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RELATIVE SEMI-METRICS

Elise Grabner[1], Gary Grabner[2 §], Kazumi Miyazaki[3], Jamal Tartir[4]
[1,2]Department of Mathematics
Slippery Rock University
1, Morrow Wag, Slippery Rock PA, 16057-1326, USA
[1]e-mail: elise.grabner@sru.edu
[2]e-mail: gary.grabner@sru.edu
[3]Department of Mathematics
Osaka Sangyo University
Osaka, 574-8530, JAPAN
e-mail: kmiyazaki@las.osaka-sandai.ac.jp
[4]Department of Mathematics
Youngstown State University
Youngstown, OH, USA
e-mail: tartir@as.ysu.edu

Abstract: In this paper we study relative properties of semi-metrizable type and give relative versions of the Alexandroff-Urysohn Metrization Theorem.

Theorem. *For a space X and $Y \subseteq X$, the following are equivalent:*

1) Y is semi-metrizable in X,

2) Fréchet in X and symmetrizable in X.

3) Y is first countable in X and semi-stratifiable in X,

4) Y is first countable in X and a semi-stratifiable subspace of X,

5) Y is first countable in X and a semi-metric subspace of X.

Theorem. *For a space X and $Y \subseteq X$, Y is Ometrizable in X if and only if there is a regular development for Y in X.*

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§Correspondence author

1. Introduction

In [1] and [2] Arhangel'skii and Gordienko introduce properties of relative symmetrizable type as a platform for a study of relative metrizability. Along the same lines we introduce properties of relative semi-metric type related to but different from the relative properties of symmetrizable type of [1] and [2]. The goal of this paper is to investigate the nature of these relative properties of semi-metrizable type and use them as a platform for finding relative versions of the Alexandroff-Urysohn Metrization Theorem.

For $Y \subseteq X$ a function $d : X \times X \to \mathbb{R}^+$ is called a symmetric on (Y, X) provided for all $x \in X$ and $y \in Y$:

$s1.$ $d(x, y) = 0$ if and only if $x = y$;
$s2.$ $d(x, y) = d(y, x)$, [1].

Clearly if d is a symmetric on (X, X) then d is a symmetric on X in the usual sense. For a space X, $Y \subseteq X$ and a symmetric d on (Y, X), for all $x \in X$ and $\epsilon > 0$ let $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ and $O_d(x, \epsilon) = int(B_d(x, \epsilon))$. Unless it will cause confusion, we will normally drop the subscript.

Recall that a space X is said to be symmetrizable with respect to a symmetric d provided $U \subseteq X$ is open if and only if for every $x \in U$ there is an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. A space X is said to be semi-metrizable provided there is a symmetric d on X such that for all $x \in X$ the collection $\{B(x, \frac{1}{n}) : n \in \mathbb{N}\}$ is a neighborhood base for x.

For a space X and $Y \subseteq X$ we say that

1 Y is *symmetrizable in* X provided there is a symmetric d on (Y, X) such that for every $U \subseteq X$, $U \cap Y = int(U) \cap Y$ if and only if for every $y \in U \cap Y$ there is an $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$.

2. Y is *semi-metrizable in* X provided there is a symmetric d on (Y, X) such that for all $x \in Y$ the collection $\{B(x, \frac{1}{n}) : n \in \mathbb{N}\}$ is a neighborhood base for x (in the space X).

3. Y is *3semi-metrizable in* X provided there is a semi-metric d on (Y, X) such that for every open set U and $x \in U$ there is an $\epsilon > 0$ with $B(x, \epsilon) \cap Y \subseteq U$.

4. Y is *Osemi-metrizable in* X provided there is a semi-metric d on (Y, X) such that for all $x \in \overline{Y}$ and $\epsilon > 0$, $x \in O(x, \epsilon)$.

5. Y is *strongly semi-metrizable in* X provided there is a symmetric d on (Y, X) which an Osemi-metric and a 3semi-metric on (Y, X).

Theorem 1. (see [3]) *The following are equivalent for a space X:*

1. *X is semi-metrizable;*
2. *X is first countable and symmetrizable;*
3. *X is Frechet and symmetrizable;*
4. *X is first countable and semi-stratifiable.*

We give several relative versions of Theorem 1. For a space X and $Y \subseteq X$, Y is semi-metrizable in X if and only if Y is first countable in X and a semi-stratifiable subspace of X, Theorem 15, if and only if Y is Fréchet in X and symmetrizable in X, Corollary 11. A subspace Y is 3semi-metrizable in X if and only if Y is first countable in X and 3semi-stratifiable in X, Theorem 19. A subspace Y is strongly semi-metrizable in X if and only if Y is strongly semi-stratifiable in X and strongly first countable in X, Theorem 26.

Suppose Y is a subset of a space X. A sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of open covers of X (of collections of open subsets of X such that, for all $n \in \mathbb{N}$, $Y \subseteq \cup\mathcal{G}_n$) is said to be a *(2-) development for Y in X* provided for all $y \in Y$ if U is an open neighborhood of y in X then there is an $n \in \mathbb{N}$ such that $st(y, \mathcal{G}_n) \subseteq U$, [6]. Clearly if Y is 2-developable in X then Y is a developable subspace of X and every point of Y has a countable local base in X. However a closed metriziable subset of a compact space X need not be $2-$ developable in X, Example 3. We say that Y is *strong developable in X* provided there is a development $\{\mathcal{G}_n : n \in \mathbb{N}\}$ for Y in X such that for all $x \in X$ and every open neighborhood U of x there is an $n \in \mathbb{N}$ such that $st(x, \mathcal{G}_n) \cap Y \subseteq U$. Such a collection is called a *strong development for Y in X*. If Y is developable in X then Y is Osemi-metrizable in X , Theorem 27, but need not be 3semi-metrizable in X, Example 5. If Y is strongly developable in X then Y is strongly semi-metrizable in X, Theorem 29.

We say that a semi-metric d on (Y, X) is a *metric on* (Y, X) (Y *is metrizable in* X) provided d satisfies the triangle inequality. Note that this version of a metric on (Y, X) different from the definition of a metric on (Y, X) given in [1]. If d is an Osemi-metric or a strong semi-metric on (Y, X) that satisfies the triangle inequality then we say d is an *Ometric* or a *strong metric on* (Y, X). A semi-metric d on (Y, X) is a *middle metric on* (Y, X) *(Y is middle metrizable* in X) provided for all $x, z \in X$ and $y \in Y$, $d(x, z) \leq d(x, y) + d(y, z)$. For a space X and $Y \subseteq X$, Y is metrizable in X if and only if there is a regular $2-$development for Y, Theorem 30. If a regular space X has a dense subspace Y which is strongly metrizable in X then X is metrizable, Theorem 37.

For a collection $\mathcal{B}$ of subsets of a set X, $x \in X$ and $Y \subseteq X$, $(\mathcal{B})_x = \{B \in \mathcal{B} : x \in B\}$ and $(\mathcal{B})_Y = \cup\{(\mathcal{B})_y : y \in Y\}$. Suppose Y a subspace of a space X.

When a set U is said to be open (closed), we mean open (closed) with respect to the topology on X even if U happens to be a subset of Y. A collection $\mathcal{B}$ of open subsets of the space X is said to be an *outerbase for Y in X* provided for every $y \in Y$ the collection $(\mathcal{B})_y$ is a local base for y in X. Throughout this paper all spaces are assumed to be T_2.

2. Basic Properties and Definitions

Theorem 2. (see [7]) *Let X be a space, and d be a symmetric on X. Then:*

(1) *X is symmetrizable with respect to d if and only if for each $H \subseteq X$ H is closed if and only if $d(x, H) > 0$ for all $x \in X \backslash H$.*

(2) *X is semi-metrizable with respect to d if and only if for each $x \in X$ and $H \subseteq X$ $x \in \overline{H}$ if and only if $d(x, H) = 0$.*

The following gives natural relative versions of these characterizations of symmetrizable and semi-metric spaces.

Theorem 3. *Suppose X is a space, $Y \subseteq X$ and d a symmetric on (Y, X). Then:*

(1) *Y is symmetrizable in X with respect to d if and only if for each $H \subseteq X$, $H \cap Y = \overline{H} \cap Y$ if and only if $d(y, H) > 0$ for all $y \in Y \backslash H$.*

(2) *Y is semi-metrizable in X with respect to d if and only if for each $y \in Y$ and $H \subseteq X$, $y \in \overline{H}$ if and only if $d(y, H) = 0$.*

For a symmetric d on (Y, X) consider the following conditions from [1]:

01) for every closed subset H of X, $d(y, H) > 0$ for all $y \in Y \backslash H$;

02) if $H \subseteq Y$ and $d(x, H) > 0$ for all $x \in X \backslash H$ then H is closed in X;

03) for every closed subset H of X, $d(x, H \cap Y) > 0$ for all $x \in X \backslash H$;

04) if $y \in Y$ and $H \subseteq X$ such that $d(y, H) > 0$ then $y \notin \overline{H}$.

In [1] they say that a symmetric on (Y, X) satisfying conditions 01), 02) and 03) *defines* Y in X and if it also satisfies condition 04) the symmetric *properly defines* Y in X. The following lemma gives alternate characterizations of conditions 01) − 04).

Lemma 4. *For a space X and $Y \subseteq X$, symmetric d on (Y, X) satisfies:*

1. *condition 01) if and only if for every open $U \subseteq X$ and each $y \in U \cap Y$ there is an $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$.*
2. *condition 02) if and only if for each $U \subseteq X$ with $Y \subseteq U$, if for all $x \in U$*

there is an $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subseteq U$ then U is open.

3. *condition* 03) *if and only if for every open $U \subseteq X$ and each $x \in U$ there is an $\epsilon > 0$ such that $B(x, \epsilon) \cap Y \subseteq U$.*
4. *condition* 04) *if and only if for each $x \in Y$ and $\epsilon > 0$, $x \in O_d(x, \epsilon)$.*

Lemma 5. *For a space X and $Y \subseteq X$, a symmetric d on (Y, X) is a semi-metric on (Y, X) if and only if d satisfies conditions* 01) *and* 04).

We say that a subspace Y is *first countable in* X provided every point of Y has a countable neighborhood base in the space X. If Y is first countable in X and for every $x \in X \backslash Y$ there is a countable collection $\{V_n : n \in \mathbb{N}\}$ of open neighborhoods of x such that for every open neighborhood U of x there is an $n \in \mathbb{N}$ with $V_n \cap Y \subseteq U$, we say that Y is *strongly first countable in* X. Clearly if Y is (strongly) semi-metrizable in X then Y is (strongly) first countable in X.

Theorem 6. *For a space X and $Y \subseteq X$, the following are equivalent:*

1. *there is a symmetric on (Y, X) satisfying conditions 01) and 04),*
2. *Y is semi-metrizable in X,*
3. *Y is first countable in X and a semi-metrizable subspace of X*

Proof. ($1 \leftrightarrow 2$) Lemma 5.

($2 \rightarrow 3$) If d is a semi-metric on (Y, X) then $d|_Y$ is a semi-metric on (Y, Y) and so Y is a semi-metric subspace of X. As noted earlier if Y is semi-metrizable in X it is first countable in X.

($3 \rightarrow 2$) Suppose $d : Y \times Y \rightarrow \mathbb{R}^+$ is a semi-metric on Y and note that for all $x \in Y$ and $H \subseteq Y$, $d(x, H) = 0$ if and only if $x \in cl_Y(H)$. For each $y \in Y$ let $\{G_n(y) : n \in \mathbb{N}\}$ be a local base for y in X such that $G_1(y) = X$, $G_{n+1}(y) \subseteq G_n(y)$ for all $n \in \mathbb{N}$ and $Y \cap G_n(y) \subseteq B_d(y, \frac{1}{n})$ for all $n \in \mathbb{N} \backslash \{1\}$. For each $x \in X \backslash Y$ and $y \in Y$ let $m(x, y) = \min\{n \in \mathbb{N} : x \notin G_n(y)\}$. Define $d^* : X \times X \rightarrow \mathbb{R}^+$ as follows:

$d^*(x, x) = 0$ for all $x \in X$,
$d^*(x, y) = d(x, y)$ for all $x, y \in Y$,
$d^*(x, y) = 1$ for all $x, y \in X \backslash Y$ with $x \neq y$,
$d^*(x, y) = d^*(y, x) = \frac{1}{m(x,y)}$ for all $x \in X \backslash Y$ and $y \in Y$.

Clearly d^* is a symmetric on X. We will show that d^* is a semi-metric on (Y, X) using Theorem 3.

Let $H \subseteq X$ and $y \in Y$. Suppose $d^*(y, H) = 0$. For each $n \in \mathbb{N}$ let $x_n \in H$ such that $d^*(x_n, y) < \frac{1}{n}$. Let $A = \{n \in \mathbb{N} : x_n \in Y\}$. If $|A| = \omega$ then $0 = d^*(y, \{x_n : n \in A\}) = d(y, \{x_n : n \in A\})$ and so $y \in cl_Y(\{x_n : n \in A\}) \subseteq \overline{H}$. Suppose that A is finite and let $m^* = \max\{A \cup \{0\}\}$ and V be an open neighborhood of y in X. Then there is an $n > m^*$ such that $G_n(y) \subseteq V$. For all $m \geq n$, since $x_m \in X \backslash Y$ and $d^*(y, x_m) < \frac{1}{m} \leq \frac{1}{n}$, $x_m \in G_n(y)$. Thus $y \in \overline{\{x_n : n \in \mathbb{N}\}} \subseteq \overline{H}$. Hence if $d^*(y, H) = 0$ then $y \in \overline{H}$.

Suppose that $y \in \overline{H}$. For all $n \in \mathbb{N}$ let $x_n \in H \cap G_n(y)$. Notice that if $n \in \mathbb{N}$ such that $x_n \in Y$ then since $G_n(y) \cap Y \subseteq B_d(y, \frac{1}{n})$, $d^*(y, x_n) = d(y, x_n) < \frac{1}{n}$. If $n \in \mathbb{N}$ such that $x_n \in X \backslash Y$ then since $x_n \in G_n(y)$, $m(x, y) > n$ and so $d^*(y, x_n) < \frac{1}{n}$. Hence $d^*(y, H) \leq d^*(y, \{x_n : n \in \mathbb{N}\}) = 0$. Thus if $y \in \overline{H}$ then $d^*(y, H) = 0$. Therefore by Theorem 3, d^* is a semi-metric on (Y, X). □

Corollary 7. *For a space X and $Y \subseteq X$, a symmetric d on (Y, X) is a* 3*semi-metric on (Y, X) if and only if it satisfies conditions* 01), 03) *and* 04).

For a space X and $Y \subseteq X$, it is readily seen that if d is a symmetric on (Y, X) satisfying the condition that for all $x \in \overline{Y}$ and $\epsilon > 0$, $x \in O(x, \epsilon)$ then d satisfies both conditions 02) and 04).

Corollary 8. *For a space X and $Y \subseteq X$, if d is a strong semi-metric (Osemi-metric) on (Y, X) then d satisfies conditions 01) through 04) (conditions* 01), 02) *and* 04)).

Before we continue let us give examples distinguishing the relative properties of semi-metrizable type discussed here.

1. There is a space X with a subspace Y with a symmetric on (Y, X) satisfying conditions 01) through 03) (Y is defined in X) but Y is not symmetrizable in X, Example 1.

2. There is a space X with a subspace Y which is both Osemi-metrizable in X and 3semi-metrizable in X but not defined in X, Example 7.

3. There is a space X with a subspace Y with a symmetric on (Y, X) satisfying conditions 01) through 04) (Y is properly defined in X) but Y is not strongly semi-metrizable in X, Example 8.

4. There is a space X with a subspace Y such that Y is Osemi-metrizable in X but not 3semi-metrizable in X, Example 3.

5. There is a space X with a subspace Y such that Y is 3semi-metrizable in X but not Osemi-metrizable in X, Example 2. In fact no symmetric on (Y, X) can satisfy both conditions 01) and 02).

6. There is a space X with a subspace Y such that Y is semi-metrizable in X but neither Osemi-metrizable in X or 3semi-metrizable in X, Example 4.

Recall that if a space X symmetrizable with respect to the symmetric d, $\{x_n : n < \omega\} \subseteq X$ *and* $x \in X$ then the following are equivalent:

- (i) $x_n \to x$;
- (ii) $d(x_n, x) \to 0$;
- (iii) for all $\epsilon > 0$, $\{x_n : n < \omega\} \backslash B(x, \epsilon)$ is finite, see [7].

This often used property of symmetrizable spaces has natural relative versions.

Lemma 9. *For a space X and $Y \subseteq X$, suppose Y is symmetrizable in X with respect to the symmetric d on (Y, X), $\{x_n : n < \omega\} \subseteq X$ and $y \in Y$. The following are equivalent:*

- (i) $x_n \to y$;
- (ii) $d(x_n, y) \to 0$;
- (iii) *for all* $\epsilon > 0$, $\{x_n : n < \omega\} \backslash B(y, \epsilon)$ *is finite.*

The following is also a relative version of this property that we will need later.

Lemma 10. *Suppose d is a symmetric on (Y, X) satisfying condition* 02) *and* 03). *For $\{y_n : n < \omega\} \subseteq Y$ and $x \in X$ the following are equivalent:*

- (i) $y_n \to x$;
- (ii) $d(y_n, x) \to 0$;
- (iii) *for all* $\epsilon > 0$, $\{y_n : n < \omega\} \backslash B(x, \epsilon)$ *is finite.*

We say that a subspace Y of X is *Fréchet in* X provided for every $A \subseteq X$ and every $y \in \overline{A} \cap Y$ there is a sequence in A converging to y. Clearly if Y is first countable in X then it is Fréchet in X. The following corollary to Theorem 6 follows from Lemma 9 and the characterization of condition 04) in Lemma 4.

Corollary 11. *For a space X and $Y \subseteq X$, Y is semi-metrizable in X if and only if Y is Fréchet in X and symmetrizable in X.*

3. First Countable Semi-Stratifiable Spaces are Semi-Metrizable

A space X is semi-stratifiable if there is a function G which assigns to each $n \in \mathbb{N}$ and closed $H \subseteq X$, an open set $G_n(H)$ containing H such that:

- (i) $H = \cap\{G_n(H) : n \in \mathbb{N}\}$;
- (ii) if $H \subseteq K$ then $G_n(H) \subseteq G_n(K)$, [7].

Equivalently, a space X is semi-stratifiable iff there exists a function

$g : \mathbb{N} \times X \to T(X)$ such that:

(*i*) *for all* $x \in X$, $\{x\} = \cap\{g_n(x) : n \in \mathbb{N}\}$;
(*ii*) *if for all* $n \in \mathbb{N}$, $y \in g_n(x_n)$ *then* $x_n \to y$, [7].

We say that Y is is 3*semi-stratifiable in* X provided there exists a function G which assigns to each $n \in \mathbb{N}$ and closed $H \subseteq X$, an open set $G_n(H)$ containing $H \cap Y$ such that:

(*i*) $\cap\{G_n(H) : n \in \mathbb{N}\} \subseteq H$;
(*ii*) if $H \subseteq K$ then $G_n(H) \subseteq G_n(K)$.

Clearly if $Y = X$ then this is just the definition of semi-stratifiable.

Lemma 12. *A subspace* Y *is 3semi-stratifiable in* X *if and only if there exists a function* $g : N \times X \to T(X)$ *such that*

(*i*) *for all* $x \in Y$, $\{x\} = \cap\{g_n(x) : n \in N\}$;
(*ii*) *if* $x \in X$ *and for all* $n \in N$, $y_n \in Y$ *such that* $x \in g_n(y_n)$ *then* $y_n \to x$.

Suppose Y is 3semi-stratifiable in X and $g : \mathbb{N} \times X \to T(X)$ satisfying the conditions of Lemma 12. Then the function $h : \mathbb{N} \times X \to T(X)$ defined by

$$h_n(x) = \begin{cases} g_n(x)\ , & \text{if } x \in Y \\ \phi\ , & \text{if } x \in X\backslash Y \end{cases}, \qquad \text{for all } n \in \mathbb{N},$$

satisfies the conditions of Lemma 12 along with the condition

(*iii*) if $y \in Y$ and for all $n \in \mathbb{N}$, $x_n \in X$ with $y \in h_n(x_n)$ then $x_n \to y$. Also the function $k : \mathbb{N} \times X \to T(X)$ defined by

$$k_n(x) = \begin{cases} g_n(x)\ , & \text{if } x \in Y \\ X\ , & \text{if } x \in X\backslash Y \end{cases}, \qquad \text{for all } n \in \mathbb{N},$$

satisfies the conditions of Lemma 12 along with the condition

(*o*) for all $x \in X$ and $n \in \mathbb{N}$, $x \in k_n(x)$.

This suggests more properties of relative semi-stratifiable type relevant to our study. For a space X and $Y \subseteq X$,

1. Y is said to be *semi-stratifiable in* X provided there exists a function $g : \mathbb{N}\times X \to \mathcal{T}(X)$ such that

(*i*) for all $x \in Y$, $\{x\} = \cap\{g_n(x) : n \in \mathbb{N}\}$;
(*iii*) if $y \in Y$ and for all $n \in \mathbb{N}$, $x_n \in X$ such that $y \in g_n(x_n)$ then $x_n \to y$.

2. Y is said to be *Osemi-stratifiable in* X provided there exists a function $g : \mathbb{N}\times X \to \mathcal{T}(X)$ such that

(*o*) for all $x \in X$ and $n \in \mathbb{N}$, $x \in g_n(x)$,
(*i*) for all $y \in Y$, $\{y\} = \cap\{g_n(y) : n \in \mathbb{N}\}$,
(*iii*) if $y \in Y$ and for all $n \in \mathbb{N}$, $x_n \in X$ such that $y \in g_n(x_n)$ then $x_n \to y$.

3. Y is said to be *strongly semi-stratifiable in* X provided there exists a function $g : \mathbb{N}\times X \to \mathcal{T}(X)$ such that:

(o) for all $x \in X$ and $n \in \mathbb{N}$, $x \in g_n(x)$,
(i) for all $y \in Y$, $\{y\} = \cap\{g_n(y) : n \in \mathbb{N}\}$,
(ii) if $x \in X$ and for all $n \in \mathbb{N}$, $y_n \in Y$ such that $x \in g_n(y_n)$ then $y_n \to x$,
(iii) if $y \in Y$ and for all $n \in \mathbb{N}$, $x_n \in X$ such that $y \in g_n(x_n)$ then $x_n \to y$.

Theorem 14. *For a space X and $Y \subseteq X$, Y is Osemi-stratifiable in X if and only if there is a function G which assigns to each $n \in N$ and closed $H \subseteq X$, an open set $G_n(H)$ containing H such that*

(i) $H \cap Y = (\cap\{G_n(H) : n \in N\}) \cap Y$;
(ii) *if* $H \subseteq K$ *then* $G_n(H) \subseteq G_n(K)$.

Clearly if Y is semi-stratifiable in X then Y is a semi- stratifiable subspace of X and points of Y are $G_\delta s$ in the space X.

Lemma 15. *For a space X and $Y \subseteq X$, Y is semi-stratifiable in X if and only if Y is a semi-stratifiable subspace and points of Y are $G_\delta s$ in the space X.*

Proof. Suppose Y is a semi-stratifiable subspace of the space X and points of Y are $G_\delta s$ in the space X. For each $y \in Y$ let $\{G_n(y) : n \in \mathbb{N}\}$ be a collection of open subsets of X such that $\{x\} = \cap\{G_n(y) : n \in \mathbb{N}\}$ and $h : \mathbb{N} \times Y \to T(X)$ such that

(i) *for all* $y \in Y$, $\{y\} = \cap\{h_n(y) : n \in \mathbb{N}\}$;
(ii) *if for all* $n \in \mathbb{N}$, $y \in h_n(x_n)$ *then* $x_n \to y$.

For each $y \in Y$ and $n \in \mathbb{N}$ let $H_n(y)$ be an open subset of X such that $h_n(y) = H_n(y) \cap Y$. For all $n \in \mathbb{N}$ and $x \in X$ let

$$g_n(x) = \begin{cases} G_n(x) \cap H_n(x)\ , & \text{if } x \in Y \\ \phi\ , & \text{otherwise.} \end{cases}$$

The function $g : \mathbb{N} \times X \to T(X)$ satisfies conditions (i) and (iii) above. □

We now can bring together Theorem 6 and Corollary 11 to give a relative version of Theorem 1.

Theorem 16. *For a space X and $Y \subseteq X$, the following are equivalent*

1) *Y is semi-metrizable in X,*
2) *Fréchet in X and symmetrizable in X.*
3) *Y is first countable in X and semi-stratifiable in X,*
4) *Y is first countable in X and a semi-stratifiable subspace of X,*
5) *Y is first countable in X and a semi-metric subspace of X.*

Corollary 16. *For a space X, if Y is a G_δ set and semi-metrizable in X then Y is 3semi-metrizable in X.*

Proof. Let $\{G'_n : n \in \mathbb{N}\}$ be a collection of open subsets of X such that $Y = \cap\{G'_n : n \in \mathbb{N}\}$ and for all $n < m$ $G'_n \supseteq G'_m$. We need to modify the symmetric constructed in Theorem 6. Suppose $d : Y \times Y \to \mathbb{R}^+$ is a semi-metric on Y For each $x \in Y$ let $\{G_n(x) : n \in \mathbb{N}\}$ be a local base for x in X such that

1. $G_n(x) \subseteq G'_n$ for all $n \in \mathbb{N}$,
2. $G_{n+1}(x) \subseteq G_n(x)$ for all $n \in \mathbb{N}$,
3. $Y \cap G_n(x) \subseteq B(x, \frac{1}{n})$ for all $n \in \mathbb{N}$.

For each $x \in X\backslash Y$ and $y \in Y$ let $m(x, y) = \min\{n \in \mathbb{N} : x \notin G_n(y)\}$. Define $d^* : X \times X \to \mathbb{R}^+$ as follows:

1. $d^*(x, x) = 0$ for all $x \in X$,
2. $d^*(x, y) = d(x, y)$ for all $x, y \in Y$,
3. $d^*(x, y) = 1$ for all $x, y \in X\backslash Y$ with $x \neq y$,
4. $d^*(x, y) = d^*(y, x) = \frac{1}{m(x,y)}$ for all $x \in X\backslash Y$ and $y \in Y$.

As in Theorem 15, d^* is a semi-metric on (Y, X). Suppose that H is a closed subset of X. Let $x \in X\backslash Y$ and $n \in \mathbb{N}$ such that $x \notin G'_n$. Then $d^*(x, H \cap Y) \geq$d$^*(x, Y) \geq \frac{1}{n} > 0$. Thus for all $x \in X\backslash Y$ $d(x, H \cap Y) > 0$. Since $H \cap Y$ is closed in Y and d is a semi-metric on Y, for all y$\in Y\backslash H$ d$^*(y, H \cap Y) = d(y, H \cap Y) > 0$. Thus d^* satisfies condition 03). □

The following lemma is essentially Theorem 3 from [1].

Lemma 17. *For a space X and $Y \subseteq X$,if Y is 3semi-metrizable in X then every subset of Y which is closed in X is a G_δ set.*

For a space X and Y a closed subset of X, if d is a semi-metric on (Y, X) then d is an Osemi-metric on (Y, X). The following theorem follows directly from this observation, Corollary 16 and Lemma 17.

Theorem 18. *For a space X and Y a closed subset of X, Y is strongly semi-metrizable in X if and only if Y is semi-metrizable in X and a G_δ set.*

Note that a closed subset of a space X can be Osemi-metrizable in X but not a G$_\delta$ set, Example 3.

In [8] Heath shows that a space X is a semi-metric space if and only if there is a function $g : \mathbb{N} \times X \to T(X)$ satisfying condition A. That is, there is a function $g : \mathbb{N} \times X \to T(X)$ such that

1. for all $x \in X$, $\{g_n(x) : n \in \mathbb{N}\}$ is a neighborhood base for x in X and for all $n \in \mathbb{N}$, $g_{n+1}(x) \subseteq g_n(x)$;
2. if $y \in X$ and for all $n \in \mathbb{N}$, $x_n \in X$ such that $y \in g_n(x_n)$ then $x_n \to y$.

For a space X and $Y \subseteq X$, we say a function $g : \mathbb{N} \times X \to T(X)$ satisfies condition $A(Y)$ provided

1(a) For all $y \in Y$, $\{g_n(y) : n \in \mathbb{N}\}$ is a local base for y in $X$and for all $n \in \mathbb{N}$ and $x \in X$, $g_{n+1}(x) \subseteq g_n(x)$.

(b) For all $x \in X$ and open neighborhoods U of x in X there is an $n \in \mathbb{N}$ such that $g_n(x) \cap Y \subseteq U$.

2(a) If $y \in Y$ and for all $n \in \mathbb{N}$, $x_n \in X$ such that $y \in g_n(x_n)$ then $x_n \to y$.

(b) If $x \in X$ and for all $n \in \mathbb{N}$, $y_n \in Y$ such that $x \in g_n(y_n)$ then $y_n \to x$.

Theorem 19. *For a space X and $Y \subseteq X$, the following are equivalent:*

1. *Y is first countable in X and 3semi-stratifiable in X;*
2. *there is a function $g : N \times X \to T(X)$ satisfying condition $A(Y)$;*
3. *Y is 3semi-metrizable in X.*

Proof. ($1 \Rightarrow 2$) For all $y \in Y$ suppose that $\{V_n(y) : n \in \mathbb{N}\}$ is a local base for y in X such that for all $n \in \mathbb{N}$, $V_{n+1}(x) \subseteq V_n(x)$. Suppose $h : \mathbb{N} \times X \to \mathcal{T}(X)$ satisfies the conditions of Lemma 12. For all $n \in \mathbb{N}$ and $x \in X$ let $g'_n(x) = \cap\{h_k(x) : k \leq n\}$ and

$$g_n(x) = \begin{cases} g'_n(x) \cap V_n(x) & \text{, if } x \in Y \\ \phi & \text{, otherwise.} \end{cases}$$

It is straightforward to show that g satisfies condition $A(Y)$ (note that in proving that g satisfies 1(b) and 2(a), the fact that $g_n(x) = \phi$ for all $x \in X \backslash Y$ is useful).

($2 \Rightarrow 3$) Suppose $g : \mathbb{N} \times X \to \mathcal{T}(X)$ satisfying condition $A(Y)$. For all $x, y \in X$ let $A(x, y) = \{n \in \mathbb{N} : x \notin g_n(y) \text{ and } y \notin g_n(x)\}$ and

$$n(x, y) = \begin{cases} 0\ , & \text{if } A(x, y) = \phi \\ \min A(x, y)\ , & \text{otherwise.} \end{cases}$$

Define $d : X \times X \to \mathbb{R}^+$ as follows:

$$d(x, y) = \begin{cases} 0\ , & \text{if } n(x, y) = 0 \\ \frac{1}{n(x,y)}\ , & \text{otherwise.} \end{cases}$$

Clearly for all $x, y \in X$, $d(x, y) = d(y, x)$ and for all $x \in X$, $d(x, x) = 0$. Suppose that $x \in X$ and $y \in Y$ with $x \neq y$. Since X is T_2 there are disjoint open subsets U and V such that $x \in U$ and $y \in V$. By condition 1(b) of $A(Y)$ there is an $m(x) \in \mathbb{N}$ such that $g_{m(x)}(x) \cap Y \subseteq U$. Since $y \in Y$ by condition 1(a) of $A(Y)$ there is an $n(y) \in \mathbb{N}$ such that $g_{n(y)}(y) \subseteq V$. Let $n^* = \max(m(x), n(y))$

and note that $x \notin g_{n^*}(y)$ and $y \notin g_{n^*}(x)$. Hence $A(x, y) \neq \phi$ and so $d(x, y) > 0$. Thus d is a symmetric on (Y, X).

Suppose $H \subseteq X$ and $y \in Y \backslash H$ with $d(y, H) = 0$. For all $n \in \mathbb{N}$ let $x_n \in H$ such that $0 \neq d(y, x_n) < \frac{1}{n}$. Then for all $n \in \mathbb{N}$ either $x_n \in g_n(y)$ or $y \in g_n(x_n)$. Let $S_1 = \{n \in \mathbb{N} : x_n \in g_n(y)\}$ and let $S_2 = \{n \in \mathbb{N} : y \in g_n(x_n)\}$.

Suppose S_2 is infinite. For each $n \in \mathbb{N}$ let $k(n) = \min\{m \in S_2 : n \leq m\}$ and $z_n = x_{k(n)}$. Then for all $n \in \mathbb{N}$ since $n \leq k(n)$, $y \in g_{k(n)}(x_{k(n)}) \subseteq g_n(x_{k(n)}) = g_n(z_n)$. Thus by condition $2(a)$, $z_n \to y$ and so $y \in \overline{H}$.

Suppose S_1 is infinite and that U is an open neighborhood of y in X. Let $n \in \mathbb{N}$ such that $g_n(y) \subseteq U$ and $m \in S_1$ with $m > n$. Then $x_m \in g_m(y) \subseteq g_n(y) \subseteq U$. Hence $U \cap H \neq \phi$. Thus $y \in \overline{H}$.

Now suppose $H \subseteq X$, $y \in Y \backslash H$ and $y \in \overline{H}$. For each $n \in \mathbb{N}$ choose an $x_n \in H \cap g_n(y)$. For each $m \in \mathbb{N}$ since $y \in Y$, $x_m \neq y$ and $x_m \in g_m(y)$, $n(x, y) > m$ and so $d(x_n, y) < \frac{1}{m}$. Therefore $d(y, H) = 0$. Hence by Theorem 3, d is a semi-metric on (Y, X).

Suppose that H is a closed subset of X, $x \in X \backslash H$ and $d(x, H \cap Y) = 0$, i.e. d does not satisfy 03). For all $n \in \mathbb{N}$ choose $y_n \in H \cap Y$ such that $d(x, y_n) < \frac{1}{n}$. Since $x \notin H$, for all $n \in \mathbb{N}$ either $x \in g_n(y_n)$ or $y_n \in g_n(x)$. Since H is a closed subset of X by condition $1(b)$ of $A(Y)$ there is an $m \in \mathbb{N}$ such that $g_m(x) \cap Y \subseteq X \backslash H$. Hence for all $n \geq m$, $y_n \notin g_n(x)$ and therefore $x \in g_n(y_n)$ for all $n \geq m$. Thus by condition $2(b)$ of $A(Y)$, $y_n \to x$ and so $x \in \overline{\{y_n : n \geq m\}} \subseteq H$, a contradiction.

$(3 \Rightarrow 1)$ Suppose d is a 3semi-metric on (Y, X). For each closed $H \subseteq X$ and $n \in \mathbb{N}$ let $G_n(H) = \cup\{O(y, \frac{1}{n}) : y \in H \cap Y\}$. Since d is a semi-metric on (Y, X), for all $y \in Y$ and $n \in \mathbb{N}$, $y \in O(y, \frac{1}{n})$. Thus for all $n \in \mathbb{N}$, $H \cap Y \subseteq G_n(H)$. Clearly if H and K are closed subsets of X and $H \subseteq K$ then $G_n(H) \subseteq G_n(K)$. Suppose H is a closed subset of X and $x \in X \backslash H$. By condition 03), $d(x, H \cap Y) > 0$. Let $n \in \mathbb{N}$ such that $d(x, H \cap Y) > \frac{1}{n}$. Then for all $y \in H \cap Y$, $x \notin O(y, \frac{1}{n})$ and so $x \notin G_n(H)$. Thus $\cap\{G_n(H) : n \in \mathbb{N}\} \subseteq H$. □

A weaker relative version of Heath's condition A can be used to characterize being semi-metrizable in X.

Theorem 20. *For a space X and $Y \subseteq X$, Y is semi-metrizable in X if and only if there is a function $g : N \times X \to T(X)$ such that:*

1. *For all $y \in Y$, $\{g_n(y) : n \in N\}$ is a local base for y in X and for all $n \in N$, $g_{n+1}(y) \subseteq g_n(y)$.*
2. *If $y \in Y$ and for all $n \in N$, $x_n \in X$ such that $y \in g_n(x_n)$ then $x_n \to y$.*

Lemma 21. *For a space X and $Y \subseteq X$, Y is 0semi-metrizable in X if and* *only if there is a function $g : N \times X \to T(X)$ such that:*

0. *For all $x \in X$ and $n \in N$, $x \in g_n(x)$.*
1. For all $y \in Y$, $\{g_n(y) : n \in N\}$ is a local base for y in X and for all $n \in N$, $g_{n+1}(y) \subseteq g_n(y)$.
2. If $y \in Y$ and for all $n \in N$, $x_n \in X$ such that $y \in g_n(x_n)$ then $x_n \to y$.

Proof. ($\Rightarrow$) Suppose d is a 0semi-metric on (Y, X). For all $x \in \overline{Y}$ and $n \in \mathbb{N}$ let $g_n(x) = O(x, \frac{1}{n})$. For all $x \in X/\overline{Y}$ and $n \in \mathbb{N}$ let $g_n(x) = X \backslash \overline{Y}$. That g satisfies conditions 0 and 1 follows directly from the fact that d is an 0semi-metric on (Y, X) and condition 2 from Lemma 9.

($\Leftarrow$) Suppose $g : \mathbb{N} \times X \to \mathcal{T}(X)$ satisfying condition 0, 1 and 2 above. For all $x, y \in X$ let $A(x, y) = \{n \in \mathbb{N} : x \notin g_n(y) \text{ and } y \notin g_n(x)\}$and

$$n(x,y) = \begin{cases} 0 \ , & \text{if } A(x,y) = \phi \\ \min A(x,y) \ , & \text{otherwise.} \end{cases}$$

Define $d : X \times X \to \mathbb{R}^+$ as follows:

$$d(x,y) = \begin{cases} 0 \ , & \text{if } n(x,y) = 0 \\ \frac{1}{n(x,y)} \ , & \text{otherwise.} \end{cases}$$

As in Theorem 19 d is a semi-metric on (Y, X) and for all $x, y \in X$, $d(x, y) = d(y, x)$. Also for all $x \in X$ and $n \in \mathbb{N}$, $x \in g_n(x) \subseteq B(x, \frac{1}{n})$. Thus d is an 0semi-metric on (Y, X). □

Notice the semi-metric on (Y, X) constructed in the proof above satisfies the condition that for all $x \in X$ and $\epsilon > 0$, $x \in O(x, \epsilon)$. This appears to be stronger than the condition in the definition of 0semi-metric on (Y, X) that for all $x \in \overline{Y}$ and $\epsilon > 0$, $x \in O(x, \epsilon)$. However if d is an 0semi-metric (strong semi-metric) on (Y, X) then the function $d' : X \times X \to \mathbb{R}^+$ defined by

$$d'(x,y) = \begin{cases} 0 & , \text{if } x, y \in X \backslash \overline{Y} \\ d(x,y) & , \text{otherwise.} \end{cases}$$

is an Osemi-metric (strong semi-metric) on (Y, X) satisfying this stronger condition.

Lemma 22. *For a space X and $Y \subseteq X$, if Y is Osemi-metrizable (strongly semi-metrizable) in X then there is an Osemi-metric (a strong semi-metric) on (Y, X) such that for all $x \in X$ and $\epsilon > 0$, $x \in O(x, \epsilon)$.*

Note that if d is an Ometric (a strong metric) on (Y, X) then the function d' constructed above need not be a metric on (Y, X). It is for this reason that we do not use this stronger condition in the definition of Osemi-metrizable in a space X.

Theorem 23. *For a space X and $Y \subseteq X$, Y is Osemi-metrizable in X if and only if Y is first countable in X and Osemi-stratifiable in X.*

Proof. ($\Rightarrow$) Construct the same function g as in Lemma 21. Then g satisfies conditions o), i) and iii) of the definition of Osemi-stratifiable in X.

($\Leftarrow$) For all $y \in Y$ suppose that $\{V_n(y) : n \in \mathbb{N}\}$ is a local base for y in X such that for all $n \in N$, $V_{n+1}(x) \subseteq V_n(x)$. Suppose $h : \mathbb{N} \times X \to \mathcal{T}(X)$ satisfies conditions o), i) and iii) of the definition of Osemi-stratifiable in X. For all $n \in \mathbb{N}$ and $x \in X$ let $g'_n(x) = \cap\{h_k(x) : k \le n\}$ and

$$g_n(x) = \begin{cases} g'_n(x) \cap V_n(x) & \text{, if } x \in Y \\ g'_n(y) & \text{, otherwise.} \end{cases}$$

Clearly g satisfies conditions 1 and 2 of Lemma 21. Suppose $y \in Y$ and for all $n \in \mathbb{N}$, $x_n \in X$ such that $y \in g_n(x_n)$. Then for all $n \in \mathbb{N}$, $y \in h_n(x_n)$. Therefore since h satisfies condition iii), $x_n \to y$. Hence g satisfies condition 3 of Lemma 21. ◻

Lemma 24. *For a space X and $Y \subseteq X$, if d is a strong semi-metric on (Y, X) then for all $x \epsilon X$, $\{\{x\} \cup (O(x, \frac{1}{n}) \cap Y) : n \in N\}$ is a countable neighborhood base for x in the subspace $\{x\} \cup Y$.*

Lemma 25. *For a space X and $Y \subseteq X$, Y is strongly semi-metrizable in X if and only if there is a function $g : N \times X \to T(X)$ satisfying property $A(Y)$ such that for all $x \in X$ and $n \in N$, $x \in g_n(x)$.*

Proof. ($\Leftarrow$) Suppose $g : \mathbb{N} \times X \to T(X)$ satisfying property $A(Y)$ such that for all $x \in X$ and $n \in \mathbb{N}$, $x \in g_n(x)$. Let d be the same 3semi-metric on (Y, X) as constructed in Theorem 19. Let $x \in X$, $n \in \mathbb{N}$ and suppose $z \in g_n(x)$. Then $n(x, z) = 0$ or $n(x, z) > n$. In either case $d(x, z) < \frac{1}{n}$. Thus $g_n(x) \subseteq B(x, \frac{1}{n})$ and so, since $g_n(x)$ is open, $x \in O(x, \frac{1}{n})$. Thus for all $x \in X$ and $\epsilon > 0$, $x \in O(x, \epsilon)$. Hence d is a strong semi-metric on (Y, X).

($\Rightarrow$) By Lemma 22 let d be a strong semi-metric on (Y, X) such that for all $x \in X$ and $\epsilon > 0$, $x \in O(x, \epsilon)$. For all $x \in X$ and $n \in \mathbb{N}$ let $g_n(x) = O(x, \frac{1}{n})$. As in Theorem 19, g satisfies property $A(Y)$ and for all $x \in X$ and $n \in \mathbb{N}$, $x \in O(x, \frac{1}{n}) = g_n(x)$. (Note that condition $1(b)$ of property $A(Y)$ follows from Lemma 24). ◻

Theorem 26. *For a space X and $Y \subseteq X$, Y is strongly semi-metrizable in X if and only if Y is strongly semi-stratifiable in X and strongly first countable in X.*

Proof (⇐) For all $y \in Y$ let $\{V_n(y) : n \in \mathbb{N}\}$ be a local base for y in X such that for all $n \in N$, $V_{n+1}(x) \subseteq V_n(x)$. For all $x \in X \setminus Y$ let $\{V_n(x) : n \in \mathbb{N}\}$ be a collection of open subsets of x such that for all $n \in N$, $V_{n+1}(x) \subseteq V_n(x)$ and for any open neighborhood U of x there is an $n \in \mathbb{N}$ with $V_n(x) \cap Y \subseteq U$. Suppose $g : \mathbb{N} \times X \to \mathcal{T}(X)$ satisfies conditions (0)-(iii) of the definition of strongly semi-stratifiable in X. For all $n \in \mathbb{N}$ and $x \in X$ let $g'_n(x) = \cap\{g_k(x) : k \leq n\}$ and

$$h_n(x) = \begin{cases} g'_n(x) \cap V_n(x) & \text{, if } x \in Y \\ g'_n(x) & \text{, otherwise.} \end{cases}$$

The function h satisfies $A(Y)$ and for all $n \in \mathbb{N}$ and $x \in X$, $x \in h_n(x)$. By Lemma 25, Y is strongly semi-metrizable in X.

(⇒) Use the same function g constructed in the proof of Lemma 25. □

4. Developable Spaces are Semi-Metrizable

Suppose X is a space, $Y \subseteq X$ and $\cup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a developement (strong developement) [2-developement] for Y in X. For all $n \in \mathbb{N}$ let $\mathcal{H}_n = \{U \subseteq X : U$ is open and $U \subseteq G$ some $G \in \mathcal{G}_n\}$. Then, as with developements in general, $\cup\{\mathcal{H}_n : n \in \mathbb{N}\}$ is a developement (strong developement) [2-developement] for Y in X such that for each $n \in \mathbb{N}$, $\mathcal{H}_n$is a base for X (base for X) [outerbase for Y in X]. The following is a modification of Heath's proof that developable spaces are semi-metrizable, [8].

Theorem 27. *For a space X and $Y \subseteq X$, if Y is developable in X then Y is Osemi-metrizable in X.*

Proof. Suppose that $\mathcal{G} = \cup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a development for Y in X such that for each $n \in \mathbb{N}$, $\mathcal{B}_n$ is a base for X. Let $\mathcal{G}_o = \{X\}$. For each $x \in X$ and $n < \omega$ choose $G_n(x) \in (\mathcal{G}_n)_x$ such that for all $n < \omega$ and $x \in X$, $G_{n+1}(x) \subseteq G_n(x)$. Note for $n < \omega$ and $x, y \in X$,

$$\begin{aligned} x \in st(y, (\mathcal{G}_n)_Y) \quad & \text{if and only if} \quad y \in st(x, (\mathcal{G}_n)_Y) \\ & \text{if and only if} \quad x, y \in G \text{ for some } G \in (\mathcal{G}_n)_Y. \end{aligned}$$

For all $x, y \in X$ define $A(x, y) = \{n < \omega : y \in st(x, (\mathcal{G}_n)_Y)\}$. For all $x, y \in X$, let $n(x, y) = \sup A(x, y)$.Notice that for all $x, y \in X$,

(a) $0 \in A(x,y)$,
(b) $A(x,y) = A(y,x),$ and so $n(x,y) = n(y,x)$,
(c) if $n \in A(x,y)$ then $[0,n] \subseteq A(x,y)$.

Define $d : X \times X \to \mathbb{R}^+$ as follows:

$$d(x,y) = \begin{cases} 0, & \text{if } n(x,y) = \omega \\ 2^{-n(x,y)}, & \text{otherwise} \end{cases} \quad \text{, for all } x,y \in X.$$

Clearly $d(x,x) = 0$ for all $x \in X$. Note that for all $x, y \in X$,

A. since $n(x,y) = n(y,x)$, $d(x,y) = d(y,x)$,
B. $d(x,y) = 0$ iff for all $n < \omega$ there is an $G \in (\mathcal{G}_n)_Y$ with $x, y \in G$,
C. if $d(x,y) \neq 0$ then $A(x,y) = [0, n(x,y)]$,
D. for all $k \in A(x,y)$, $d(x,y) \leq 2^{-k}$.

Also note since X is T_1 and $\mathcal{G}$ is a development for Y in X,

E. for all $x \in X$ and $y \in Y$, $d(x,y) = 0$ iff $x = y$.

Observations A and E imply that d is a symmetric on (Y,X).

Let $x \in \overline{Y}$ and $\epsilon > 0$. Since $x \in \overline{Y}$, for all $m < \omega$ $G_m \in (\mathcal{G}_{n+1})_Y$. Choose $n < \omega$ such that $2^{-n} < \epsilon$ and note that $B(x, 2^{-n}) \subseteq B(x,\epsilon)$. Suppose $y \in G_{n+1}(x)$. If $n(x,y) = \omega$ then $d(x,y) = 0$ and so $y \in B(x,\epsilon)$. Suppose $n(x,y) = m < \omega$. Since $y \in G_{n+1}(x) \in (\mathcal{G}_{n+1})_Y$, $n+1 \in A(x,y)$ and so $d(x,y) = 2^{-m} \leq 2^{-(n+1)} < 2^{-n}$. Thus $y \in B(x,\epsilon)$. Hence $G_{n+1}(x) \subseteq B(x,\epsilon)$. Since $G_n(x)$ is an open neighborhood of x in X, $x \in int(B(x,\epsilon)) = O(x,\epsilon)$. Hence for all $x \in \overline{Y}$ and $\epsilon > 0$, $O(x,\epsilon)$ is an open neighborhood of x in X.

Suppose $y \in Y$. Then for all $n < \omega$, $B(y, 2^{-n}) \subseteq st(y, \mathcal{G}_n)$. Since $\{st(y,\mathcal{G}_n) : n < \omega\}$ is a local base for y in X. The collection $\{B(y,\epsilon) : \epsilon > 0\}$ is a local base for y in X. Hence Y is Osemi-metrizable in X. □

Using essentially the same construction as in Theorem 27 we get the following result. Note that in the proof of Theorem 30 we will construct this semi-metric on (Y,X).

Theorem 28. *For a space X and $Y \subseteq X$, if Y is $2-$ developable in X then it is a semi-metrizable in X.*

Theorem 29. *For a space X and $Y \subseteq X$, if Y is strongly developable in X then Y is strongly semi-metrizable in X.*

Proof. Suppose that $\mathcal{G} = \cup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a strong development for Y in X such that for each $n \in \mathbb{N}$, $\mathcal{G}_n$ is a base for X. Let $\mathcal{G}_o = \{X\}$. Define a symmetric $d : X \times X \to \mathbb{R}^+$ as in Theorem 27. We need only show that d has property 03) of a symmetric on (Y,X).

Suppose that H is a closed subset of X and $x \in X \backslash H$. There is an $m \in \mathbb{N}$

such that $st(x, \mathcal{G}_m) \cap Y \subseteq X \backslash H$. Suppose $y \in Y \cap H$. Then $y \notin st(x, \mathcal{G}_m)$ and so $x \notin st(y, \mathcal{G}_m)$. Since $y \in Y, st(y, \mathcal{G}_m) = st(y, (\mathcal{G}_m)_Y)$. Hence $n(x, y) < m$ and so $d(x, y) > 2^{-m}$. Therefore $d(x, H \cap Y) \geq 2^{-m} > 0$ and so d has property 03) of a symmetric on (Y, X). □

5. Relative Versions of the Alexandroff-Urysohn Metrization Theorem

A (2-) development $\{\mathcal{H}_n : n < \omega\}$ for Y in X is said to be *regular* [Y*- regular*] provided for all $n < \omega$ and $H, H' \in (\mathcal{H}_{n+1})_Y$ with $H \cap H' \neq \phi$ [$H \cap H' \cap Y \neq \phi$] there is a $K \in \mathcal{H}_n$ with $H \cup H' \subseteq K$. .

We proceed along the lines suggested in [9] and Frink's proof, [5], in the proof of the following relative version of the Alexandroff-Urysohn Metrization Theorem.

Theorem 30. *For a space X and $Y \subseteq X$, Y is metrizable in X if and only if there is a regular $2-$development for Y in X.*

Proof. ($\Rightarrow$) Suppose d is a metric on (Y, X). Let $\mathcal{H}_o = \{X\}$ and for all $0 < n < \omega$ let $\mathcal{H}_n = \{O(x, 3^{-n}) : x \in Y\}$. Since for all $y \in Y$ the collection $\{B(y, 3^{-n}) : n \in \mathbb{N}\}$ is a neighborhood base for y in X it is readily seen that $\{\mathcal{H}_n : n < \omega\}$ is a 2- development for Y in X. Suppose $x, z \in Y$ such that $O(x, 3^{-(n+1)}) \cap O(z, 3^{-(n+1)}) \neq \phi$. Let $w \in O(x, 3^{-(n+1)}) \cap O(z, 3^{-(n+1)})$. Then $d(x, z) \leq d(x, w) + d(w, z) < 3^{-(n+1)} + 3^{-(n+1)} = 2/3^{(n+1)}$. Let $y \in O(z, 3^{-(n+1)})$. Then $d(x, y) \leq d(x, z) + d(z, y) < 2 \cdot 3^{-(n+1)} + 3^{-(n+1)} = 3^{-n}$ and so $y \in B(x, 3^{-n})$. Thus $O(x, 3^{-(n+1)}) \cup O(z, 3^{-(n+1)}) \subseteq B(x, 3^{-n})$ and so $O(x, 3^{-(n+1)}) \cup O(z, 3^{-(n+1)}) \subseteq O(x, 3^{-n}) \in \mathcal{H}_n$. Thus $\{\mathcal{H}_n : n < \omega\}$ is a regular 2- development for Y in X.

($\Leftarrow$) Suppose that $\{\mathcal{H}_n : n < \omega\}$ is a regular $2-$ development for Y in X with $\mathcal{H}_o = \{X\}$ and such that for each $n \in \mathbb{N}$, $\mathcal{H}_n$ is an outerbase for Y in X. For each $y \in Y$ and $n < \omega$ choose $H_n(y) \in (\mathcal{H}_n)_y$ such that for all $n < \omega$ and $y \in Y$, $H_{n+1}(y) \subseteq H_n(y)$. For every $x, y \in X$, let $A(x, y) = \{n < \omega : y \in st(x, (\mathcal{H}_n)_Y)\}$ and $n(x, y) = \sup A(x, y)$. Define $g : X \times X \to \mathbb{R}$ by

$$g(x, y) = \begin{cases} 0, & \text{if } n(x, y) = \omega \\ 2^{-n(x,y)}, & \text{otherwise} \end{cases} \quad \text{for all } x, y \in X.$$

Then proceeding as in Theorem 27 we can show that g is a semi-metric on (Y, X) and that for all $x, y \in X$,

A. since $n(x,y) = n(y,x)$, $g(x,y) = g(y,x)$,
B. $g(x,y) = 0$ if and only if for all $n < \omega$ there is a $H \in (\mathcal{H}_n)_Y$ with $x, y \in H$,
C. if $g(x,y) \neq 0$ then $A(x,y) = [0, n(x,y)]$,
D. for all $k \in A(x,y)$, $g(x,y) \leq 2^{-k}$.

Also note since X is T_1 and $\mathcal{H}$ is a $2-$ development for Y in X,

E. for all $x \in X$ and $y \in Y$, $g(x,y) = 0$ if and only if $x = y$.

(Note if $\{\mathcal{H}_n : n < \omega\}$ is a development for Y in X then g is an Osemi-metric on (Y, X).)

Claim 1. *Suppose that $x, y, z \in X$, $0 < \epsilon$, $g(x,y) < \epsilon$ and $g(y,z) < \epsilon$. Then $g(x,z) < 2\epsilon$. In particular $g(x,z) \leq 2g(x,y)$ or $g(x,z) \leq 2g(y,z)$.*

Proof. Case 1. Suppose $g(x,y) = 0 = g(y,z)$.

Let $n < \omega$. By note A there are sets $H, H' \in (\mathcal{H}_{n+1})_Y$ such that $x, y \in H$ and $y, z \in H'$. Let $K \in \mathcal{H}_n$ such that $H \cup H' \subseteq K$. Since $x, z \in K$, $n \in A(x,z)$. Thus $n(x,z) = \omega$ and so $g(x,z) = 0 < 2\epsilon$.

Case 2. Suppose that $g(x,y) = 0$ and $0 < g(y,z) < \epsilon$.

Since $g(x,y) = 0$, $A(x,y) = \omega$. Since $0 < g(y,z)$, $n(y,z) < \omega$. Let $k = n(y,z)$ and note that $k \in A(y,z)$. If $k = 0$ then $g(x,z) \leq 1 = 2^{-0} = g(y,z) < 2g(y,z) < 2\epsilon$.

Suppose $k > 0$. Let $H, H' \in (\mathcal{H}_k)_Y$ such that $x, y \in H$ and $y, z \in H'$. Let $K \in \mathcal{H}_{k-1}$ such that $H \cup H' \subseteq K$. Then, as in Case 1, $x, z \in K \in (\mathcal{H}_{k-1})_Y$ and hence $g(x,z) \leq 2^{-(k-1)} = 2 \cdot 2^{-k} = 2g(y,z) < 2\epsilon$.

(Same for $0 < g(x,y) < \epsilon$ and $g(y,z) = 0$.)

Case 3. Suppose $0 < g(x,y) < \epsilon$ and $0 < g(y,z) < \epsilon$.

Let $k = \min\{n(x,y), n(y,z)\}$. Notice that $k < \omega$ and $2^{-k} < \epsilon$. If $k = 0$ then $g(x,z) \leq 1 = 2^{-k} < \epsilon < 2\epsilon$.

Suppose $0 < k$. Since $k \in A(x,y) \cap A(y,z)$, let $H, H' \in (\mathcal{H}_k)_Y$ such that $x, y \in H$ and $y, z \in H'$. Let $K \in \mathcal{H}_{k-1}$ such that $H \cup H' \subseteq K$. Then $x, z \in K \in (\mathcal{H}_{k-1})_Y$ and so $n(x,z) > k - 1$. Hence by notes B and D, $g(x,z) \leq 2^{-(k-1)} \leq 2 \cdot 2^{-k} < 2\epsilon$.

Notice that in either case ($k = 0$ or $k > 0$), $g(x,z) \leq 2 \cdot 2^{-k}$. Hence $g(x,z) \leq 2g(x,y)$ or $g(x,z) \leq 2g(y,z)$. □

Claim 2. If $a, x_1, x_2, ...x_n, b \in X$ then

$$g(a,b) \leq 2g(a,x_1) + 4g(x_1,x_2) + ... + 4g(x_{n-1},x_n) + 2g(x_n,b).$$

Proof. The same as in Frink's proof, [5]. □

Define $d : X \times X \to \mathbb{R}$ as follows:

$d(a,b) = \inf\{g(a,x_{1)+} \sum_{i=1}^{n-1} g(x_i,x_{i+1}) + g(x_n,b) : n < \omega$ and $x_1, x_2, .., x_n \in X\}$ for all $a, b \in X$.

Claim 3. *For all $a, b, c \in X$, $d(a,c) \leq d(a,b) + d(b,c)$.*

Using Claim 2 the following is readily established.

Claim 4. *For all $a, b \in X$, $g(a,b)/4 \leq d(a,b) \leq g(a,b)$.*

Suppose $x \in X$ and $y \in Y$. Since $g(x,y) = g(y,x)$, $d(x,y) = d(y,x)$. Also since $g(x,y)/4 \leq d(x,y)$ and $g(x,y) = o$ if and only if $x = y$, $d(x,y) = 0$ if and only if $x = y$. Hence d is a symmetric on (Y,X).

Suppose that $y \in Y$ and $\epsilon > 0$. Since g is a semi-metric on (Y,X), $y \in O_g(y,\epsilon) \subseteq B_g(y,\epsilon)$. Since, by Claim 4, for all $x \in X$, $d(x,y) \leq g(x,y)$, $B_g(y,\epsilon) \subseteq B_d(y,\epsilon)$ and so $y \in O_d(y,\epsilon)$.

Suppose $y \in Y$ and U is an open neighborhood of y in X. Let $n \in \mathbb{N}$ such that $B_g(y,\frac{1}{n}) \subseteq U$. Since for all $x \in X$, $g(x,y)/4 \leq d(x,y)$ and thus $g(x,y) \leq 4d(x,y)$, $y \in O_d(y,\frac{1}{4n}) \subseteq B_d(x,\frac{1}{4n}) = \{x \in X : d(x,y) < \frac{1}{4n}\} \subseteq \{x \in X : g(x,y) < \frac{1}{n}\} = B_g(y,\frac{1}{n}) \subseteq U$. Thus the collection $\{B_d(y,\frac{1}{n}) : n \in \mathbb{N}\}$ is a neighborhood base for y in X. Hence d is a semi-metric on (Y,X). This along with Claim 3 establishes that d is a metric on (Y,X). □

Theorem 31. *For a space X and $Y \subseteq X$, Y is Ometrizable in X if and only if there is a regular development for Y in X.*

Proof. (⇒) Suppose d is an Ometric on (Y,X). Let $\mathcal{H}_o = \{X\}$ and for all $0 < n < \omega$ let $\mathcal{H}_n = \{O(x,3^{-n}) : x \in \overline{Y}\} \cup \{X \backslash \overline{Y}\}$. Then $\{\mathcal{H}_n : n < \omega\}$ is a regular development for Y in X.

(⇐) If we proceed as in Theorem 27, the function g is an Osemi-metric on (Y,X) having the same properties as g in Theorem 30. Thus d is a metric on (Y,X). Suppose that $x \in \overline{Y}$ and $\epsilon > 0$. Then $x \in O_g(x,\epsilon)$. Since by Claim 4, for all $y \in X$, $d(x,y) \leq g(x,y)$, and so $x \in O_g(x,\epsilon) \subseteq B_g(x,\epsilon) \subseteq B_d(x,\epsilon)$. Hence $x \in O_d(x,\epsilon)$. Thus d is an Ometric on (Y,X). □

Corollary 32. *For a space X and $Y \subseteq X$, Y is strongly metrizable in X if and only if there is a regular strong development for Y in X.*

Proof. (⇒) Suppose d is a strong metric on (Y,X). Let $\mathcal{H}_o = \{X\}$ and for all $0 < n < \omega$ let $\mathcal{H}_n = \{O(x,3^{-n}) : x \in \overline{Y}\} \cup \{X \backslash \overline{Y}\}$. Then, as in Theorem 31, $\{\mathcal{H}_n : n < \omega\}$ is a regular development for Y in X.

Let $x \in X \backslash Y$ and U be an open neighborhood of x. Since d satisfies condition 03) there is and $\epsilon > 0$ such that $B_d(x,\epsilon) \cap Y \subseteq U$. Let $n \in \mathbb{N}$ such that $3^{-n} <$

ϵ and note that $B_d(x, 3^{-n}) \cap Y \subseteq U$. Suppose $y \in \overline{Y}$ such that $x \in O_d(y, 3^{-(n+1)})$. If $z \in O_d(y, 3^{-(n+1)})$ then $d(x, z) \leq d(x, y) + d(y + z) < 3^{-(n+1)} + 3^{-(n+1)} < 3^{-n}$. Thus $O_d(y, 3^{-(n+1)}) \subseteq B_d(x, 3^{-n})$ and so $st(x, 3^{-(n+1)}) \cap Y \subseteq U$. Hence $\{\mathcal{H}_n : n < \omega\}$ is a regular strong development for Y in X.

($\Leftarrow$) Proceeding as in Theorem 31, we only need show that the Ometric d on (Y, X) determined by the strong development $\{\mathcal{H}_n : n \in \mathbb{N}\}$ for Y in X also satisfies condition 03) of a symmetric on (Y, X). Suppose that H is a closed subset of X and that $x \in X \backslash H$. In Theorem 29 we show that the symmetric g on (Y, X) satisfies condition 03). Thus $g(x, H \cap Y) > 0$ and so $d(x, H \cap Y) \geq g(x, H \cap Y)/4 > 0$. Hence d meets condition 03) of a symmetric on (Y, X). □

Corollary 33. *For a space X and Y a dense subset of X, if Y is Ometrizable (strongly metrizable) in X then there is an Ometric (strong metric) d on (Y, X) which is also a pseudometric on X.*

Proof. Suppose that $\{\mathcal{H}_n : n < \omega\}$ is a regular (strong)development for Y in X with $\mathcal{H}_o = \{X\}$ and such that for each $n \in \mathbb{N}$, $\mathcal{H}_n$ is a base for X. Construct an Ometric (strong metric) d on (Y, X) as in Theorem 31 (Corollary 32). Then d satisfies the triangle inequality and as noted after Claim 4 in Theorem 30, for all $x, y \in X$, $d(x, y) = d(y, x)$. Since Y is dense in X and for all $n < \omega$ the collection $\mathcal{H}_n$ covers X, for all $x \in X$ and $n < \omega$ there is an $H_n \in (\mathcal{H}_n)_Y = \mathcal{H}_n$ with $x \in H_n$. Thus by observation B in Theorem 30, for all $x \in X$, $g(x, x) = 0$ and so $d(x, x) = 0$. □

Theorem 34. *For a space X and $Y \subseteq X$, Y is middle metrizable in X if and only if there is a $Y-$regular $2-$ development for Y in X.*

Proof. ($\Rightarrow$) Suppose d is a middle metric on (Y, X). Let $\mathcal{H}_o = \{X\}$ and for all $0 < n < \omega$ let $\mathcal{H}_n = \{O(x, 3^{-n}) : x \in Y\}$. Then $\{\mathcal{H}_n : n < \omega\}$ is a 2-development for Y in X. Suppose $x, z \in Y$ such that $O(x, 3^{-(n+1)}) \cap O(z, 3^{-(n+1)}) \cap Y \neq \phi$. Let $w \in O(x, 3^{-(n+1)}) \cap O(z, 3^{-(n+1)}) \cap Y$. Then as in the proof of Theorem 30, we can show that $O(x, 3^{-(n+1)}) \cup O(z, 3^{-(n+1)}) \subseteq O(x, 3^{-n}) \in \mathcal{H}_n$. Thus $\{\mathcal{H}_n : n < \omega\}$ is a $Y-$ regular 2- development for Y in X.

($\Leftarrow$) Suppose that $\{\mathcal{H}_n : n < \omega\}$ is a $Y-$ regular $2-$ development for Y in X with $\mathcal{H}_o = \{X\}$ and such that for each $n \in \mathbb{N}$, $\mathcal{H}_n$ is an outerbase for Y in X. Proceed as in the proof of Theorem 30, to construct the same semi-metric g on (Y, X). Then for all $a, b \in X$ let $d(a, b) = \inf\{g(a, x_1)+ \sum_{i=1}^{n-1} g(x_i, x_{i+1}) + g(x_n b) : n < \omega$ and $x_1, x_2, .., x_n \in Y\}$. Then $d : X \times X \to \mathbb{R}$ is a middle metric on (Y, X). □

If X is metrizable with respect to the metric d then for all $x \in X$ and $\epsilon > 0$ $B(x, \epsilon) :\}$ is an open neighborhood of x.

Theorem 35. *For a space X and $Y \subseteq X$,*

1. *if d is a middle metric on (Y, X) then for all $y \in Y$, and $\epsilon > 0$,* $B(y, \epsilon) \cap Y = O(y, \epsilon) \cap Y$.
2. *if d is a metric on (Y, X). Then for all $x \in X$ and $\epsilon > 0$,* $B(x, \epsilon) \cap Y = O(x, \epsilon) \cap Y$.

Lemma 36. *For a space X and Y a dense subset of X,*

1. *if d is an Ometric on (Y, X) then for all $x \in X$ and $\epsilon > 0$* $B(x, \epsilon) = O(x, \epsilon)$.
2. *if X is regular and d is a strong metric on (Y, X) then for all $x \in X$ the collection $\{B(x, \epsilon) : \epsilon > 0\}$ is an open neighborhood base for x.*

Proof. 1) Let $x \in X$ and $\epsilon > 0$. To show $B(x, \epsilon)$ is open we proceed in the usual manner. Suppose $z \in B(x, \epsilon)$ and let $\delta = d(x, z)$. Let $y \in B(z, \epsilon - \delta)$. Then $d(x, y) \leq d(x, z) + d(z, y) < \delta + \epsilon - \delta = \epsilon$. Thus $y \in O(y, \epsilon - \delta) \subseteq B(Y, \epsilon - \delta) \subseteq B(x, \epsilon)$. Hence $B(x, \epsilon)$ is open.

2) Let $x \in X$ and U is an open neighborhood of x. Let V be an open neighborhood of x such that $\overline{V} \subseteq U$. Suppose for all $n \in \mathbb{N}$ $B(x, \frac{1}{n}) \nsubseteq \overline{V}$. This implies that for all $n \in \mathbb{N}$, $B(x, \frac{1}{n}) \setminus \overline{V}$ is a nonempty open set. For $n \in \mathbb{N}$ let $y_n \in Y \cap (B(x, \frac{1}{n}) \setminus \overline{V})$. Then $d(y_n, x) \to 0$ but $y_n \nrightarrow x$ a contradiction of Lemma 10. □

The following is an easy consequence of the Alexandroff-Urysohn Metrization Theorem.

Theorem 37. *If a regular space X has a dense subspace Y which is strongly metrizable in X then X is metrizable.*

Proof. Use a regular strong development for Y in X to construct a strong metric d on (Y, X) as in Corollary 32. As noted in Corollary 33 d is also a pseudometric on X. By Lemma 36, X is pseudometrizable with respect to d and since X is T_1, d is a metric on X. □

6. Examples

Example 1. Let X be the one point compactification of an uncountable discrete space. Let $*$ be the one nonisolated point of X and $Y = \{*\}$.

1. There is a symmetric on (Y, X) defining Y in X (satisfies conditions 01), 02) and 03)) but no symmetric on (Y, X) satisfies condition 04).
2. The subspace Y is not first countable in X ($\{*\}$ is not a G_δ set).
3. The subspace Y is not symmetrizable in X.

1. Define $d : X \times X \to \mathbb{R}^+$ as follows:

$$d(x,y) = \begin{cases} 0 \,, & \text{if } x, y \in X \backslash \{*\} \text{ or } x = * = y \\ 1 \,, & \text{otherwise.} \end{cases}$$

Then d is a symmetric on (Y, X) which defines Y in X. Since $* \in \overline{H}$ for every infinite $H \subseteq X$ and $X \backslash Y$ is uncountable no symmetric on (Y, X) can satisfy condition 04).

3. Suppose d is a symmetric on (Y, X). For all $n \in \mathbb{N}$ let $A_n = \{x \in X \backslash Y : d(*, x) > \frac{1}{n}\}$. Since $\cup\{A_n : n \in \mathbb{N}\} = X \backslash Y$ is uncountable. There is an $n \in \mathbb{N}$ such that A_n is infinite. Let $\{x_n\}$ be any sequence of distinct members of A_n. Then $x_n \to *$ but $d(x_n, *) \nrightarrow 0$. Hence, by Lemma 9, Y is not symmetrizable in X with respect to symmetric d.

Example 2. Let $X = \omega_1$ with the order topology and Y the set of all non-limit ordinals in ω_1. Notice that Y is first countable in X and any symmetric on (Y, X) will satisfy condition 04).

1. There is a function $g : \mathbb{N} \times X \to \mathcal{T}(X)$ satisfying property $A(Y)$ and thus Y is 3 semi-metrizable in X.
2. The subspace Y is not Osemi-metrizable in X since no symmetric on (Y, X) satisfying 01) can also satisfy 02).

1. For all $n \in \mathbb{N}$ and $\alpha < \omega_1$ let

$$g_n(\alpha) = \begin{cases} \{\alpha\}, & \text{if } \alpha \in Y \\ \phi, & \text{otherwise.} \end{cases}$$

The function $g : \mathbb{N} \times X \to \mathcal{T}(X)$ satisfying property $A(Y)$. 2. Suppose d is a symmetric on X satisfying condition 01). For each $\alpha \in Y$, $X \backslash \{\alpha\}$ is closed in X and so by condition 01), $d(\alpha, X \backslash \{\alpha\}) > 0$. Let $n(\alpha) \in \mathbb{N}$ such that $d(\alpha, X \backslash \{\alpha\}) > \frac{1}{n(\alpha)}$. Note that for all $\alpha \in Y$ and $\beta \in X \backslash \{\alpha\}$, $d(\alpha, \beta) \geq \frac{1}{n(\alpha)}$. Since Y is an uncountable set there is an $m \in \mathbb{N}$ and an uncountable $H \subseteq Y$ such that for all $\alpha \in H$, $n(\alpha) = m$. For all $\alpha \in X \backslash H$, $d(x, H) \geq \frac{1}{m} > 0$ but no uncountable (infinite) subset of Y is closed in X. Hence d does not satisfy property 02).

Example 3. Let X be Alexandorff's double circle ([4] Example 3.1.26). Let $X = C_1 \cup C_2$ where $C_i = \{(r_1, r_2) \in \mathbf{R}^2 : r_1^2 + r_2^2 = i\}$ for $i = 1, 2$. For every $z \in C_1$ and $j \in \mathbb{N}$, let $U_j(z) = V_j(z) \cup p(V_j(z) - \{z\})$ where$V_j(z)$ is the arc of C_1 centered at z with length $\frac{2}{j}$ and p is the projection of C_1 to C_2 from the origin $O = (0, 0)$. Points of C_2 are isolated and for all $z \in C_1$ the collection $\{U_j(z) : j = 1, 2, \cdots\}$ is a neighborhood base for z. The space X is compact and first countable and C_1 is a closed metrizable subspace of X but not a G_δ subset of X, [4]. For $x, y \in C_1$, let $\theta(x, y)$ be length of the shortest arc between x and y on C_1.

1. C_1 is O semi-metrizable in X but since C_1 is not a G_δ subset of X it is not 3semi-metrizable in X.
2. Since C_1 does not have a countable outerbase in X, it is not $2-$ developable in X.
3. There is a symmetric on (C_2, X) such that for all $x \in X$ and $\epsilon > 0$, $x \in O(x, \epsilon)$.
4. No symmetric on (C_2, X) can satisfy both conditions 01) and 02).

1. We define a symmetric d on (C_1, X) as follows:

$$d(x,y) = d(y,x) = \begin{cases} 0, & \text{if } x = y, \\ \theta(x,y), & \text{if } x, y \in C_1 \\ \theta(x, p^{-1}(y)), & \text{if } x \in C_1, y \in C_2, y \neq p(x), \\ \pi, & \text{otherwise.} \end{cases}$$

Then for all $y \in C_1$ and $j \in \mathbb{N}$, $B_d(y, \frac{1}{j}) = U_j(y)$. Thus for every $y \in C_1$, $\{B_d(y, \frac{1}{j}) : j = 1, 2, \cdots\}$ is a neighborhood base of y in X and d satisfies condition O). Thus d is an Osemi-metric on (C_1, X).

3. The symmetric d above is a symmetric on (C_2, X) satisfying the desired condition.

4. Note that C_2 is not an F_σ subset of X. Suppose d_1 be a symmetric on (C_2, X) satisfying the conditions 01) and 02). Assume that d_1 satisfies condition 01). For every $z \in C_2$ let $H_z = X - \{z\}$ then H_z is closed in X because every point of C_2 is isolated. By condition 01), $d_1(z, H_z) > 0$. For every $z \in C_2$ let $j_z \in \mathbb{N}$ such that $d_1(z, H_z) > \frac{1}{j_z}$. For every $j \in \mathbb{N}$ put $K_j = \{z \in C_2 : j = j_z\}$ then $X - K_j = X - \cup\{\{z\} : z \in K_j\} = \cap\{X - \{z\} : z \in K_j\} = \cap\{H_z : z \in K_j\}$. Therefore for every $x \in X - K_j$, $d_1(x, K_j) > \frac{1}{j}$, and so by condition 02), K_j is closed in X. Then $C_2 = \cup\{K_j : j = 1, 2, \cdots\}$, a contradiction, since C_2 is not F_σ subset of X.

Example 4. Let $X_1 = \omega_1$ with the order topology and Y_1 be the set of isolated points of X_1. Let X_2 be the space given in Example 2 and $Y_2 = C_1$. Let X be the topological sum of X_1 and X_2 and $Y = Y_1 \cup Y_2$.

1. Y is semi-metrizable in X.
2. Y is neither Osemi-metrizable or 3semi-metrizable in X.

1. Let d_1 be a 03)− semi-metric on (Y_1, X_1), see Example 3, and let d_2 be an 02)− semi-metric on (Y_2, X_2), see Example 2. Define a symmetric d on (Y, X) as follows:

$$d(x,y) = \begin{cases} d_1(x,y), & \text{if } x,y \in X_1, \\ d_2(x,y), & \text{if } x,y \in X_2, \\ 1, & \text{otherwise.} \end{cases}$$

Since both d_1 and d_2 satisfy conditions 01) and 04) so does d. Thus d is a semi-metric on (Y, X).

2. Suppose that d is a symmetric on (Y, X). Notice that if d satisfies any of the conditions 01) to 04) then $d|_{X_1}$ is a symmetric on (Y_1, X_1) and $d|_{X_2}$ is a symmetric on (Y_2, X_2) satisfying the same conditions. Thus since Y_1 is not Osemi-metrizable in X_1, Y is not Osemi-metrizable in X. Since Y_2 is not 3semi-metrizable in X_1, Y is not 3semi-metrizable in X.

Example 5. Let $X = \omega_1$ and Y be the set of all limit ordinals in ω_1. Define a topology on X by letting points of $X \backslash Y$ be isolated and for all $\alpha \in Y$ basic open sets are of the form $((\beta, \alpha)\backslash Y) \cup \{\alpha\}$. Notice that Y is a closed discrete non G_δ subset of the Hausdorff space X.

1. Y is developable in X.
2. No semi-metric on (Y, X) can satisfy condition 03).

Example 6. Let $X = \mathbb{R} \times [0, \infty)$ with the tangent disk topology with points above the $x-$axis isolated and $Y = \mathbb{R} \times \{0\}$. Since X is developable Y is strongly developable in X. Note Y is a closed discrete G_δ subset of X. For the same reason that X is not paracompact, no development for Y in X can be regular.

Although Y is not metrizable in X consider the following function $d : X \times X \to \mathbb{R}^+$:

i. $d((a,b),(c,e)) = 0,$ for all $a, c \in \mathbb{R}$ and $b, e \in (0, \infty)$
ii. $d((a,0),(c,0)) = 1,$ for all $a, c \in \mathbb{R}$
iii. $d((a,0),(c,e)) = d((c,e),(a,0)) = \sqrt{\frac{(a-c)^2+(b-e)^2}{2e}}$ for all $a, c \in \mathbb{R}$ and $e \in (0, \infty)$.

Note that $d((a,0),(c,e))$ is the radius of the circle tangent to the $x-$axis at $(a, 0)$ passing through (c, e). The function d is a strong semi-metric on (Y, X). Although d does not satisfy the triangle inequality it is a middle metric on (Y, X).

Example 7. Let $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ be a collection of uncountable subsets of ω_1 such that

1) for all $\alpha < \beta < \omega_1$ $A_\alpha \neq A_\beta$ and there is a $\gamma < \omega_1$ such that $A_\gamma \subseteq A_\alpha \cap A_\beta$.

2) if $\mathcal{A}' \subseteq \mathcal{A}$ is infinite then $\cap\mathcal{A}' = \phi$.

Also assume that for all $n < \omega$, $A_{n+1} \subset A_n$. Let $X = [0, \omega_1]$ and topologize X as follows:

$a)$ points of ω_1 are isolated;

$b)$ basic neighborhoods of the point ω_1 are of the form $A_\alpha \cup \{\omega_1\}$ where $\alpha < \omega_1$.

The space X is regular. Let $Y = \omega_1$.

1. The subspace Y is Osemi-metrizable in X.

2. Since Y consists of isolated points of X, it is 3semi-metrizable in X.

3. No symmetric on (Y, X) defines Y in X.

1. For each $\alpha < \omega_1$let $n(\alpha) = \min\{k < \omega : \alpha \notin A_k\}$. For all $\alpha, \beta \leq \omega_1$ let

$$d(\alpha, \beta) = d(\beta, \alpha) = \begin{cases} 0, & \text{if } \alpha = \beta; \\ 1, & \text{if } \alpha \neq \beta \text{ and } \alpha, \beta < \omega_1; \\ \frac{1}{n(\alpha)}, & \text{if } \alpha < \omega_1 \text{ and } \beta = \omega_1. \end{cases}$$

Then d is an Osemi-metric on (Y, X).

3. Suppose that $h : X \times X \to \mathbb{R}^+$ is a symmetric on (Y, X) which defines Y in X. Since every countable subset of ω_1 is closed in X, by condition 03) for every $\alpha < \omega_1$ there is an $n_\alpha \in \mathbb{N}$ such that $h(\omega_1, [0, \alpha]) > \frac{1}{n_\alpha}$. Thus there is an $n^* \in \mathbb{N}$ and an uncountable $S \subseteq \omega_1$ such that, for all $\alpha \in S$, $h(\omega_1, [0, \alpha]) > \frac{1}{n^*}$. Hence $h(\omega_1, Y) = h(\omega_1, \cup\{[0, \alpha] :\in S\}) > \frac{1}{n^*} > 0$. Hence by condition 02), Y is closed in X, which it is not. Therefore no symmetric on (Y, X) can define Y in X.

Example 8. (see [4] Example 1.6.19) Let $X = \{0\} \cup (\cup\{X_n : n \in \mathbb{N}\})$ where for all $n \in \mathbb{N}$, $X_n = \{\frac{1}{n}\} \cup \{\frac{1}{n} + \frac{1}{j} : j = n^2, n^2 + 1, ...\}$. Note that for $i, j \in \mathbb{N}$ if $i \neq j$ then $X_i \cap X_j = \phi$.For convenience let $A_n = \{n^2, n^2 + 1, ...\}$ for each $n \in \mathbb{N}$. The topology on X is defined as follows:

$i)$ All points of $\cup\{X_n \backslash \{\frac{1}{n}\} : n \in \mathbb{N}\}$are isolated.

$ii)$ For each $n \in \mathbb{N}$ a basic neighborhood of $\frac{1}{n}$ is of the form $B(n, j) = \{\frac{1}{n}\} \cup \{\frac{1}{n} + \frac{1}{k} : k = j, j + 1, ...\}$ some $j \in A_n$.

$iii)$ A basic neighborhood of 0 is of the form $\{0\} \cup (\cup\{B(i, j_i) : i \geq n\})$ for some $n \in \mathbb{N}$ and choice of $j_i \in A_i$ for each $i \geq n$.

As noted in [4] the space X is a countable perfectly normal sequential space which is not Fréchet and the point 0 does not have a countable local base. Let

$Y = X \backslash \{0\}$.

1. There is a symmetric on (Y, X) satisfying conditions 01)-04).
2. If $g : X \times \mathbb{N} \to \mathcal{T}(X)$ satisfying condition $A(Y)$ then there is an $n \in \mathbb{N}$ such that $0 \notin g_n(0)$. Thus Y is not strongly semi-metrizable in X. and Y is not strongly first countable in X.

1. Define a symmetric d on (Y, X) as follows:

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}, \text{ for all } x, y \in X \backslash \{\tfrac{1}{n} : n \in \mathbb{N}\} \text{ and}$$

$$d(\tfrac{1}{n}, x) = d(x, \tfrac{1}{n}) = \begin{cases} \frac{1}{n}, & \text{if } x = 0 \\ |\frac{1}{n} - \frac{1}{k}|, & \text{if } x = \frac{1}{k} \text{ some } k \in \mathbb{N} \\ \frac{1}{j}, & \text{if } x = \frac{1}{n} + \frac{1}{j} \text{ some } j \in A_n \\ 1, & \text{otherwise} \end{cases}, \quad \begin{matrix} \text{for all } n \in \mathbb{N} \\ \text{and } x \in X. \end{matrix}$$

The symmetric d on (Y, X) satisfying conditions 01)-04).

2. Suppose that $g : \mathbb{N} \times X \to \mathcal{T}(X)$ satisfies condition $A(Y)$. Note that if $0 \in g_n(0)$ for all $n \in \mathbb{N}$ then by condition $1(b)$ of $A(Y)$ the collection $\{\{0\} \cup g_n(0) : n \in \mathbb{N}\}$ would be a local base for 0 in X, but 0 does not have a countable local base.

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