcal{A}(U_\alpha) \longrightarrow \mathcal{A}(U_\beta)$, for all $\alpha, \beta \in \mathfrak{A}$. If $\Phi : \mathcal{A} \longrightarrow \mathcal{A}$ is a $\mathbb{K}$-algebra sheaf isomorphism, then the diagram

$$\begin{array}{ccc} \mathcal{A}(U_\alpha) & \xrightarrow{\Phi\restriction_{\mathcal{A}(U_\alpha)}} & \mathcal{A}(U_\alpha) \\ \rho_{U_\alpha U_\beta}\downarrow & & \downarrow\rho_{U_\alpha U_\beta} \\ \mathcal{A}(U_\beta) & \xrightarrow{\Phi\restriction_{\mathcal{A}(U_\beta)}} & \mathcal{A}(U_\beta) \end{array}$$

commutes. If $R : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ is an invertible endomorphism of $\mathbb{K}$-algebra sheaves, then we have the commutative diagram

$$\begin{array}{ccc} \mathcal{A}(U_\alpha) \otimes \mathcal{A}(U_\alpha) & \xrightarrow{R_{U_\alpha}} & \mathcal{A}(U_\alpha) \otimes \mathcal{A}(U_\alpha) \\ \rho_{U_\alpha U_\beta}\otimes\rho_{U_\alpha U_\beta}\downarrow & & \downarrow\rho_{U_\alpha U_\beta}\otimes\rho_{U_\alpha U_\beta} \\ \mathcal{A}(U_\beta) \otimes \mathcal{A}(U_\beta) & \xrightarrow{R_{U_\beta}} & \mathcal{A}(U_\beta) \otimes \mathcal{A}(U_\beta) \end{array}$$

Now write $R_{U_\alpha} = \sum_{j=1}^{l} a_j \otimes b_j$, for every $a_j, b_j \in \mathcal{A}(U_\alpha)$ and for the $\mathbb{K}$-algebra sheaf isomorphism $S : \mathcal{A} \longrightarrow \mathcal{A}^{op}$ we consider its restriction to U_α,

$$S_{U_\alpha}\colon \mathcal{A}(U_\alpha) \longrightarrow \mathcal{A}^{op}(U_\alpha).$$

Then the conditions characterizing the quantum $\mathbb{K}$-algebra sheaf $(\mathcal{A}, R, S)$ restricted to $U_\alpha \in Ob(Op(M))$ can be formulated as follows:

(i) $R^{-1}_{U_\alpha} = (S_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)})(R_{U_\alpha})$,

(ii) $R_{U_\alpha} = (S_{U_\alpha} \otimes S_{U_\alpha})(R_{U_\alpha})$ and the quantum Yang-Baxter equation

$(iii) \quad (R_{U_\alpha})_{12}(R_{U_\alpha})_{13}(R_{U_\alpha})_{23} = (R_{U_\alpha})_{23}(R_{U_\alpha})_{13}(R_{U_\alpha})_{12}$.

Next apply $(\Phi_{U_\alpha} \otimes \Phi_{U_\alpha} \otimes \Phi_{U_\alpha})$ to both sides of (iii) to show that (iii) holds for $(R_\Phi)_{U_\alpha}$. Since for any point $p \in U_\alpha$, all sheaf data is local and contained in the stalk, the sheaf isomorphism $\mathcal{A}(U_\alpha) \cong \mathcal{A}_p$ implies that condition (iii) holds for R_Φ. Now apply $(\Phi_{U_\alpha} \otimes \Phi_{U_\alpha})$ to both sides of (i) and (ii) together with the fact that $\Phi_{U_\alpha} S_{U_\alpha} = (R_\Phi)_{U_\alpha} \Phi_{U_\alpha}$, and the statement immediately above to deduce that (i) and (ii) hold for R_Φ. We use the fact that $\Phi : \mathcal{A} \longrightarrow \mathcal{A}$ is a $\mathbb{K}$-algebra sheaf morphism to deduce that $S\colon \mathcal{A} \longrightarrow \mathcal{A}^{op}$ is a $\mathbb{K}$-algebra sheaf morphism. It is then clear that $(\mathcal{A}, R_\Phi, S_\Phi)$ is a quantum $\mathbb{K}$-algebra sheaf and so $\Phi\colon (\mathcal{A}, R, S) \longrightarrow (\mathcal{A}, R_\Phi, S_\Phi)$ is an isomorphism of quantum $\mathbb{K}$-algebra sheaves.

For the proof of part (2), if $\Phi\colon \mathcal{A}(U_\alpha) \longrightarrow \mathcal{A}(U_\alpha)^{op}$ is a $\mathbb{K}$-algebra presheaf isomorphism then sheafification implies $\Phi : \mathcal{A} \longrightarrow \mathcal{A}^{op}$ is a $\mathbb{K}$-algebra sheaf isomorphism. Thus if we define the $\mathbb{K}$-algebra sheaf morphism $\Psi : \mathcal{A} \longrightarrow \mathcal{A}$ as the composite $\mathbb{K}$-algebra sheaf morphism $\Psi = \Phi \circ S$, then by part (1), $(\mathcal{A}, R_\Psi, S_\Psi)$ is a quantum $\mathbb{K}$-algebra sheaf and hence $\Psi\colon (\mathcal{A}, R, S) \longrightarrow (\mathcal{A}, R_\Psi, S_\Psi)$ is an isomorphism of quantum $\mathbb{K}$-algebra sheaves. Thus by computation we get

$$R_\Psi = (\Psi \otimes \Psi)(R) = (\Phi \otimes \Phi)(S \otimes S)(R) = (\Phi \otimes \Phi)(R) = R_\Phi$$

and

$$S_\Psi = \Psi S \Psi^{-1} = (\Phi S) S (\Phi S)^{-1} = \Phi S \Phi^{-1} = S_\Phi,$$

which says that $(\mathcal{A}, R_\Psi, S_\Psi) = (\mathcal{A}, R_\Phi, S_\Phi)$. Therefore, (2) follows from part (1), which completes the proof of the theorem. □

Let $\mathcal{A} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$ and $\mathfrak{QYB}(\mathcal{A})$ be the set of all solutions $R \in \mathcal{A} \otimes \mathcal{A}$ of the quantum Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{3.1}$$

We can reformulate this in terms of $\mathbb{K}$-algebra pre-sheaves by requiring that for any $U_\alpha \in Ob(Op(M))$ we write $R_{U_\alpha} = \sum_{j=1}^{l} a_j \otimes b_j \in \mathcal{A}(U_\alpha) \otimes \mathcal{A}(U_\alpha)$ such that

$$(R_{st})_{U_\alpha} \in \mathcal{A}(U_\alpha) \otimes \mathcal{A}(U_\alpha) \otimes \mathcal{A}(U_\alpha)$$

for $1 \leq s < t \leq l$, which are in turn defined by

$$(R_{12})_{U_\alpha} := R_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)} = \sum_{j=1}^{l} a_j \otimes b_j \otimes 1_{\mathcal{A}(U_\alpha)},$$

$$(R_{23})_{U_\alpha} := 1_{\mathcal{A}(U_\alpha)} \otimes R_{U_\alpha} = \sum_{j=1}^{l} 1_{\mathcal{A}(U_\alpha)} \otimes a_j \otimes b_j$$

and

$$(R_{13})_{(U_\alpha)} := (1_{\mathcal{A}(U_\alpha)} \otimes \tau_{\mathcal{A}(U_\alpha)\mathcal{A}(U_\alpha)})(R_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)}) = \sum_{j=1}^{l} a_j \otimes 1_{\mathcal{A}(U_\alpha)} \otimes b_j$$

with $\tau_{\mathcal{A}(U_\alpha)\mathcal{A}(U_\alpha)} \colon \mathcal{A}(U_\alpha) \otimes \mathcal{A}(U_\alpha) \longrightarrow \mathcal{A}(U_\alpha) \otimes \mathcal{A}(U_\alpha)$ the twist $\mathbb{K}$-algebra presheaf morphism. Then the quantum Yang-Baxter equation (3.1) takes the form

$$(R_{12})_{U_\alpha}(R_{13})_{U_\alpha}(R_{23})_{U_\alpha} = (R_{23})_{U_\alpha}(R_{13})_{U_\alpha}(R_{12})_{U_\alpha}.$$

We also set $R^{op}_{U_\alpha} := \tau_{\mathcal{A}(U_\alpha)\mathcal{A}(U_\alpha)}(R_{U_\alpha}) := \sum_{j=1}^{l} b_j \otimes a_j$. Let $Aut_{\mathcal{G}}(\mathcal{A}(U_\alpha))$ be the group of all $\mathbb{K}$-algebra presheaf automorphisms $\Phi \colon \mathcal{A}(U_\alpha) \longrightarrow \mathcal{A}(U_\alpha)$ such that either $\Phi \colon \mathcal{A}(U_\alpha) \longrightarrow \mathcal{A}(U_\alpha)$ or $\Phi \colon \mathcal{A}(U_\alpha) \longrightarrow \mathcal{A}(U_\alpha)^{op}$ is a $\mathbb{K}$-algebra presheaf isomorphism. Theorem (3.3) implies that $Aut_{\mathcal{G}}\mathcal{A}(U_\alpha)$ acts on $\mathfrak{QYB}(\mathcal{A}(U_\alpha))$, i.e. the action

$\mu_{\mathcal{G}} \colon Aut_{\mathcal{G}}(\mathcal{A}(U_\alpha)) \times \mathfrak{QYB}(\mathcal{A}(U_\alpha)) \longrightarrow \mathfrak{QYB}(\mathcal{A}(U_\alpha)) \colon (\Phi, R_{U_\alpha}) \longmapsto \mu_{\mathcal{G}}(\Phi, R_{U_\alpha})$

is given by $\mu_{\mathcal{G}}(\Phi, R_{U_\alpha}) := (R_\Phi)_{U_\alpha}$, for all $\Phi \in Aut_{\mathcal{G}}(\mathcal{A}(U_\alpha))$ and $R_{U_\alpha} \in \mathfrak{QYB}(\mathcal{A}(U_\alpha))$. If $(\mathcal{A}(U_\alpha), R_{U_\alpha}, S_{U_\alpha})$ and $(\mathcal{A}(U_\alpha), \tilde{R}_{U_\alpha}, \tilde{S}_{U_\alpha})$ are two quantum $\mathbb{K}$-algebra presheaf structures on $\mathcal{A}(U_\alpha)$, then we say that $(\mathcal{A}(U_\alpha), R_{U_\alpha}, S_{U_\alpha})$ is equivalent to $(\mathcal{A}(U_\alpha), \tilde{R}_{U_\alpha}, \tilde{S}_{U_\alpha})$ written $(\mathcal{A}(U_\alpha, R_{U_\alpha}, S_{U_\alpha}) \approx (\mathcal{A}(U_\alpha), \tilde{R}_{U_\alpha}, \tilde{S}_{U_\alpha})$ if and only if R_{U_α} and $\tilde{R}_{U_\alpha}$ are by definition in the same $Aut_{\mathcal{G}}(\mathcal{A}(U_\alpha))$-orbit. On the other hand Theorems (3.4) and (3.1) imply that, if either $(\mathcal{A}(U_\alpha), R_{U_\alpha}, S_{U_\alpha})$ or $(\mathcal{A}(U_\alpha), \tilde{R}_{U_\alpha}, \tilde{S}_{U_\alpha})$ is minimal then R_{U_α} and $\tilde{R}_{U_\alpha}$ lie in the same orbit and hence

$$(\mathcal{A}(U_\alpha), R_{U_\alpha}, S_{U_\alpha}) \approx (\mathcal{A}(U_\alpha), \tilde{R}_{U_\alpha}, \tilde{S}_{U_\alpha}).$$

Observe that if $R_{U_\alpha} \in \mathfrak{QYB}(\mathcal{A}(U_\alpha))$ then $R^{op}_{U_\alpha} \in \mathfrak{QYB}(\mathcal{A}(U_\alpha))$. Therefore, there is a natural action of the multiplicative group $\mathbb{Z}_2 = (z)$ of order 2 on $\mathfrak{QYB}(\mathcal{A}(U_\alpha))$ determined by $z \cdot R_{U_\alpha} = R^{op}_{U_\alpha}$, for all $R_{U_\alpha} \in \mathfrak{QYB}(\mathcal{A}(U_\alpha))$ and $\mathfrak{QYB}(\mathcal{A}(U_\alpha))$ is closed under $\mathbb{K}^*$-multiplication with $\mathbb{K}^*$, the multiplicative group of units of $\mathbb{K}$. Applying sheafification functor to each step gives the corresponding $\mathbb{K}$-algebra sheaf form of the results on $\mathfrak{QYB}(\mathcal{A})$. Now that we have Theorem 3 in place, we can apply it to obtain the following classification result for quantum $\mathbb{K}$-algebra sheaf structures on any category, $Assoc\mathcal{A}lg_{\mathbb{K}}$-$\mathcal{S}h_M$, of associative $\mathbb{K}$-algebra sheaves.

Theorem 3.5. *Let $\mathcal{A} \in Ob(Assoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$ be such that $R, \tilde{R} \in \mathfrak{QYB}(\mathcal{A})$ lie in the same $Aut_{\mathcal{G}}(\mathcal{A}(U_\alpha)) \times \mathcal{Z}_2$-orbit. Further, suppose that $(\mathcal{A}, \tilde{R}, \tilde{S})$ is a quantum $\mathbb{K}$-algebra sheaf structure on $\mathcal{A}$.*

Then there exists a quantum $\mathbb{K}$-algebra sheaf structure $(\mathcal{A}, R, S)$ on $\mathcal{A}$ such that $(\mathcal{A}, \tilde{R}, \tilde{S}) \approx (\mathcal{A}, R, S)$ or $(\mathcal{A}, \tilde{R}, \tilde{S}) \approx (\mathcal{A}, R^{op}, S^{-1})$.

Proof. For any $\Phi \in Aut_{\mathcal{G}}(\mathcal{A}(U_\alpha))$ with $U_\alpha \in Ob(Op(M))$, we have either

(i) $R_{U_\alpha} = (\tilde{R}_\Phi)_{U_\alpha}$ or

(ii) $R_{U_\alpha} = (\tilde{R}_\Phi)^{op}_{U_\alpha}$.

In case (i), Theorem 3.1 implies that there is a quantum $\mathbb{K}$-algebra presheaf structure $(\mathcal{A}(U_\alpha), R_{U_\alpha}, S_{U_\alpha})$ on $\mathcal{A}(U_\alpha)$ $\mathbb{K}$-algebra presheaf isomorphic to

$$(\mathcal{A}(U_\alpha), \tilde{R}_{U_\alpha}, \tilde{S}_{U_\alpha}).$$

Case (ii) follows from the fact that $R^{op} = \tilde{R}_\Phi$. Hence there exists a quantum $\mathbb{K}$-algebra presheaf structure $(\mathcal{A}(U_\alpha), R^{op}_{U_\alpha}, T_{U_\alpha})$ on $\mathcal{A}(U_\alpha)$ $\mathbb{K}$-algebra presheaf isomorphic to $(\mathcal{A}(U_\alpha), \tilde{R}_{U_\alpha}, \tilde{S})$. But $(\mathcal{A}(U_\alpha), (R^{op})^{op}, T^{-1}_{U_\alpha}) = (\mathcal{A}(U_\alpha), R_{U_\alpha}, S_{U_\alpha})$ is a quantum $\mathbb{K}$-algebra presheaf with $S_{U_\alpha} = T^{-1}_{U_\alpha}$. Sheafification then implies the result in both cases. This completes the proof of the theorem. ☐

Theorem 3.6. *Let* $\mathcal{A} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$ *and*

$$R\colon \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$$

be an invertible $\mathbb{K}$*-algebra sheaf endomorphism such that* $(\mathcal{A}, R)$ *is a* $\mathbb{K}$*-algebra sheaf. Then* R *is a Yang-Baxter structure for* $\mathcal{A}$ *if and only if*

$$R(1 \otimes a) = a \otimes 1,\ \ R(a \otimes 1) = 1 \otimes a$$

and

$$R(m \otimes 1_{\mathcal{A}}) = (1_{\mathcal{A}} \otimes m)(R_{12})(R_{23}),\ \ R(1_{\mathcal{A}} \otimes m) = (m \otimes 1_{\mathcal{A}})(R_{23})(R_{12}),$$

as maps from $\mathcal{A}^{\otimes 3}$ *to* $\mathcal{A}^{\otimes 2}$*, for all* $a \in \mathcal{A}(U_\alpha)$ *where* $U_\alpha \in Ob(Op(M))$ *with the inclusion maps* $\imath_{\beta\alpha}\colon U_\beta \hookrightarrow U_\alpha$ *if and only if* $U_\beta \subset U_\alpha$ *in* $Op(M)$*.*

Proof. Let $U_\alpha \in Ob(Op(M))$ such that at presheaf level,

$$(R_{12})_{U_\alpha}(R_{23})_{U_\alpha} = (R_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)})(1_{\mathcal{A}(U_\alpha)} \otimes R_{U_\alpha}),$$

$$(R_{23})_{U_\alpha}(R_{12})_{U_\alpha} = (1_{\mathcal{A}(U_\alpha)} \otimes R_{U_\alpha})(R_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)}),$$

$$1_{\mathcal{A}(U_\alpha)} \otimes ((R_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)})(1_{\mathcal{A}(U_\alpha)} \otimes R_{U_\alpha})) = 1_{\mathcal{A}(U_\alpha)} \otimes (R_{12})_{U_\alpha}(R_{23})_{U_\alpha},$$

and

$$((R_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)})(1_{\mathcal{A}(U_\alpha)} \otimes R_{U_\alpha})) \otimes 1_{\mathcal{A}(U_\alpha)} = (R_{12})_{U_\alpha}(R_{23})_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)}.$$

Multiplying $(R_{12})_{U_\alpha}(R_{23})_{U_\alpha}$ on the left by $(1_{\mathcal{A}(U_\alpha)} \otimes m_{U_\alpha})$, we obtain

$$(1_{\mathcal{A}(U_\alpha)} \otimes m_{U_\alpha})(R_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)})(1_{\mathcal{A}(U_\alpha)} \otimes R_{U_\alpha}) = R_{U_\alpha}(m_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)})$$

which is one of the quasi-triangularity conditions. Similarly, multiplying $(R_{23})_{U_\alpha}$ $(R_{12})_{U_\alpha}$ on the left by $(m_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)})$, we get $(m_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)})(1_{\mathcal{A}(U_\alpha)} \otimes R_{U_\alpha})(R_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)}) = R_{U_\alpha}(1_{\mathcal{A}(U_\alpha)} \otimes m_{U_\alpha})$ which is the remaining quasi-triangularity condition.

Conversely if,

$$1_{\mathcal{A}(U_\alpha)} \otimes ((R_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)})(1_{\mathcal{A}(U_\alpha)} \otimes R_{U_\alpha})) = 1_{\mathcal{A}(U_\alpha)} \otimes (R_{12})_{U_\alpha}(R_{23})_{U_\alpha}$$

and

$$((R_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)})(1_{\mathcal{A}(U_\alpha)} \otimes R_{U_\alpha})) \otimes 1_{\mathcal{A}(U_\alpha)} = (R_{12})_{U_\alpha}(R_{23})_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)},$$

then

$$\begin{aligned}
&(m_{U_\alpha} \otimes m_{U_\alpha})(1_{\mathcal{A}(U_\alpha)} \otimes (R_{12})_{U_\alpha}(R_{23})_{U_\alpha}(R_{12})_{U_\alpha}(R_{23})_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)}) = \\
&(1_{\mathcal{A}(U_\alpha)}\otimes m_{U_\alpha})(m_{U_\alpha}\otimes 1_{\mathcal{A}(U_\alpha)})(1_{\mathcal{A}(U_\alpha)}\otimes(R_{12})_{U_\alpha}(R_{23})_{U_\alpha})((R_{12})_{U_\alpha}(R_{23})_{U_\alpha}\otimes 1_{\mathcal{A}(U_\alpha)}) \\
&= (1_{\mathcal{A}(U_\alpha)} \otimes m_{U_\alpha})(R_{12})_{U_\alpha}(R_{23})_{U_\alpha})(1_{\mathcal{A}(U_\alpha)} \otimes m_{U_\alpha}) \\
&= R_{U_\alpha}(m_{U_\alpha} \otimes m_{U_\alpha}).
\end{aligned}$$

Applying sheafification functor to each step gives the corresponding sheaf form of the statements. □

Theorem 3.7. *Let $(\mathcal{A}, \Delta, \epsilon)$ be a nonassociative (i.e. there exists $\Phi \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ so that*

$$(1_{\mathcal{A}(U_\alpha)} \otimes \Delta) \circ \Delta = \Phi((\Delta \otimes 1_{\mathcal{A}(U_\alpha)}) \circ \Delta)\Phi^{-1}))$$

sheaf $\mathbb{K}$-bialgebra and $R \in End_{\mathbb{K}}(\mathcal{A} \otimes \mathcal{A})$ an invertible endomorphism. Furtheremore, assume that $S \colon \mathcal{A} \longrightarrow \mathcal{A}^{op}$ is a sheaf $\mathbb{K}$-algebra isomorphism. Then:

(1) *If S is an antipode for the sheaf $\mathbb{K}$-algebra $\mathcal{A}$, we have*

$$R^{-1} = (S \otimes 1_{\mathcal{A}}(U_\alpha))(R) = (1_{\mathcal{A}(U_\alpha)} \otimes S^{-1})(R).$$

(2) *If R_{U_α} generates $\mathcal{A}(U_\alpha)$ such that $(\mathcal{A}_{[R]})(U_\alpha) = \mathcal{A}(U_\alpha)$ and $R^{-1} = (S \otimes 1_{\mathcal{A}(U_\alpha)})(R) = (1_{\mathcal{A}(U_\alpha)} \otimes S)(R)$ we deduce that $S \colon \mathcal{A} \longrightarrow \mathcal{A}^{op}$ is an antipode for the sheaf $\mathbb{K}$-algebra $\mathcal{A}$.*

(3) *If $(\Delta^{op}(a)(R) = R(\Delta(a))$ for all $a \in \mathcal{A}$, the condition $(\Delta^{op}(a))(R) = R(\Delta(a))$ holds for all $a \in \mathcal{A}$ such that the conditions:*

(i) $(\Delta \otimes 1_{\mathcal{A}(U_\alpha)})(R) = R_{13}R_{23}$ *and*

(ii) $(1_{\mathcal{A}(U_\alpha)} \otimes \Delta)(R) = R_{13}R_{12}$ *hold.*

(4) *If R_{U_α} generates $\mathcal{A}(U_\alpha)$ i.e. $\mathcal{A}(U_\alpha) = (\mathcal{A}_{[R]})(U_\alpha)$ satisfies the quantum Yang-Baxter condition:*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

we have $(\Delta^{op}(a))(R) = R\Delta(a)$ for all $a \in \mathcal{A}$.

Proof. Suppose $U_\alpha \in Ob(Op(M))$ and let $l \in \mathbb{N}$ be as small as possible such that the subsets $\{a_j\}_{1 \le j \le l}$, $\{b_j\}$ of $A(U_\alpha)$ are linearly independent. We can express $R_{U_\alpha} = \sum_{j=1}^{l} a_j \otimes b_j \in \mathcal{A}(U_\alpha) \otimes \mathcal{A}(U_\alpha)$. R_{U_α} generates $\mathcal{A}(U_\alpha)$, i.e. $\mathcal{A}(U_\alpha) = \mathcal{A}_{[R_{U_\alpha}]}(U_\alpha)$ and the sets $\{a_j\}_{1\le j\le l}$, $\{b_j\}$ generate $\mathcal{A}(U_\alpha)$ as a sheaf $\mathbb{K}$-algebra. We can now reformulate the conditions (i) and (ii) as (iii) $\Delta(a_j)\otimes b_j = \sum_{j,k} a_j \otimes a_k \otimes b_j b_k$ and (iv) $\sum a_j \otimes \Delta(b_j) = \sum a_j a_k \otimes b_k \otimes b_j$ respectively.

We easily see that (v) $\sum \epsilon(a_j)b_j = 1$ and (vi) $\sum a_j\epsilon(b_j) = 1$.

To see (v) note that if we let $\delta = \sum \epsilon(a_j)b_j = (\epsilon \otimes 1_{\mathcal{A}(U_\alpha)})(R)$, then δ is invertible since R_{U_α} is invertible and $\epsilon\colon \mathcal{A}(U_\alpha) \longrightarrow \mathbb{K}$ is a sheaf $\mathbb{K}$-algebra morphism. Similarly if we apply $\epsilon\otimes\epsilon 1_{\mathcal{A}(U_\alpha)}$ to both sides of $\sum \Delta(a_j)\otimes b_j = \sum a_j\otimes a_k \otimes b_j b_k$ we obtain $\delta = \delta^2$ which implies that $\delta = 1$, the identity section of the sheaf $\mathbb{K}$-algebra $\mathcal{A}(U_\alpha)$. Hence we obtain the claim (v).

Similarly using

$$\sum a_j \otimes \Delta(b_j) = \sum a_j a_k \otimes b_k \otimes b_j,$$

we deduce the proof of (vi). To prove part (i), we assume that $S\colon \mathcal{A}(U_\alpha) \longrightarrow \mathcal{A}^{op}(U_\alpha)$ is an antipode for the sheaf $\mathbb{K}$-algebra $\mathcal{A}$. Then (iii) and (v) imply that

$$\begin{aligned}\sum a_j S(a_k)\otimes b_j b_k &= \sum a_{j1}\otimes S(a_{j2})\otimes b_j \\ &= \sum \epsilon(a_j)1\otimes b_j = \sum 1\otimes \epsilon(a_j)b_j = 1\otimes 1.\end{aligned}$$

Thus $(S\otimes 1_{\mathcal{A}(U_\alpha)})(R_{U_\alpha})$ is a right inverse to R_{U_α} and so $R_{U_\alpha}^{-1} = (S\otimes 1_{\mathcal{A}(U_\alpha)})(R_{U_\alpha})$. Now suppose $S\colon \mathcal{A}(U_\alpha) \longrightarrow \mathcal{A}(U_\alpha)^{op}$ is a sheaf $\mathbb{K}$-algebra isomorphism. It is immediate that $\sum a_{(2)}S^{-1}(a_{(1)}) = \epsilon(a)1$ for all $a \in \mathcal{A}(U_\alpha)$. Then by (iv) and (vi) using similar argument as above imply that $R_{U_\alpha}^{-1} = (S\otimes 1_{\mathcal{A}(U_\alpha)})(R_{U_\alpha})$ and hence $R_{U_\alpha}^{-1} = (1_{\mathcal{A}(U_\alpha)}\otimes S^{-1})(R_{U_\alpha})$.

For part (b) assume R_{U_α} generates $\mathcal{A}(U_\alpha)$ such that $\mathcal{A}(U_\alpha) = (\mathcal{A}_{[R]})(U_\alpha)$ and

$$R_{U_\alpha}^{-1} = (S\otimes 1_{\mathcal{A}(U_\alpha)})(R_{U_\alpha}) = (1_{\mathcal{A}(U_\alpha)}\otimes S^{-1})(R_{U_\alpha}).$$

Formula $R_{U_\alpha}^{-1} = (S\otimes 1_{\mathcal{A}(U_\alpha)})(R_{U_\alpha})$ is shown by an easy computation that:

$$\sum a_{j(1)}S(a_{j(2)})\otimes b_j = \sum a_j S(a_j)\otimes b_j b_k = 1\otimes 1.$$

Use the formula $R_{U_\alpha}^{-1} = (1_{\mathcal{A}(U_\alpha)}\otimes S^{-1})(R_{U_\alpha})$ and (iv) so that by a simple calculation we obtain:

$$\sum a_j\otimes S^{-1}(b_{j(1)})b_{j(2)} = \sum a_j a_k \otimes S^{-1}(b_j)b_k = 1\otimes 1.$$

Thus $\sum a_j\otimes b_{j(1)}S(b_{j(2)}) = \sum a_j a_k\otimes S^{-1}(b_j)b_k = 1\otimes 1$. Therefore $\sum a_j\otimes b_{j(1)}S(b_{j(2)}) = 1\otimes 1$, since $S\colon \mathcal{A}(U_\alpha)\longrightarrow \mathcal{A}(U_\alpha)^{op}$ is a sheaf $\mathbb{K}$-algebra isomorphism.

Let $\xi\colon \mathcal{A}(U_\alpha) \longrightarrow \mathcal{A}(U_\alpha)\colon a \longmapsto \xi(a) := a_{(1)}S(a_{(2)})$ be a linear endomorphism. Since the a_j's and the b_j's form linearly independent sets, we deduce that $\xi(a_j), \xi(b_j) \in \mathbb{K}.1$ for all j from (vii) and $(viii)$ respectively. Thus $\xi(a_j)b_j = 1 = a_j\xi(b_j)$. Since $\xi\colon \mathcal{A}(U_\alpha) \longrightarrow \mathcal{A}(U_\alpha)$ is a linear endomorphism, we can write

$$\xi(ab) = (ab)_{(1)}S(ab)_{(2)} = a_{(1)}b_{(1)}S(a_{(2)})S(b_{(2)})$$
$$= a_{(1)}S(a_{(2)})b_{(1)}S(b_{(2)}) = a_{(1)}S(a_{(2)})\xi(b) = a_{(1)}\xi(b)S(a_{(2)}),$$

where we have used $S(x_{(1)})x_{(2)} \in \mathcal{A}(U_\alpha)^{op}$. Hence the set $\{a \in \mathcal{A}(U_\alpha)\colon \xi(a) \in \mathbb{K}.1\}$ is a sheaf $\mathbb{K}$-subalgebra of $\mathcal{A}(U_\alpha)$, which is $(\mathcal{A}_{[R]})(U_\alpha)$ since $\mathcal{A}(U_\alpha) = (\mathcal{A}_{[R]})(U_\alpha)$. Therefore $\xi\colon \mathcal{A}(U_\alpha) \longrightarrow \mathcal{A}(U_\alpha)$ is a sheaf $\mathbb{K}$-subalgebra map. From (v) and (vi) $\sum \epsilon(a_j)b_j = 1 = \sum a_j\epsilon(b_j)$ and the independence of the sets of the a_j's and the b_j's together with $\xi(a_j)b_j = 1 = a_j\xi(b_j)$ we deduce that $\xi(a_j) = \epsilon(a_j)$ and $\xi(b_j) = \epsilon(b_j)$ for all j. Noting that the sheaf $\mathbb{K}$-algebra maps ξ and ϵ agree on generators, we then have $\xi = \epsilon$. In a like manner, we get $S(a_{(1)}a_{(2)} = \epsilon(a)1$ for all $a \in \mathcal{A}(U_\alpha)$ which proves part (b). Consider the following:

$$\Delta^{op}(a) = \sum a"_k \otimes a'_j, R = \sum a_j \otimes b_j.$$

$$\Delta^{op}(a)(R) = (\sum a''_k \otimes a'_j)(\sum a_j \otimes b_j) = \sum a''_k a_j \otimes a'_k b_j.$$

$$\Delta^{op}(a)(R) \otimes b_j = \sum a''_k a_j \otimes a'_k b_j \otimes b_k b_j.$$

$\Delta(a_j) = \sum a_{j(1)} \otimes a_{j(2)}$. $(R)(\Delta(a_j)) = (\sum a_j \otimes b_j)(\sum a_{j(1)} \otimes a_{j(2)}) = \sum a_j a_{j(1)} \otimes b_j a_{(2)}$. $(R)(\Delta(a_j)) \otimes b_k = (\sum a_j a_{j(1)} \otimes b_j a_{(2)}) \otimes b_j = \sum\sum a_j a_{j(1)} \otimes b_j a_{(2)} \otimes b_{(2)} b_k$. From (iv) we obtain

$$a_k \otimes (\Delta^{op}(a)(R) = \sum a_k a_j \otimes b_k a_l \otimes b_l a_k$$

and $a_j \otimes (R)(\Delta(b_k)) = \sum a_j a_k \otimes a_k b_l \otimes b_l b_k$. Part (c) is now established.
To prove part (d), we assume R_{U_α} satisfies

$$(R_{12})_{U_\alpha}(R_{13})_{U_\alpha}(R_{23})_{U_\alpha} = (R_{23})_{U_\alpha}(R_{13})_{U_\alpha}(R_{12})_{U_\alpha}$$

and $(R_{U_\alpha}$ generates $\mathcal{A}(U_\alpha)$. Therefore the a_j's and the b_j's satisfy the equation

$$(\Delta^{op}(a)(R)_{U_\alpha} = R_{U_\alpha}(\Delta(a)), a \in \mathcal{A}(U_\alpha)$$

whose solutions form a sheaf $\mathbb{K}$-subalgebra of $\mathcal{A}(U_\alpha)$. Since $\mathcal{A}(U_\alpha) = (\mathcal{A}_{[R]})_{U_\alpha}$, part (d) follows. □

Corollary 3.8. *Let $(\mathcal{H}, R)$ be a non-associative quasi-triangular Hopf $\mathbb{K}$-algebra on M with the antipode $S : \mathcal{H} \longrightarrow \mathcal{H}^{op}$. Then $(\mathcal{H}, R, S)$ is a quantum K-algebra sheaf.*

We end this section with a theorem which establishes an important connection between the representation of braid groups and Yang-Baxter $\mathbb{K}$-algebra and coalgebras sheaves.

Theorem 3.9. *Let $\mathcal{A} \in Ob(\mathcal{C}ohAssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$ and $R\colon \mathcal{A}\otimes\mathcal{A} \longrightarrow \mathcal{A}\otimes\mathcal{A}$ an invertible $\mathbb{K}$-algebra sheaf endomorphism such that the pair $(\mathcal{A}, R)$ is a Yang-Baxter $\mathbb{K}$-algebra sheaf. Further, assume that R is a Yang-Baxter structure on $\mathcal{A}$, such that for any $k \in \mathbb{N}$, with $k \geq 1$, the $\mathbb{K}$-algebra sheaf multiplication morphisms: $m_k\colon \mathcal{A}^{\otimes k} \longrightarrow \mathcal{A}$ are defined by $m_k(a_1 \otimes \cdots \otimes a_k) := a_1 a_2 \cdots a_k$ for every $a_1, \cdots, a_k \in \mathcal{A}(U_\alpha)$ with $U_\alpha \in Ob(Op(M))$ and the inclusion maps $\imath_{\beta\alpha}\colon U_\beta \hookrightarrow U_\alpha$ if and only if $U_\beta \subset U_\alpha$ in $Op(M)$. Then given any $k, l \in \mathbb{N}$ with $k, l \geq 1$*

$$R(a_1 \cdots a_k \otimes b_1 \cdots \otimes b_l) = (m_l \otimes m_k)\pi(s_{kl})(a_1 \otimes \cdots a_k \otimes b_1 \otimes \cdots \otimes b_l),$$

for all $a_1, \cdots, a_k, b_1, \cdots, b_l \in \mathcal{A}(U_\alpha)$, where $\pi\colon \mathbb{B}_{k+l} \longrightarrow End(\mathcal{A}^{\otimes(k+l)})$ is the representation of the braid group $\mathbb{B}_{k+l}$ on $\mathcal{A}^{\otimes(k+l)}$ determined by R.

Proof. Let $U_\alpha \in Ob(Op(M))$ and the inclusion maps $\imath_{\beta\alpha}\colon U_\beta \hookrightarrow U_\alpha$, if $U_\beta \subset U_\alpha$ in $Op(M)$. We consider the $\mathbb{K}$-algebra presheaf multiplication morphisms $(m_k)_{(U_\alpha)}\colon \mathcal{A}(U_\alpha)^{\otimes k} \longrightarrow \mathcal{A}(U_\alpha)$ such that the diagram

$$\begin{array}{ccc} \mathcal{A}(U_\alpha)^{\otimes k} & \xrightarrow{(m_k)_{(U_\alpha)}} & \mathcal{A}(U_\alpha) \\ {\scriptstyle (\rho_{U_\alpha U_\beta})^{\otimes k}}\downarrow & & \downarrow{\scriptstyle \rho_{U_\alpha U_\beta}} \\ \mathcal{A}(U_\beta)^{\otimes k} & \xrightarrow{(m_k)_{(U_\beta)}} & \mathcal{A}(U_\beta) \end{array}$$

is commutative.

We will prove

$$R_{U_\alpha}((m_k)_{(U_\alpha)} \otimes (m_l)_{(U_\alpha)}) = ((m_l)_{(U_\alpha)} \otimes (m_k)_{(U_\alpha)})\pi(s_{kl})_{U_\alpha} \tag{3.2}$$

by induction for all $k, l, \in \mathbb{N}$ with $k, l \geq 1$ where

$$R_{U_\alpha} : \mathcal{A}(U_\alpha) \otimes \mathcal{A}(U_\alpha) \longrightarrow \mathcal{A}(U_\alpha) \otimes \mathcal{A}(U_\alpha)$$

is the invertible $\mathbb{K}$-algebra presheaf endomorphism. Observe that if $k, l = 1$, the result is immediate i.e. equation (3.2) implies simply that $R_{U_\alpha} = R_{U_\alpha}$. Assume inductively that:

$$R_{U_\alpha}((m_k)_{(U_\alpha)} \otimes (m_l)_{(U_\alpha)}) = ((m_l)_{(U_\alpha)} \otimes (m_k)_{(U_\alpha)})\pi(s_{kl})_{U_\alpha}$$

holds for all $k \leq K$ and $l \leq L$. It is then enough to prove that for such k, l, we obtain

$$R_{U_\alpha}((m_{k+1})_{(U_\alpha)} \otimes (m_l)_{(U_\alpha)}) = ((m_l)_{(U_\alpha)} \otimes (m_{k+1})_{(U_\alpha)})\pi(s_{k+1,l})_{U_\alpha}\,. \tag{3.3}$$

Similarly

$$R_{U_\alpha}((m_k)_{(U_\alpha)} \otimes (m_{l+1})_{(U_\alpha)}) = (m_{l+1})_{(U_\alpha)} \otimes (m_k)_{(U_\alpha)})\pi(s_{k,l+1})_{U_\alpha}\,. \qquad (3.4)$$

We only prove equation (3.3) as equation (3.4) is analogous. By Theorem (3.6) we have

$$R_{U_\alpha}((m_{k+1})_{(U_\alpha)}\otimes(m_l)_{(U_\alpha)}) = R((m_k)_{(U_\alpha)}\otimes 1_{\mathcal{A}(U_\alpha)})(1_{\mathcal{A}(U_\alpha)}\otimes(m_k)_{(U_\alpha)}\otimes(m_l)_{(U_\alpha)})$$
$$= (1_{\mathcal{A}_(U_\alpha)} \otimes (m_1)_{(U_\alpha)})(R_{23}R_{12})(1_{\mathcal{A}(U_\alpha)} \otimes (m_k)_{(U_\alpha)} \otimes (m_l)_{(U_\alpha)})$$
$$= (1_{\mathcal{A}(U_\alpha)}\otimes(m_1)_{(U_\alpha)})(1_{\mathcal{A}(U_\alpha)}\otimes(R_{U_\alpha}))(R_{U_\alpha}\otimes 1_{\mathcal{A}(U_\alpha)})(1_{\mathcal{A}(U_\alpha)}\otimes(m_k)_{(U_\alpha)}\otimes(m_l)_{(U_\alpha)}),$$

so by the induction hypothesis

$$R_{U_\alpha}((m_{k+1})_{(U_\alpha)} \;\otimes\; (m_l)_{(U_\alpha)}) = (1_{\mathcal{A}(U_\alpha)} \otimes (m_1)_{(U_\alpha)})$$
$$\times (R_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)})(1_{\mathcal{A}(U_\alpha)} \otimes (m_k)_{(U_\alpha)} \otimes (m_l)_{(U_\alpha)})(1_{\mathcal{A}(U_\alpha)} \otimes \pi(s_{kl})),$$

and the inductive hypothesis also implies that

$$R_{U_\alpha}(1_{\mathcal{A}} \otimes (m_k)_{(U_\alpha)}) = ((m_k)_{U_\alpha} \otimes 1_{\mathcal{A}(U_\alpha)}\pi(s_{kl})),$$

hence

$$R_{U_\alpha}((m_{k+1})_{(U_\alpha)} \otimes (m_l)_{(U_\alpha)})$$
$$= (1_{\mathcal{A}(U_\alpha)} \otimes (m_1)_{(U_\alpha)})((m_l)_{(U_\alpha)} \otimes (m_k)_{(U_\alpha)})(\pi(s_{1l}) \otimes 1_{\mathcal{A}})(1_{\mathcal{A}} \otimes \pi(s_{kl}))$$
$$= ((m_l)_{(U_\alpha)} \otimes (m_{k+1})_{(U_\alpha)})\pi(s_{k+1,l})$$

as desired. Applying sheafification at each step, we obtain the sheaf form of the results. □

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