International Journal of Pure and Applied Mathematics
Volume 49 No. 3 2008, 303-308

ON THE CONNECTION BETWEEN CHARACTERIZATION THEOREMS OF POLYA AND SKITOVICH-DARMOIS

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Abstract: The two basic theorems of characterization of Gaussian (normal) distributions are those of Polya's and Skitovich-Darmais'. The first of this two can be proved by the elementary methods based on the Levy Continuity Theorem, while the second one uses non-elementary and non-statistical tools from the theory of analytic functions. We give an elementary proof of Sticovih-Darmais' Theorem for the case of two random variables in each of the two linear forms. The proof is based on the reduction of Sticovih-Darmais' Theorem to Polya's Theorem.

AMS Subject Classification: 60E15, 60E07
Key Words: characterization of Gaussian distribution, Sticovih-Darmais' Theorem, Polya's Theorem

1. Introduction

The non-parametric problem of finding unknown distribution functions is among of principal problems of mathematical statistics. Clearly, characterization theorems for various classes of distributions are very important for the solution of this problem since they give necessary and sufficient conditions for a distribution to belong to a given class of distributions. Besides, these theorems have other statistical applications and independent interest as well. It is also clear that characterization theorems for Gaussian (normal) distributions are the most

Received: August 14, 2008

important ones because of the special role of the Gaussian law in mathematical statistics and probability theory.

The research on characterizing properties of Gaussian distributions has a long history going back to Maxwell's investigations in statistical theory of ideal gas. J. Maxwell, as a physicist, was interested in the distribution of velocities of ideal gas moleculas. Using physical considerations and reasonings, in 1859 he obtained the distribution which is now well-known in statistics as Pearson's χ^2-distribution. To obtain this result J. Maxwell first showed that if X_1 and X_2 are independent random variables such that $Y_1 = X_1 \cos\alpha + X_2 \sin\alpha$ and $Y_2 = -X_1 \sin\alpha + X_2 \cos\alpha$ (where α is not a multiple of $\frac{\pi}{2}$) are also independent then X_1 and X_2 both are Gaussian random variables thus having obtained the very first characterization theorem for Gaussian distributions. Specifically probabilistic studies were initiated by D. Polya [6], M. Kac [3] and S.N. Bernstein [1] who proved first mathematical theorems in the area of characterization of Gaussian distributions.

We give now the brief sketch of their results.

M. Kac gave the mathematical proof of J. Maxwell's result for the case in which X_1 and X_2 are symmetric random variables with now other restriction of generality [1]. S.N. Bernstein considered essentially the same problem for the particular case $\alpha = \frac{\pi}{4}$ in which the linear forms are $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$, and proved the result assuming the equality of variances of X_1 and X_2 and also the existence of their densities [2] which additionally were supposed to be non-zero everywere.

The result of D. Polya is of another kind of the characterization theorems. The formulation is as follows:

Polya's Theorem. (see D. Polya, [6]) *Let ξ be a random variable with finite variance and $\xi_1, \xi_2, \dots, \xi_m$ $(m > 1)$ be independent copies of ξ. Let also $\alpha_1, \alpha_2, \dots, \alpha_m$ be non-zero real numbers such that $\alpha_1^2 + \alpha_2^2 + \dots + \alpha_m^2 = 1$. If the linear form $a_1\xi_1 + a_2\xi_2 + \dots + a_m\xi_m$ has the same distribution as ξ then ξ is Gaussian random variable.*

Remark. The restriction of the existence of the second moment in this theorem was removed first by R.G. Laha and L. Lukacs [5].

After the results of the first stage "an impressive number of authors contributed improvements and variants, sometimes by rather deep methods. The development culminated in a result of Skitovich" (W. Feller, [2], p. 79). This result, usually referred as Skitovich-Darmois' Theorem, is formulated as follows:

Skitovich-Darmois' Theorem. (see [6]) *Let $L_1 = \alpha_1 X_1 + \alpha_2 X_2 + \dots +$*

$\alpha_n X_n$ *and* $L_2 = \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_n X_n$, $n \geq 2$, *be linear forms of mutually independent random variables (not necessarily identically distributed). If* L_1 *and* L_2 *are also independent then those of* X_j *for which* $\alpha_j \beta_j \neq 0$ *are Gaussian.*

The essential contribution to the general theory of characterization problems of mathematical statistics, including the case of Gaussian distribution, has been made by Yu.V. Linnik, C.R. Rao and other investigators (see [4] for more information).

The result of Skitovich and, especially, the later results of Yu. V. Linnik and C.R. Rao were proved using special non-elementary and non-statistical tools from the theory of analytic functions.

In our short paper [7] an elementary and simple proof of the theorem of the first stage, namely the characterization theorem of Polya, is found in a general form without any a priori restriction on the random variables under consideration. It turned out that the existence of the variance can be derived using only natural necessary (and hence non-restrictive) assumptions of the theorem itself. It is plausible that some other known and new results in this area can be proved using the idea of reducing the problem to the Polya's characterization theorem. As an example of this possibility we give now an elementary and simple proof of Skitovich-Darmois theorem for the particular case of two linear forms with two random variables in each.

Theorem. *Let* ξ_1 *and* ξ_2 *be independent random variables* (*not necessarily identically distributed*) *and* a_1, a_2, b_1, b_2 *be non-zero real numbers. If the random variables* $\eta_1 = a_1\xi_1 + a_2\xi_2$ *and* $\eta_2 = b_1\xi_1 + b_2\xi_2$ *are also independent then* ξ_1 *and* ξ_2, *and hence* η_1 *and* η_2, *are Gaussian random variables.*

Proof. Without loss of generality we can assume that $|a_1 b_1| = |a_2 b_2|$. Indeed, if $|a_1 b_1| \neq |a_2 b_2|$ we could consider the linear forms $\eta_1 = a_1\xi_1 + a_2\sqrt{c}\xi_2'$ and $\eta_2 = b_1\xi_1 + b_2\sqrt{c}\xi_2'$, where $c = \frac{|a_1 b_1|}{|a_2 b_2|}$ and $\xi_2' = \frac{\xi_2}{\sqrt{c}}$. Nothing will be changed this way: η_1 and η_2 will remain the same and ξ_2' will be independent of ξ_1. Therefore, in what follows in the linear forms η_1 and η_2 we assume that $|a_1 b_1| = |a_2 b_2|$.

Denote the characteristic functions of ξ_1, ξ_2 and (η_1, η_2) by χ_1, χ_2 and $\chi_{(\eta_1,\eta_2)}$ respectively. Using independence of ξ_1 and ξ_2 we get the equality

$$\chi_{(\eta_1,\eta_2)}(t_1, t_2) = Ee^{i[(a_1\xi_1 + a_2\xi_2)t_1 + (b_1\xi_1 + b_2\xi_2)t_2]}$$
$$= \chi_1(a_1 t_1 + b_1 t_2)\chi_2(a_2 t_1 + b_2 t_2).$$

If we use independence of η_1 and η_2 at first and then that of ξ_1 and ξ_2, we get

$$\chi_{(\eta_1,\eta_2)}(t_1, t_2) = \chi_1(a_1 t_1)\chi_2(a_2 t_1)\chi_1(b_1 t_2)\chi_2(b_2 t_2)$$

and therefore the following equality holds:

$$\chi_1(a_1t_1 + b_1t_2)\chi_2(a_2t_1 + b_2t_2) = \chi_1(a_1t_1)\chi_2(a_2t_1)\chi_1(b_1t_2)\chi_2(b_2t_2).$$

Denoting $a_1t_1+b_1t_2 = x$ and $a_2t_1+b_2t_2 = y$ and solving this simple algebraic system we come to the following equality in which Δ denotes the determinant of the system, i.e. $\Delta = a_1b_2 - a_2b_1$,

$$\chi_1(x)\chi_2(y) = \chi_1\left(\frac{a_1b_2x - a_1b_1y}{\Delta}\right)\chi_2\left(\frac{a_2b_2x - a_2b_1y}{\Delta}\right) \times \chi_1\left(\frac{-a_2b_1x + a_1b_1y}{\Delta}\right)\chi_2\left(\frac{-a_2b_2x + a_1b_2y}{\Delta}\right).$$

Note that $\Delta \neq 0$. Indeed, if $a_1b_2 = a_2b_1$ then $a_1b_2\xi_1 + a_2b_2\xi_2 = a_2b_1\xi_1 + a_2b_2\xi_2$ i.e. $b_2\eta_1 = a_2\eta_2$ in contradiction with independence of η_1 and η_2. Putting in the equality above first $y = 0$ and then $x = 0$ we get, respectively, the following two equalities:

$$\chi_1(x)=\chi_1\left(\frac{a_1b_2}{\Delta}x\right)\chi_2\left(\frac{a_2b_2}{\Delta}x\right)\chi_1\left(\frac{-a_2b_1}{\Delta}x\right)\chi_2\left(\frac{-a_2b_2}{\Delta}x\right), \tag{1}$$

$$\chi_2(x)=\chi_1\left(\frac{-a_1b_1}{\Delta}x\right)\chi_2\left(\frac{-a_2b_1}{\Delta}x\right)\chi_1\left(\frac{a_1b_1}{\Delta}x\right)\chi_2\left(\frac{a_1b_2}{\Delta}x\right). \tag{2}$$

As we notice above, without loss generality we can suppose that $|a_1b_1| = |a_2b_2|$. Consider now the two possible cases: 1. $a_1b_1 = -a_2b_2$ and 2. $a_1b_1 = a_2b_2$.

Taking into account independence of ξ_1 and ξ_2, denoting $\chi = \chi_1\chi_2$ and multiplying equalities (1) and (2), we get for the first case the equality

$$\chi(x) = \chi\left(\frac{a_1b_2}{\Delta}x\right)\chi\left(\frac{-a_2b_1}{\Delta}x\right)\chi\left(\frac{a_1b_1}{\Delta}x\right)\chi\left(\frac{-a_1b_1}{\Delta}x\right) \tag{3}$$

which means that $\chi(x) = \chi(\alpha_1x)\chi(\alpha_2x)\chi(\alpha_3x)\chi(\alpha_4x)$ with $\Sigma\alpha_i^2 = 1$ and by virtue of Polya's Theorem we get that $\xi_1 + \xi_2$ is Gaussian random variable. Now using Cramer's Theorem we see that the proof is finished for the case $a_1b_1 = -a_2b_2$.

Now we consider the case $a_1b_1 = a_2b_2$. Let us note that in this case $a_1b_2a_2b_1 = (a_1b_1)^2$ and hence a_1b_2 and a_2b_1 have the same sign. Therefore, $max(|a_1b_2|, |a_2b_1|) > |a_1b_2 - a_2b_1| = |\Delta|$. Without loss of generality we may assume $|a_1b_2| = max(|a_1b_2|, |a_2b_1|)$. Using equality (3) we get $|\chi(x)| \leq |\chi\left(\frac{a_1b_2}{\Delta}x\right)|$ and so $|\chi(x)| \geq |\chi(\alpha x)|$ with $\alpha = \frac{\Delta}{a_1b_2}$, $|\alpha| < 1$. From this inequality it follows that for all x we have

$$|\chi(x)| \geq |\chi(\alpha x)| \geq |\chi(\alpha^2x)| \geq \cdots \geq |\chi(\alpha^m x)| \cdots \geq |\chi(0)| = 1,$$

and the sum $\xi_1 + \xi_2$ is constant and therefore both of the summands are constants.

Remark. If one of the coefficients a_1, a_2, b_1, b_2 is zero, say $a_2 = 0$, then the two forms η_1 and η_2 can be written as $\eta_1 = \xi_1$ and $\eta_2 = \xi_1 + b\xi_2$. Equality (3) shows for this case that

$$\chi(x) = \chi(x)\chi\left(\frac{x}{b}\right)\chi\left(-\frac{x}{b}\right).$$

Canceling $\chi(x)$ out of this equality in a sufficiently small neighborhood of zero we get the relation

$$1 = \chi\left(\frac{x}{b}\right)\chi\left(-\frac{x}{b}\right)$$

which shows that the sum $\xi_1+\xi_2$, and hence both of the summands are constants (because the modulus of the characteristic function of the sum is 1 in some neighborhood of zero). Therefore if one of the coefficients a_1, a_2, b_1, b_2, is zero then ξ_1 and ξ_2 will still be Gaussian (unlike the case of $n > 2$). If two or more of these coefficients are zero then we get trivial situations in which the random variables may be or may not be Gaussian.

It would be interesting if the approach of this note could be successful for the case of linear forms with more than two random variables.

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