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(ω)BITOPOLOGICAL SPACES

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Abstract: Considering two increasing sequences of topologies on a set, we introduce the notion of a (ω)bitopological space. Different types of compactness are defined and studied.

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1. Introduction

A set X equipped with an increasing sequence $\mathcal{T} = \{\mathcal{T}_n\}$ of topologies on it is called a (ω)*topological space* (Bose and Tiwari [1]) and is denoted by $(X, \mathcal{T})$. A set G is called (ω)*open* if $G \in \bigcup_n \mathcal{T}_n$, and a set F is (ω)*closed* if $X - F$ is (ω)open. The union and intersection of a finite number of (ω)open sets are (ω)open. The union of a countable number of (ω)open sets may not be (ω)open. These are called $(\sigma\omega)$*open* sets. Since the union of an arbitrary number of $(\mathcal{T}_n)$open sets is $(\mathcal{T}_n)$open, the union of an arbitrary number of (ω)open sets is expressible as the union of a countable number of (ω)open sets, and hence it is a $(\sigma\omega)$open set. Similarly, $(\delta\omega)$*closed* sets are defined as the intersection of countable number of (ω)closed sets. Kelly [4] initiated the study of the notion of a bitopological

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space which is a set equipped with two topologies. Further works on bitopology were done by Lane [7], Kim [5], Fletcher, Hoyle III and Patty [3], Reilly [10], Cooke and Reilly[2] and others.

By considering two increasing sequences of topologies on a set, we define a (ω)bitopological space as follows.

Definition 1.1. Let X be a nonempty set. If $\mathcal{P} = \{\mathcal{P}_n\}$ and $Q = \{Q_n\}$ be two sequences of topologies on X with $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ and $Q_n \subset Q_{n+1}$ for all n, then X is called a (ω)*bitopological space* and is denoted by $(X, \mathcal{P}, Q)$ or $(X, \{\mathcal{P}_n\}, \{Q_n\})$.

We study some properties of (ω)bitopological spaces concerning compactness, local compactness and paracompactness.

If $\mathcal{P} = Q$, then $(X, \mathcal{P}, Q)$ is the (ω)topological space $(X, \mathcal{P})$. If $\mathcal{P}_n = \mathcal{P}_m$ and $Q_n = Q_m$ for all $n, m \in N$, then $\mathcal{P} = \{\mathcal{P}_n\}$ and $Q = \{Q_n\}$ may be considered as topologies on X, and so $(X, \mathcal{P}, Q)$ becomes a bitopological space.

Throughout the paper, unless otherwise mentioned, X denotes the (ω) bitopological space $(X, \mathcal{P}, Q)$ where $\mathcal{P} = \{\mathcal{P}_n\}$ and $Q = \{Q_n\}$ are two increasing sequences of topologies on X. For any set $A \subset X$, $(\tau)\text{cl}A$ denotes the closure of A with respect to the topology τ on X and $\tau|A$ denotes the subspace topology of τ on A. If $\mathcal{A}$ is a class of subsets of X, then $\mathcal{T}(\mathcal{A})$ denotes the smallest topology on X containing $\mathcal{A}$. R and N denote the set of real numbers and the set of natural numbers respectively, and l, m, n, etc. denote elements of N. For definitions and notations which are not explained here, the reader is referred to Willard [11].

A set $\in \underset{n}{\cup}\mathcal{P}_n$ (resp. $\underset{n}{\cup}Q_n$) is said to be $(\omega_{\mathcal{P}})$*open* (resp. (ω_Q)*open*). Similarly, we define $(\sigma\omega_{\mathcal{P}})$*open sets*, $(\delta\omega_{\mathcal{P}})$*closed sets* etc.

The (ω)topological space $(X, \mathcal{P})$ is called $(\omega_{\mathcal{P}})$*compact* [1] if every $(\omega_{\mathcal{P}})$open cover of X has a finite subcover. If $(X, \mathcal{P})$ is $(\omega_{\mathcal{P}})$compact, then any $(\omega_{\mathcal{P}})$closed subset of X is $(\omega_{\mathcal{P}})$compact. $(X, \mathcal{P})$ is said to be *locally* $(\omega_{\mathcal{P}})$*compact* [1] if for each point $x \in X$, there exists an n such that for some $(\mathcal{P}_n)$open neighbourhood U of x, $(\mathcal{P}_n)\text{cl}U$ is $(\omega_{\mathcal{P}})$compact. If $\mathcal{T}$ is a topology on X, then a collection $\mathcal{C}$ of subsets of X is said to be $(\mathcal{T})$*locally finite* if each $x \in X$ has a $(\mathcal{T})$open neighbourhood intersecting only a finite number of elements of $\mathcal{C}$. $\mathcal{C}$ is said to be σ-$(\mathcal{T})$*locally finite* if $\mathcal{C} = \overset{\infty}{\underset{k=1}{\cup}} \mathcal{C}_k$, where each $\mathcal{C}_k$ is a $(\mathcal{T})$locally finite collection. $(X, \mathcal{P})$ is said to be $(\omega_{\mathcal{P}})$*paracompact* [1] if each $(\omega_{\mathcal{P}})$open cover of X has a $(\mathcal{P}_n)$locally finite $(\mathcal{P}_n)$open refinement for some n. For a set $A \subset X$, $(\omega_{\mathcal{P}})\text{cl}A$ is the intersection of all $(\omega_{\mathcal{P}})$closed sets containing A. It is clear that $(\omega_{\mathcal{P}})\text{cl}A$

is $(\delta\omega_{\mathcal{P}})$closed set.

$(X, \mathcal{P})$ is said to be $(\omega_{\mathcal{P}})$*Hausdorff* [1] if for every pair of distinct points x and y of X, there exists an n such that for some $U, V \in \mathcal{P}_n$, we have $x \in U$, $y \in V$ and $U \cap V = \phi$.

If there is no scope of confusion, we omit the subscript from $(\omega_{\mathcal{P}})$open, $(\omega_{\mathcal{P}})$closed, $(\omega_{\mathcal{P}})$compact, etc.

We require the following theorem.

Theorem 1.1. (Theorem 3.6, [1]) *If the (ω)topological space $(X, \mathcal{P})$ is (ω)Hausdorff and $K \subset X$ is (ω)compact, then K is a $(\delta\omega)$closed set.*

2. Pairwise (ω)Compactness and Pairwise Local (ω)Compactness

A cover $\mathcal{U}$ of X is said to be *pairwise* (ω)*open* if $\mathcal{U} \subset (\underset{n}{\cup}\mathcal{P}_n) \cup (\underset{n}{\cup}Q_n)$ and $\mathcal{U}$ contains at least one nonempty set from each of $\underset{n}{\cup}\mathcal{P}_n$ and $\underset{n}{\cup}Q_n$. X is said to be *pairwise* (ω)*compact* if every pairwise (ω)open cover of X has a finite subcover.

$\mathcal{P}$ is said to be *locally* (ω)*compact with respect to* Q if for each point $x \in X$, there exists an n such that for some $(\mathcal{P}_n)$open set G containing x, $(Q_n)\mathrm{cl}G$ is pairwise (ω)compact. X is said to be *pairwise locally* (ω)*compact* if $\mathcal{P}$ and Q are locally (ω)compact with respect to each other.

X is said to be *pairwise* (ω)*Hausdorff* if for each pair of distinct points x and y of X, there exists an n such that for some $U \in \mathcal{P}_n$ and $V \in Q_n$, we have $x \in U$, $y \in V$ and $U \cap V = \phi$. $\mathcal{P}$ is said to be (ω)*regular with respect to* Q if for each $x \in X$ and each $(\omega_{\mathcal{P}})$closed set A with $x \notin A$, there exists an n such that for some $U \in \mathcal{P}_n$ and $V \in Q_n$, we have $x \in U$, $A \subset V$ and $U \cap V = \phi$. X is said to be *pairwise* (ω)*regular* if $\mathcal{P}$ is (ω)regular with respect to Q, and Q is (ω)regular with respect to $\mathcal{P}$. It is easy to see that $\mathcal{P}$ is (ω)regular with respect to Q iff for each point $x \in X$ and a $(\omega_{\mathcal{P}})$open set G with $x \in G$, there exists an n such that for some $(\mathcal{P}_n)$open set H, we have $x \in H \subset (Q_n)\mathrm{cl}H \subset G$. X is said to be *pairwise* (ω)*normal* if for each pair of a $(\omega_{\mathcal{P}})$closed set A and a (ω_Q)closed set B with $A \cap B = \phi$, there exists an n such that for some $U \in Q_n$ and $V \in \mathcal{P}_n$, we have $A \subset U$, $B \subset V$ and $U \cap V = \phi$. Equivalently, X is pairwise (ω)normal if for any (ω_Q)closed set A and any $(\omega_{\mathcal{P}})$open set G with $A \subset G$, there exists an n such that for some $(\mathcal{P}_n)$open set U, we have $A \subset U \subset (Q_n)\mathrm{cl}U \subset G$.

$\mathcal{P}$ is said to be (ω) R_1 *with respect to* Q if for any two points $x, y \in X$ with

$x \notin (\omega_{\mathcal{P}})\mathrm{cl}\{y\}$, there exists an n such that for some $U \in \mathcal{P}_n$ and $V \in Q_n$, we have $x \in U$, $y \in V$ and $U \cap V = \phi$. X is said to be *pairwise* $(\omega)\, R_1$ if $\mathcal{P}$ is $(\omega)\, R_1$ with respect to Q and Q is $(\omega)\, R_1$ with respect to $\mathcal{P}$.

Now we give some examples of (ω)bitopological spaces. We denote by τ_n the power set of $\{1, 2, \ldots, n\}$, and by σ_n the power set of $\{-1, -2, \ldots, -n\}$.

Example 2.1. Let $\mathcal{T}_1$ be the topology of countable complements on R and $\mathcal{T}_2$ be the discrete topology on R. We write $\mathcal{P}_n = \mathcal{T}_1 \cup \tau_n$ and $Q_n = \{\phi, R\} \cup \{G \cap (-n,\ n) | G \in \mathcal{T}_2\}$. Then the (ω)bitopological space $(R, \{\mathcal{P}_n\}, \{Q_n\})$ is pairwise (ω)Hausdorff but it is not a pair of (ω)Hausdorff spaces.

Example 2.2. Let $\mathcal{U}$ denote the usual topology on R. If $\mathcal{P}_n = \mathcal{T}(\mathcal{U} \cup \tau_n)$ and $Q_n = \mathcal{T}(\mathcal{U} \cup \sigma_n)$, then the (ω)bitopological space $(R, \{\mathcal{P}_n\}, \{Q_n\})$ is pairwise (ω)Hausdorff, and both $(R, \{\mathcal{P}_n\})$ and $(R, \{Q_n\})$ are (ω)Hausdorff.

Example 2.3. Let $\mathcal{T}$ be the discrete topology on R, and $\mathcal{U}$ be the usual topology on R. If $\mathcal{P}_n = \{\phi, R\} \cup \{G \cap (-n,\ n) | G \in \mathcal{U}\}$ and $Q_n = \{\phi, R\} \cup \{G \cap (-n,\ n) | G \in \mathcal{T}\}$, then $(R, \{\mathcal{P}_n\}, \{Q_n\})$ is pairwise (ω)Hausdorff but for no n, the bitopological space $(R, \mathcal{P}_n, Q_n)$ is pairwise Hausdorff ($Kelly$, [4]). Both $(R, \{\mathcal{P}_n\})$ and $(R, \{Q_n\})$ are (ω)Hausdorff.

Example 2.4. Let $\mathcal{S}_1$ denote the left hand topology on R generated by the base $\{\phi\} \cup \{(-\infty, x) | x \in R\}$, $\mathcal{S}_2$ denote the right hand topology on R generated by the base $\{\phi\} \cup \{(x, \infty) | x \in R\}$. If $\mathcal{P}_n = \{\phi, R\} \cup \{G \cap (-\infty,\ n) | G \in \mathcal{S}_1\}$ for all n and $Q_n = \{\phi, R\} \cup \{G \cap (-n, \infty) | G \in \mathcal{S}_2\}$, then the (ω)bitopological space $(R, \{\mathcal{P}_n\}, \{Q_n\})$ is pairwise (ω)compact, and hence pairwise locally (ω)compact. But $(R, \{\mathcal{P}_n\})$ and $(R, \{Q_n\})$ are not (ω)compact. Even they are not locally (ω)compact.

Example 2.5. Let X be the set of all non-negative real numbers, $\mathcal{T}$ be the usual topology on X and $\mathcal{P}_n = \{\phi, X\} \cup \{G \cap [0, n) | G \in \mathcal{T}\}$. Also let $Q_n = \{\phi\} \cup \{G \cup (n, \infty) | G \in \mathcal{T}\}$. Then the (ω)bitopological space $(X, \{\mathcal{P}_n\}, \{Q_n\})$ is pairwise (ω)Hausdorff and pairwise (ω)compact. But $\{\mathcal{P}_n\} \neq \{Q_n\}$.

Theorem 2.1. *(i) If X is pairwise (ω)compact and K is a $(\delta\omega_{\mathcal{P}})$closed or a $(\delta\omega_Q)$closed subset of X, then K is pairwise (ω)compact.*

(ii) X is pairwise (ω)compact iff each $(\delta\omega_{\mathcal{P}})$closed (resp. $(\delta\omega_Q)$closed) proper subset of X is (ω_Q)compact (resp. $(\omega_{\mathcal{P}})$compact).

The proof is straightforward and is omitted.

Corollary 2.1. *If X is pairwise (ω)compact, and if the (ω)topological spaces $(X, \mathcal{P})$ and (X, Q) are $(\omega_{\mathcal{P}})$Hausdorff and (ω_Q)Hausdorff respectively,* *then each* ($\sigma\omega_{\mathcal{P}}$)*open* (*resp.* ($\sigma\omega_Q$)*open*) *subset of* X *is* ($\sigma\omega_Q$)*open* (*resp.* ($\sigma\omega_{\mathcal{P}}$)*open*).

Proof. Let A be a ($\sigma\omega_{\mathcal{P}}$)open proper subset of X. Then $B = X - A$ is ($\delta\omega_{\mathcal{P}}$)closed. Hence by Theorem 2.1, B is (ω_Q)compact. Since (X, Q) is (ω_Q)Hausdorff, by Theorem 1.1, B is ($\delta\omega_Q$)closed, and hence A is ($\sigma\omega_Q$)open. □

Theorem 2.2. *Let* X *be pairwise* (ω)*Hausdorff. If* $K \subset X$ *is* ($\omega_{\mathcal{P}}$)*compact* (*resp.* (ω_Q)*compact*), *then* K *is* ($\delta\omega_Q$)*closed* (*resp.* ($\delta\omega_{\mathcal{P}}$)*closed*).

Proof. Suppose K is ($\omega_{\mathcal{P}}$)compact proper subset of X. If $x \in K$ and $y \in X - K$, then there exists an $n_{xy} \in N$ such that for some $U_{xy} \in \mathcal{P}_{n_{xy}}$ and $V_{xy} \in Q_{n_{xy}}$, we have $x \in U_{xy}$, $y \in V_{xy}$ and $U_{xy} \cap V_{xy} = \phi$. The collection $\{U_{xy} | x \in K\}$ forms a ($\omega_{\mathcal{P}}$)open cover of K. Therefore there exists a finite subcover $\{U_{x_1y}, U_{x_2y}, \ldots, U_{x_ky}\}$ of the cover $\{U_{xy} | x \in K\}$ of K. If $U_y = \bigcup_{i=1}^{k} U_{x_iy}$ and $V_y = \bigcap_{i=1}^{k} V_{x_iy}$, then $K \subset U_y$, $y \in V_y$ and $U_y \cap V_y = \phi$. Now $X - K \subset \cup\{V_y | y \in X - K\} \subset \{X - U_y | y \in X - K\} \subset X - K$. Thus $X - K = \cup\{V_y | y \in X - K\}$, and so $X - K$ is ($\sigma\omega_Q$)open. Therefore K is ($\delta\omega_Q$)closed. □

Corollary 2.2. *If* X *is pairwise* (ω)*Hausdorff, and if the* (ω)*topological spaces* $(X, \mathcal{P})$ *and* (X, Q) *are* ($\omega_{\mathcal{P}}$)*compact and* (ω_Q)*compact respectively, then each* ($\sigma\omega_{\mathcal{P}}$)*open* (*resp.* ($\sigma\omega_Q$)*open*) *subsets of* X *are* ($\sigma\omega_Q$)*open* (*resp.* ($\sigma\omega_{\mathcal{P}}$)*open*).

Theorem 2.3. *If* X *is pairwise* (ω)*compact and pairwise* (ω)*Hausdorff, then* X *is pairwise* (ω)*regular.*

The proof is similar to Theorem 12 in [3].

Theorem 2.4. *If* X *is pairwise* (ω)*compact, and either* $\mathcal{P}$ *is* (ω)*regular with respect to* Q *or* Q *is* (ω)*regular with respect to* $\mathcal{P}$, *then* X *is pairwise* (ω)*normal.*

The proof is similar to Theorem 13 in [3].

Theorem 2.5. *Let* X *be pairwise* (ω) R_1. *If* $K \subset X$ *is* ($\omega_{\mathcal{P}}$)*compact* (*resp.* (ω_Q)*cmpact*), *then* $(\omega_Q)\mathrm{cl}K = \bigcup_{x \in K} (\omega_Q)\mathrm{cl}\{x\}$ (*resp.* $(\omega_{\mathcal{P}})\mathrm{cl}K = \bigcup_{x \in K} (\omega_{\mathcal{P}})\mathrm{cl}\{x\}$).

Proof. Suppose K is ($\omega_{\mathcal{P}}$)compact. Let $y \notin \bigcup_{x \in K} (\omega_Q)\mathrm{cl}\{x\}$ so that $y \notin (\omega_Q)\mathrm{cl}\{x\}$ for all $x \in K$. Then there exists an $n_x \in N$ such that for some $U_{n_x} \in \mathcal{P}_{n_x}$ and $V_{n_x} \in Q_{n_x}$, we have $x \in U_{n_x}$, $y \in V_{n_x}$ and $U_{n_x} \cap V_{n_x} = \phi$. Since K is ($\omega_{\mathcal{P}}$)compact, the ($\omega_{\mathcal{P}}$)open cover $\{U_{n_x} | x \in K\}$ of K has a finite subcover

$U_{n_{x_1}}, U_{n_{x_2}}, \ldots, U_{n_{x_k}}$. If $U = \overset{k}{\underset{i=1}{\cup}} U_{n_{x_i}}$, $V = \overset{k}{\underset{i=1}{\cap}} V_{n_{x_i}}$ and $n = \max\{n_{x_1}, n_{x_2}, ...,$ $n_{x_k}\}$, then U is $(\mathcal{P}_n)$open, V is (Q_n)open, $K \subset U$, $y \in V$ and $U \cap V = \phi$. Then $X-V$ is (Q_n)closed, $y \notin X-V$ and $K \subset X-V$. Therefore $y \notin (Q_n)\mathrm{cl}K$, and so $y \notin (\omega_Q)\mathrm{cl}K$. Hence $(\omega_Q)\mathrm{cl}K \subset \underset{x\in K}{\cup} (\omega_Q)\mathrm{cl}\{x\}$. Also $\underset{x\in K}{\cup} (\omega_Q)\mathrm{cl}\{x\} \subset (\omega_Q)\mathrm{cl}K$ is obviously true. □

Theorem 2.6. *Suppose X is pairwise (ω)Hausdorff. Then $\mathcal{P}$ is locally (ω)compact with respect to Q iff for each point $x \in X$ and $(\omega_{\mathcal{P}})$open set G containing x, there exists an n such that for some $(\mathcal{P}_n)$open set U containing x, we have $(Q_n)\mathrm{cl}U \subset G$ and $(Q_n)\mathrm{cl}U$ is pairwise (ω)compact.*

Proof. We only need to prove the 'only if' part. Let $x \in X$ and G be a $(\omega_{\mathcal{P}})$open set with $x \in G$. There exists an l such that for some $V \in \mathcal{P}_l$ with $x \in V$, we have $K = (Q_l)\mathrm{cl}V$ is pairwise (ω)compact. So the subspace $(K,$ $\{\mathcal{P}_n|K\}$, $\{Q_n|K\})$ is pairwise (ω)compact and pairwise (ω)Hausdorff. Therefore by Theorem 2.3, it is pairwise (ω)regular, and hence there is an m such that for some $W \in \mathcal{P}_m|K$ with $x \in W$, we have $(Q_m|K)\mathrm{cl}W \subset G \cap K$. Let $W = U$ $\cap K$ with $U \in \mathcal{P}_m$. If $H = U \cap V$ and $n = \max\{l, m\}$, then $H \in \mathcal{P}_n$, $x \in H$ and $(Q_n)\mathrm{cl}H = ((Q_n)\,\mathrm{cl}H) \cap K = (Q_n|K)\mathrm{cl}H$. Therefore by Theorem 2.1, $(Q_n)\mathrm{cl}H$ is pairwise (ω)compact. Moreover $(Q_n)\mathrm{cl}H \subset (Q_n|K)\mathrm{cl}W \subset G$. □

Corollary 2.3. *If X is pairwise (ω)Hausdorff and $\mathcal{P}$ is locally (ω)compact with respect to Q, then $\mathcal{P}$ is (ω)regular with respect to Q. Hence if X is pairwise (ω)Hausdorff and pairwise locally (ω)compact, then it is pairwise (ω)regular.*

Theorem 2.7. *Suppose X is pairwise (ω)Hausdorff. If $\mathcal{P}$ is locally (ω)compact with respect to Q, then for a $(\omega_{\mathcal{P}})$compact subset K of X, there is a $(\mathcal{P}_n)$open set G such that $(Q_n)\mathrm{cl}G$ is pairwise (ω)compact and $K \subset G$.*

Proof. Let $x \in K$. Then there exists an $n_x \in N$ such that for some $U_{n_x} \in \mathcal{P}_{n_x}$with $x \in U_{n_x}$, $(Q_{n_x})\mathrm{cl}U_{n_x}$ is pairwise (ω)compact. The collection $\{U_{n_x}|x \in K\}$ is an $(\omega_{\mathcal{P}})$open cover of K. So there exists a finite subcover $U_{n_{x_1}}$, $U_{n_{x_2}}, \ldots, U_{n_{x_k}}$ of K. If $n = \max\{n_{x_1}, n_{x_2}, \ldots, n_{x_k}\}$ and $G = \overset{k}{\underset{i=1}{\cup}} U_{n_{x_i}}$, then $G \in \mathcal{P}_n$ and $K \subset G$. For each i, $(Q_n)\mathrm{cl}U_{n_{x_i}}$ is a (ω_Q)closed subset of $(Q_{n_{x_i}})\mathrm{cl}U_{n_{x_i}}$, and hence by Theorem 2.1, it is pairwise (ω)compact. Therefore $(Q_n)\mathrm{cl}G = \overset{k}{\underset{i=1}{\cup}} (Q_n)\mathrm{cl}U_{n_{x_i}}$ is pairwise (ω)compact. □

Theorem 2.8. *Let $\mathcal{P}$ be locally (ω)compact with respect to Q. If F is a (ω_Q)closed subset of X, then $\{\mathcal{P}_n|F\}$ is locally (ω)compact with respect to $\{Q_n|F\}$. If, in addition, X is pairwise (ω)Hausdorff, then for any $(\omega_{\mathcal{P}})$open* *subset G of X, $\{\mathcal{P}_n|G\}$ is locally (ω)compact with respect to $\{Q_n|G\}$.*

Proof. Let $x \in F$. There exists an n such that for some ($\mathcal{P}_n$)open set H containing x, $C = (Q_n)\text{cl}H$ is pairwise (ω)compact. Since F is (ω_Q)closed, by Theorem 2.1, $F \cap C$ is pairwise (ω)compact. Also $F \cap H$ is ($\mathcal{P}_n|F$)open, $x \in F \cap H$ and $(Q_n|F)\text{cl}(F \cap H) = F \cap (Q_n)\text{cl}(F \cap H) \subset F \cap C$.

If X is pairwise (ω)Hausdorff, then by Corollary 2.3, $\mathcal{P}$ is (ω)regular with respect to Q. Therefore if $y \in G$, then there exists an l such that for some $U \in \mathcal{P}_l$, we have $y \in U \subset (Q_l)\text{cl}U \subset G$. By Theorem 2.6, there exists an m such that for some $V \in \mathcal{P}_m$, we have $y \in V \subset (Q_m)\text{cl}V \subset U$ and $(Q_m)\text{cl}V$ is pairwise (ω)compact. Then $V \cap G \in \mathcal{P}_m|G$, $y \in V \cap G$ and $(Q_m|G)\text{cl}(V \cap G) = (Q_m|G)\text{cl}V = (Q_m)\text{cl}V$ is pairwise (ω)compact. □

3. Pairwise (ω)Paracompactness

A refinement $\mathcal{V}$ of a pairwise (ω)open cover $\mathcal{U}$ is said to be (n)*parallel refinement* if it satisfies the following condition: $V \in \mathcal{V}$ is ($\mathcal{P}_n$)open (resp. (Q_n)open) if V is a subset of a ($\omega_{\mathcal{P}}$)open (resp. (ω_Q)open) set $U \in \mathcal{U}$. A collection $\mathcal{U}$ of subsets of X is said to be *pairwise* (n)*locally finite* if for each $x \in X$, there exist a ($\mathcal{P}_n$)open and a (Q_n)open neighbourhood of x, each intersecting a finitely many $U \in \mathcal{U}$. X is said to be *pairwise* (ω)*paracompact* if for every pairwise (ω)open cover of X, there exists a pairwise (m)locally finite (n)parallel refinement for some m, $n \in N$.

The (ω)bitopological space (X, $\{\mathcal{P}_n\}$, $\{Q_n\}$) of Example 2.5 is pairwise (ω)Hausdorff and pairwise (ω)compact and hence pairwise (ω)paracompact. But $\{\mathcal{P}_n\} \neq \{Q_n\}$.

Theorem 3.1. *If X is pairwise (ω)paracompact, and if A is a ($\omega_{\mathcal{P}}$)closed or a (ω_Q)closed subset of X, then A is pairwise (ω) paracompact. If A is a proper subset of X, and A is ($\omega_{\mathcal{P}}$)closed (resp. (ω_Q)closed), then A is (ω_Q)paracompact (resp. ($\omega_{\mathcal{P}}$)paracompact).*

The proof is omitted.

Theorem 3.2. *If X is pairwise (ω)Hausdorff and pairwise (ω)paracompact, then X is pairwise (ω)regular.*

Proof. It is sufficient to show that $\mathcal{P}$ is (ω)regular with respect to Q. Let $x \in X$ and F be a ($\omega_{\mathcal{P}}$)closed set with $x \notin F$. Since X is pairwise (ω)Hausdorff, for each $y \in F$, there exists an $n_y \in N$ such that for some $U_y \in \mathcal{P}_{n_y}$ and $V_y \in Q_{n_y}$, we have $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \phi$. The collection $\mathcal{V} = \{V_y |\ y \in F\} \cup \{X - F\}$ is a pairwise (ω)open cover of X. Since X is pairwise (ω)paracompact, $\mathcal{V}$ has, for some m, $n \in N$, a pairwise (m)locally finite (n)parallel refinement $\mathcal{W}$. Let $V = \cup\{W \in \mathcal{W} |\ W \cap F \neq \phi\}$. Then $V \in Q_n$. Again there exists a $(\mathcal{P}_m)$open neighbourhood H of x intersecting only a finite number of sets W_1, W_2, $\ldots$, W_k of $\mathcal{W}$ such that for each $i = 1, 2, \ldots, k$, $W_i \cap F \neq \phi$. Let $W_i \subset V_{y_i}$, $y_i \in F$, $i = 1, 2, \ldots, k$. Then $U = H \cap (\bigcap_{i=1}^{k} U_{y_i}) \in \mathcal{P}_l$ and $V \in Q_l$ where $l = \max\{m, n, n_{y_1}, \ldots, n_{y_k}\}$. Also we have $x \in U$, $F \subset V$ and $U \cap V = \phi$. □

Using the above result, we prove the following theorem.

Theorem 3.3. *If X is pairwise (ω)Hausdorff and pairwise (ω)paracompact, then it is pairwise (ω)normal.*

Proof. Let A be a $(\omega_{\mathcal{P}})$closed set and B be a (ω_Q)closed set with $A \cap B = \phi$. Let $x \in B$. By Theorem 3.2, X is pairwise (ω)regular. Therefore there exists an $n_x \in N$ such that for some $U_{n_x} \in Q_{n_x}$ and $V_{n_x} \in \mathcal{P}_{n_x}$, we have $A \subset U_{n_x}$, $x \in V_{n_x}$ and $U_{n_x} \cap V_{n_x} = \phi$. For some n_1, $n_2 \in N$, let $\mathcal{E}$ be a (n_1)locally finite (n_2)parallel refinement of the pairwise (ω)open cover $\{V_{n_x} | x \in B\} \cup \{X - B\}$ of X. Let $V = \cup\{E \in \mathcal{E} | E \cap B \neq \phi\}$. Then $V \in \mathcal{P}_{n_2}$ and $B \subset V$. For each $y \in A$, there exists a (Q_{n_1})open neighbourhood D_y of y that intersects a finite number of sets $E_1(y)$, $\ldots$, $E_k(y)$ of $\mathcal{E}$ with $E_i(y) \cap B \neq \phi$. Let $E_i(y) \subset V_{x_i}$, $i = 1, 2, \ldots, k$ and $G_y = D_y \cap (\bigcap_{i=1}^{k} U_{x_i})$. Then $G_y \in Q_l$ and $y \in G_y$ where $l = \max\{n_1, n_{x_1}, n_{x_2}, \ldots, n_{x_k}\}$. Also $G_y \cap V = \phi$. For some n_3, $n_4 \in N$, let $\mathcal{H}$ be a (n_3)locally finite (n_4)parallel refinement of the pairwise (ω)open cover $\{G_y | y \in A\} \cup \{X - A\}$ and let $U = \cup\{H \in \mathcal{H} | H \cap A \neq \phi\}$. Then $U \in Q_{n_4}$, $U \cap V = \phi$ and $A \subset U$. If $n = \max\{n_2, n_4\}$, then $U \in Q_n$, $V \in \mathcal{P}_n$. □

4. β-Pairwise (ω)Paracompactness

Raghavan and Reilly [9] introduced the notions of α-, β-, γ- and δ-pairwise paracompactness of a bitopological space and proved a δ-pairwise paracompactness version of the Michael's [8] characterization of paracompactness of regular topological spaces. But unfortunately, the proof is not correct (Kovár [6], p. 396). Here we introduce β-pairwise (ω)paracompactness of a (ω)bitipological space and prove an analogue of the Michael's Theorem for this. The corresponding result for pairwise (ω)paracompactness remains as an open question.

X is said to be *β-pairwise (ω)paracompact* if for any m, every $(\mathcal{P}_m)$open (resp. (Q_m)open) cover of X has, for some n, a $(\mathcal{T}(\mathcal{P}_n \cup Q_n))$open $(\mathcal{P}_n)$locally finite (resp. (Q_n)locally finite) refinement.

Theorem 4.1. *Let the bitopological space $(X, \mathcal{P}_n, Q_n)$ be pairwise regular (see Kelly, [4]) for each $n \in N$. Then the following statements are equivalent.*

(i) X is β-pairwise (ω)paracompact.

(ii) For any m, every $(\mathcal{P}_m)$open (resp. (Q_m)open) cover of X has, for some n, a σ-$(\mathcal{P}_n)$locally finite (resp. σ-(Q_n)locally finite) $(\mathcal{T}(\mathcal{P}_n \cup Q_n))$open refinement.

(iii) For any m, every $(\mathcal{P}_m)$open (resp. (Q_m)open) cover of X has, for some n, a $(\mathcal{P}_n)$locally finite (resp. (Q_n)locally finite) refinement (not necessarily open or closed in any sense).

(iv) For any m, every $(\mathcal{P}_m)$open (resp. (Q_m)open) cover of X has, for some n, a $(\mathcal{P}_n)$locally finite (resp. (Q_n)locally finite) $(\mathcal{T}(\mathcal{P}_n \cup Q_n))$closed refinement.

Proof. $(i) \Rightarrow (ii)$. Trivial.

$(ii) \Rightarrow (iii)$. Similar to the proof in Michael [8].

$(iii) \Rightarrow (iv)$. Let $\mathcal{U}$ be a $(\mathcal{P}_m)$open cover of X. For each $x \in X$, choose $U \in \mathcal{U}$ such that $x \in U$. Since $(X, \mathcal{P}_m, Q_m)$ is pairwise regular and U is $(\mathcal{P}_m)$open, there exists a $(\mathcal{P}_m)$open set W_x such that $x \in W_x \subset (Q_m)\mathrm{cl}W_x \subset U$. Then $\{W_x | x \in X\}$ is a $(\mathcal{P}_m)$open cover of X. Therefore it has, for some l, a $(\mathcal{P}_l)$locally finite refinement $\mathcal{H}$. If $n = \max\{l, m\}$ and if for $H \in \mathcal{H}$, $H \subset W_x$, then $(\mathcal{T}(\mathcal{P}_n \cup Q_n))\mathrm{cl}H \subset (\mathcal{T}(\mathcal{P}_n \cup Q_n))\mathrm{cl}W_x \subset (\mathcal{T}(\mathcal{P}_m \cup Q_m))\mathrm{cl}W_x \subset (Q_m)\mathrm{cl}W_x \subset U$. Since $\mathcal{P}_l \subset \mathcal{P}_n$, it follows that $\mathcal{H}$ is $(\mathcal{P}_n)$locally finite. Therefore for $x \in X$, there exists a $(\mathcal{P}_n)$open set O containing x such that $O \cap H = \phi$ and hence $O \cap (\mathcal{P}_n)\mathrm{cl}H = \phi$ for all but finite number of sets $H \in \mathcal{H}$. Thus except for a finite number of sets $H \in \mathcal{H}$, $O \cap (\mathcal{T}(\mathcal{P}_n \cup Q_n))\mathrm{cl}H = \phi$. It now follows that $\{(\mathcal{T}(\mathcal{P}_n \cup Q_n))\mathrm{cl}H | H \in \mathcal{H}\}$ is $(\mathcal{T}(\mathcal{P}_n \cup Q_n))$closed $(\mathcal{P}_n)$locally finite refinement of $\mathcal{U}$. Similarly, we get a desired refinement of a (Q_m)open cover of X.

$(iv) \Rightarrow (i)$. Let $\mathcal{U}$ be a $(\mathcal{P}_m)$open cover of X. By (iv), there exists a $(\mathcal{P}_n)$locally finite refinement $\mathcal{W}$ of $\mathcal{U}$. For each $x \in X$, there exists a $(\mathcal{P}_n)$open neighbourhood G_x which intersects only a finite number of sets of $\mathcal{W}$. The collection $\{G_x | x \in X\}$ is a $(\mathcal{P}_n)$open cover of X. Therefore by (iv), it has a $(\mathcal{P}_l)$locally finite $(\mathcal{T}(\mathcal{P}_l \cup Q_l))$closed refinement $\mathcal{D}$. For $W \in \mathcal{W}$, write

$$H(W) = X - \cup\{D \in \mathcal{D} | D \cap W = \phi\}. \tag{1}$$

Since $\mathcal{D}$ is $(\mathcal{P}_l)$locally finite, it is $(\mathcal{T}(\mathcal{P}_l \cup Q_l))$locally finite, and therefore the union $\cup\{D \in \mathcal{D} | D \cap W = \phi\}$ is $(\mathcal{T}(\mathcal{P}_l \cup Q_l))$closed. Hence by (1), $H(W)$ is $(\mathcal{T}(\mathcal{P}_l \cup Q_l))$open. Also $W \subset H(W)$. Since a set in $\mathcal{D}$ intersects $H(W)$ if it does W and since $\mathcal{D}$ is a $(\mathcal{P}_l)$locally finite refinement of $\{G_x|x \in X\}$, it follows that $\mathcal{H} = \{H(W)\,|W \in \mathcal{W}\}$ is a $(\mathcal{P}_l)$locally finite $(\mathcal{T}(\mathcal{P}_l \cup Q_l))$open cover of X. For each $W \in \mathcal{W}$, choose $U \in \mathcal{U}$ such that $W \subset U$. Then $\{U \cap H(W)\,|W \in \mathcal{W}\}$ is a $(\mathcal{P}_j)$locally finite $(\mathcal{T}(\mathcal{P}_j \cup Q_j))$open refinement of $\mathcal{U}$, where $j = \max\{m, l\}$. Similarly, we obtain a necessary refinement of a (Q_m)open cover of X. Hence X is β-pairwise (ω)paracompact. □

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