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AN ORTHOGONAL TYPE FORMULA JACOBI POLYNOMIALS AND COMPUTATION OF RELATED SPECIAL FUNCTIONS

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Abstract: The most commonly used methods for computing special functions are those based on polynomials, rational approximation, or series expansions. When there exist recurrence relations, the higher order functions are usually calculated using a recursion method. In this paper, a new orthogonal type formula for the Jacobi polynomials is given with elementary multiplier functions used for computation of special functions $f(x, a, b)$.

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1. Introduction

The Jacobi polynomials are orthogonal polynomials [1, p. 285, (5) and (9)] over the interval $(-1, 1)$ with respect to the weight function $(1-x)^a (1+x)^b$, if $\operatorname{Re} a > -1, \operatorname{Re} b > -1$. In fact in order to make the weight function non-negative and integrable, we assume $\operatorname{Re} a > -1, \operatorname{Re} b > -1$. However, many of the formal results are valid without these restrictions [2, p. 274]. This remark of Luke [2] is true in case of the results given in this paper (cf [3]).

In this paper, we present a new orthogonal type formula for the Jacobi

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polynomials over the interval $(1,\infty)$ with respect to the multiplier function $(x-1)^{a+1}(x+1)^b$, if $\operatorname{Re} a > -2, \operatorname{Re} b > 0, \operatorname{Re}(a+b) < -(m+n+2)$. In order to make the weight function non-negative and integrable, we have $\operatorname{Re} a > -2, \operatorname{Re} b > 0, \operatorname{Re}(a+b) < -(m+n+2)$. The Jacobi polynomials are defined by the relation [4, p. 170 (16)]:

$$P_n^{(a,b)}(x) = \frac{(1+a)_n}{n\,!} 2F_1\left(-n, n+a+b+1; 1+a, \frac{1-x}{2}\right). \tag{1}$$

Provided a is not a negative integer. The following formulas are required in the proof. The modified form of the integral [1, p. 201, (6)]:

$$\int_1^\infty (x-1)^{w-1}(x+1)^b\, dx = 2^{w+b}\frac{\Gamma(w)\Gamma(-b-w)}{\Gamma(-b)},$$

$$\operatorname{Re} b < 0, \operatorname{Re} w > 0, \quad \operatorname{Re}(w+b) < 0. \tag{2}$$

The Saalschutz's Theorem [5, p. 188, (3)]:

$$3F_2\left[\begin{matrix} -n, a, b & ; 1 \end{matrix}\right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}. \tag{3}$$

The modified form of the relation is (see [5, p. 3 (4)])

$$\Gamma(1+a-n) = \frac{(-1)^n \Gamma(1+a)}{(-a)\, n}. \tag{4}$$

The following formula is (see [5, p. 3 (6)])

$$\sin xz = \frac{\pi}{\Gamma(z)\Gamma(z-1)}. \tag{5}$$

2. Orthogonal Type Formula

Theorem. *The orthogonal type formula to be established is*

$$\int_1^\infty (x-1)^{a+1}(x+1)^b p_m^{(a,b)}(x)\, p_n^{(a,b)}(x)\, dx = 0 \ \ \textit{if} \ \ m = n-1, n\,, \tag{6}$$

$$\frac{(1+a)_m}{m!}\frac{(1+a)_n}{n!}\int_1^\infty (x-1)^{a+1}(x+1)^b$$

$$\times\, 2^{F_1(-m,m+a+b+1;1+a,\frac{1-x}{2})} 2^{F_1(-n,n+a+b+1;1+a,\frac{1-x}{2})} dx$$

$$\begin{cases} = \frac{2^{a+b+2}\Gamma(1+a+n)\Gamma(1+b+n)\Gamma(a+b+2n-1)\sin\pi(1+b)}{(n-1)!\Gamma(a+b+n)\Gamma(a+b+2n+2)\sin\pi(2+a+b)} & (\textit{if } m=n-1) \\ = \frac{2^{a+b+2}\Gamma(1+a+n)\Gamma(a+b+n)}{n!\Gamma(1+a=b+2n)\Gamma(1+a=b+n)}\cdot\frac{\sin\pi(1+b)}{\sin\pi(3+a+b)}\left\{\frac{n(n+a)}{(2n+a+b)}\right. & \\ \left.+\frac{(n+1)(n+a+1)}{(2n+a+b+2)}\right\} \qquad (\textit{if } m=n) & \\ = \frac{2^{a+b+2}\Gamma(2+a+n)\Gamma(2+b+n)\Gamma(a+b+2n+1)\sin\pi(1+b)}{n!\Gamma(a+b+n+1)\Gamma(a+b+2n+4)\sin\pi(2+a+b)} & (\textit{if } \ m=n+1)\,, \end{cases} \tag{7}$$

where $Re\,a > -2$, $Re\,b < 0$, $Re\,(a+b) < -(2+m+n)$, *provided* $(1+b)$, *and* $(3+a+b)$ *are nonintegrals.*

Proof. In view of (1), the integral (2) can be written as

$$= \frac{(1+a)_m}{m!}\frac{(1+a)_n}{n!}\sum_{r=0}^{m}\frac{(-m)_r(m+a+b+1)_r(-1)^r}{(1+a)_r r!2^r}$$

$$\times\sum_{u=0}^{n}\frac{(-n)_u(n+a+b+1)_u(-1)^u}{(1+a)_u u!\,2^u}x\int_1^\infty (x-1)^{a+1+r+u}\,(x+1)^b\,dx\,. \tag{8}$$

Evaluating the last integral in (11) with the help of the integral (2) then using (4) and simplifying, the right hand side of (11) becomes

$$\frac{(1+a)_m\,(1+a)_n}{m!\,n!}\frac{2^{a+b+2}}{\Gamma(-b)}\sum_{r=0}^{m}\frac{(-m)_r\,(m+a+b+1)_r\,(-1)^r}{(1+a)_r\ r!}x\Gamma(a+r+2)$$

$$\Gamma(-2-a-b-r)\ 3F_2\begin{bmatrix} -n & n+a+b+1 & 2+a+r \\ 1+a & 3+a+b+r & \end{bmatrix}. \tag{9}$$

Now applying Saalschutz's Theorem (3) to (12), we get

$$\frac{(1+a)_m\,2^{a+b+2}}{m!\,n!\,\Gamma(-b-n)}$$

$$\sum_{r=0}^{m}\frac{(-m)_r\,(-1-r)_n\,(1+a+b+m)_r\,\Gamma(a+r+2)\,\Gamma(-2-a-b-r-n)}{(1+a)_r\ r!}$$

$$\times(-1)^r\,. \tag{10}$$

If $r < n-1$, the numerator of (13) vanishes and since r runs from 0 to m, it follows that (13) vanishes when $m < n-1$. By symmetry, it will be seen that (13) also vanishes when $n < m$. Now, it is clear that for $m \neq n-1$, $n, n+1$ all terms of the series (13) vanish, which proves (6). When $m = n-1$, using the standard result $(-n)_n = (-1)nn!, (1.4)$, we obtain

$$\int_1^\infty (x-1)^{a+1}\,(x+1)^b\,p_{n-1}^{(a,b)}(x)\,p_n^{(a,b)}(x)\,dx =$$

$$\frac{2^{a+b+2}\Gamma\left(1+a+n\right)\Gamma\left(1+b+n\right)\Gamma\left(a+b+2n-1\right)\Gamma\left(2+a+b\right)\Gamma\left(-1-a-b\right)}{(n-1)!\,\Gamma\left(a+b+n\right)\Gamma\left(a+b+2n+2\right)\Gamma\left(1+b\right)\Gamma\left(-b\right)}\,. \qquad (11)$$

By applying (5) to (14), we get (7). When $m = n$, using the standard results like $(-n)_{n-1} = (-1)^{n-1}n!$ and $(-n-1)_n = (-1)^n\,(n+1)!$ and adding the resulting two terms $(r = n-1, n)$ and simplifying, we have

$$\int_1^\infty (x-1)^{a+1}\,(x+1)^b \left\{p_n^{(a,b)}\right\}^2 dx$$
$$= \frac{2^{a+b+2}\Gamma\left(a+n+1\right)\Gamma\left(1+b+n\right)\Gamma\left(-2-a-b\right)\Gamma\left(3+a+b\right)}{n!\,\Gamma\left(a+b+n+1\right)\,\left(a+b+2n+1\right)\,\Gamma\left(-b\right)\Gamma\left(1+b\right)}$$
$$\left[\frac{(n+1)(a+n+1)}{a+b+2n+2}+\frac{n(n+a)}{a+b+2n}\right] \qquad (12)$$

On applying (5) to (15) we obtain (8). Similarly, we can find the value of the integral (9). □

3. Finite Fourier Jacobi Series

Based on the relations (7) and (8), we can obtain expansion of arbitrary polynomials, or functions in general, in finite series of Jacobi polynomials. Specially, if $f(x)$ is a suitable function defined for all x, we consider expansion in the from

$$f(x) = \sum_{m=0}^{n} A_m\,(x+1)^{-m}\,p_m^{(a,b)}\,(x)\,dx\,, \qquad (13)$$

where the Fourier coefficients are given by

$$A_m = \frac{m!\Gamma\left(1+a+b+m\right)\Gamma\left(1-b\right)\Gamma\left(1+b\right)}{2^{a+b}\Gamma\left(1+a+m\right)\Gamma\left(1+b+m\right)\Gamma\left(-a-b\right)\Gamma\left(1+a+b\right)}$$
$$x\int_1^\infty f\left(x\right)(x-1)^a\,(x+1)^{m+b-1}\,p_m^{(a,b)}\,(x)\,dx\,. \qquad (14)$$

4. Computation of the Values of Special Function

In view of (13), consider

$$f(n,a,b) = I_{n-1,n}\,J_{n-1} + I_n, n\,J_n + I_{n+1,}n\,J_{n+1}\,, \qquad (15)$$

where $I_{n-1,n}$, $I_{n,n}\,and\,I_{n+1,n}$ are defined as follows:

To solve the above system (15), we shall take

$$I_{n-1,n} = \frac{2^{a+b+2}\Gamma(1+a+n)\Gamma(1+b+n)\Gamma(a+b+2n-1)\sin\pi(1+b)}{(n-1)!\,\Gamma(a+b+n)\Gamma(a+b+2n+2)\,\sin\pi(2+a+b)} \quad \text{if} \quad m = n-1 \tag{16}$$

$$I_{n,n} = \frac{2^{a+b+2}\Gamma(1+a+n)\Gamma(a+b+n)}{n!\Gamma(1+a=b+2n)\Gamma(1+a=b+n)} \cdot \frac{\sin\pi(1+b)}{\sin\pi(3+a+b)} \left[\frac{n(n+a)}{(2n+a+b)} + \frac{(n+1)(n+a+1)}{(2n+a+b+2)}\right] \quad \text{if} \quad m = n \tag{17}$$

$$I_{n+1,n} = \frac{2^{a+b+2}\Gamma(2+a+n)\Gamma(2+b+n)\Gamma(a+b+2n+1)\,\sin\pi(1+b)}{n!\Gamma(a+b+n+1)\Gamma(a+b+2n+4)\,\sin\pi(2+a+b)} \quad \text{if} \quad m = n+1 \tag{18}$$

$$f(n,a,b) = \frac{2^{a+b+2}\Gamma(1+a+n)\Gamma(1+b+n)\Gamma(a+b+2n-1)\,\sin\pi(1+b)}{(n-1)!\,\Gamma(a+b+n)\,\Gamma(a+b+2n+2)\,\sin\pi(2+a+b)}. \tag{19}$$

We assume that $a = 0$, $b = -\frac{1}{2}$ and $n = 1, 2, \ldots$. If we truncate to the value of $n = 10$ (for example) then the system (15) can be written as,

$$\begin{bmatrix} f(1,0) \\ f(2,0) \\ f(3,0) \\ \vdots \\ f(12,0) \end{bmatrix} = \begin{bmatrix} I_{0,1} & I_{1,1} & I_{2,1} & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ 0 & I_{0,1} & I_{0,1} & I_{0,1} & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & I_{23} & I_{33} & I_{43} & 0 & \vdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & I_{9,10} & I_{10,10} & I_{11,10} \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ \vdots \\ J_{12} \end{bmatrix}$$

which satisfies the following formula:

$$J_i = \frac{1}{I_{i,i+1}}\left[f(i+1,a,b) - I_{i+1,i+1}\,J_{i+1} - I_{i+2,i+1}\,J_{i+2}\right];\ i = 9, 8, \ldots, 0 \tag{20}$$

where

$$f(i+1,0,-1/2) = \frac{-2^{3/2}(i+1)(i+1/2)}{(2i+1/2)(2i+3/2)(2i+5/2)},$$

$$I_{j+1,j+1} = \frac{-2^{3/2}}{\pi(2i+3/2)}\left[\frac{(i+1)^2}{(2i+3/2)} + \frac{(i+2)}{(2i+7/2)}^2\right],$$

$$I_{j+2,j+1} = \frac{-2^{3/2}(i+3/2)(i+2)}{(2i+5/2)(2i+7/2)(2i+9/2)}, \qquad \text{and}$$

$$I_{j+2,j+1} = \frac{-2^{3/2}(i+1)(i+1/2)}{(2i+5/2)(2i+3/2)(2i+1/2)}.$$

A program of evaluating the coefficients can be written using *Matlab*. Since the *Matlab* does not provide index of zero, so we are going to shift only the index of the formula (20) to

$$j_{i+1} = \frac{1}{I_i + 1, i+2}[f(i+2,a,b) - I_{i+2,i+2}.j_{i+2} - J_{i+3,i+2}J_{i+3}]\,, \qquad i = 0,1,\ldots,8. \quad (21)$$

$J_0 \approx J_1, J_1 \approx J_2$ ect. But the formulas for I's, f and J do not change. The formula (21) is a 3-term recurrence relation for the coefficients J_{i+1}. All nine coefficients $J_{i+1}(i = 0,\ldots,8)$ are determined in terms of J_{10}, J_{11}. Noting that our truncation scheme means that $J_{12} = J_{13} \approx 0$, hence from equation (21) by putting $n = 9, 10$ We have the following equations sufficient to determine J_{10}, J_{11}:

$$J_{11} = \frac{1}{I_{11,12}} f(12,a,b)\,, \quad J_{10} = \frac{1}{I_{10,11}}\left[f(11,a,b) - I_{11,11}J_{11}\right]. \qquad (22)$$

Thus the system (20) with (22) given all $J_1,\ldots,J_9$ are determined numerically and *self-consistently.*

5. Conclusion

A finite Fourier Jacobi series expansion is given for arbitrary polynomial function. A truncation scheme is adopted and the 3-term recurrence relation for the expansion coefficients is solved self-consistently.

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