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MULTIPLE DOMAIN DECOMPOSITION FOR BITSADZE-SAMARSKI NONLOCAL BOUNDARY VALUE PROBLEM

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Abstract: In the present paper the author considers Bitsadze-Samarski nonlocal boundary value problem to quasilinear elliptic equation. The domain decomposition into multiple subdomains for two-dimensional case is done, the sequence of auxiliary problems is constructed and the convergence of their solutions to the solution of the source problem is proved. The rate of convergence is established and the a priori estimate of the convergence is obtained.

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1. Introduction

Consider Bitsadze-Samarski nonlocal boundary value problem to quasilinear partial differential equation

$$u = f(x,\, y,\, u), \quad (x,\, y) \in \Omega = (a,\, b) \times (0,\, 1)\,, \tag{1}$$

$$u\,(a, y\,) = \phi\,(y)\,, \quad u\,(b,\, y) = u\,(\xi,\, y)\,, \quad y \in [0,\, 1], \tag{2}$$

$$u\,(x, 0\,) = \psi_0(x), \quad u\,(x, 1) = \psi_1(x), \quad x \in [a,\, b]\,, \tag{3}$$

where f, φ, ψ_0, ψ_1 are given functions, u is unknown function, ξ is a fixed point from $(a,\, b)$ interval.

The linear version of the problem (1)-(3) was investigated in many works (see, for example Bitsadze [2], Bensusan [1]). Multiple domain decomposition

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for linear version of the problem (1)-(3) was made by the author in work [3].

In the present work we consider the questions of existence and uniqueness of the solution of problem (1)-(3), then make the decomposition of initial domain into n subdomains, construct the sequence of auxiliary problems and prove the convergence of their solutions to the solution of the problem (1)-(3).

It is true the following

Theorem 1. *Let functions f, φ, ψ_0, ψ_1 be continuous and let f have continuous nonnegative partial derivative on argument u. Then the problem (1)-(3) may have only one solution.*

Theorem 2. *Let functions f, φ, ψ_0, ψ_1 be continuous and let f have continuous nonnegative partial derivative on argument u. If the properties of function f provide existence of the solution of the following boundary value problem to (1) equation*

$$u = f(x, y, u), \quad (x, y) \in \Omega,$$

$$u(a, y) = \phi(y), \quad u(b, y) = g(y), \quad y \in [0, 1],$$

$$u(x, 0) = \psi_0(x), \quad u(x, 1) = \psi_1(x), \quad x \in [a, b],$$

for any continuous function g, consistent with ψ_0 and ψ_1, then exists unique solution of the problem (1)-(3).

Consider the decomposition of the domain Ω into n (n is any positive integer) subdomains as follows

$$\overline{\Omega} = \cup_{i=1}^{n} \overline{\Omega}_i, \quad \Omega_i = (a_i, b_i) \times (0, 1),$$

where

$$a_i < a_{i+1} < b_i, \quad a_{i+1} < b_i < b_{i+1}, \quad i = 1, 2, \dots, n-1,$$

$$a_1 = a, \quad b_n = b.$$

Let i_ξ be the number of the interval, such that $\xi \in \left(a_{i_\xi}, b_{i_\xi}\right)$. Assume that $i_\xi < n$.

Put on the subdomains Ω_i the following problems

$$v_{ik} = q_{ik}(x, y) v_{ik}, \quad (x, y) \in \Omega_i, \tag{4}$$

$$i = 1, 2, \dots, n; \quad k = 1, 2, \dots,$$

$$v_{ik}(a_i, y) = v_{i-1,k}(a_i, y), \quad y \in [0, 1], \tag{5}$$

$$i = 2, 3, \dots, n; \quad k = 1, 2, \dots,$$

$$v_{ik}(b_i, y) = v_{i+1,k-1}(b_i, y), \quad y \in [0, 1], \tag{6}$$

$$i = 1, 2, \dots, n-1; \quad k = 2, 3, \dots$$

$$v_{1,k}(a_1, y) = 0, \quad v_{nk}(b_n, y) = v_{i_\xi, k}(\xi, y), \quad y \in [0, 1], \tag{7}$$

$$k = 1, 2, \dots,$$

$$v_{ik}(x, 0) = 0, \quad v_{ik}(x, 1) = 0, \quad x \in [a_i, b_i], \tag{8}$$

$$i = 1, 2, \dots, n; \quad k = 1, 2, \dots,$$

$$v_{i,1}(b_i, y) = r_i(y), \quad y \in [0, 1], \quad i = 1, 2, \dots, n-1, \tag{9}$$

where $r_1, r_2, \dots, r_{n-1}, q_{1,k}, q_{2,k}, \dots, q_{nk}, \quad k = 1, 2, \dots,$ are given functions and

$$r_1(0) = r_1(1) = r_2(0) = r_2(1) = \dots = r_{n-1}(0) = r_{n-1}(1) = 0.$$

Enter the following notations:

$$\alpha_i(x) = \frac{b_i - x}{b_i - a_i}, \quad \beta_i(x) = \frac{x - a_i}{b_i - a_i},$$

$$\underline{Q}_i = \underline{Q}_{i-1}\, \alpha_i(a_{i+1}) + \beta_i(a_{i+1}), \quad i = 2, 3, \dots, n-1,$$

$$\overline{Q}_i = \underline{Q}_i\, \alpha_{i+1}(b_i) + \beta_{i+1}(b_i), \quad i = 1, 2, \dots, n-1,$$

where $\underline{Q}_1 = \beta_1(a_2)$. Let $Q = \max\limits_{1 \le i \le n-1} \overline{Q}_i$. It is easy to show that $0 < Q < 1$.

It is true the following

Theorem 3. *If functions $q_{1,k}, q_{2,k}, \dots, q_{nk}, \quad k = 1, 2, \dots,$ are continuous and nonnegative in domains $\overline{\Omega}_1, \overline{\Omega}_2, \dots, \overline{\Omega}_n$ respectively and functions $r_1, r_2, \dots, r_{n-1}$ are continuous in $[0, 1]$ segment, then sequences $(v_{1k})_{k=1}^{\infty}$, $(v_{2k})_{k=1}^{\infty}, \dots, (v_{nk})_{k=1}^{\infty}$ of the solutions of problems (4)-(9) uniformly converge to zero in domains $\overline{\Omega}_1, \overline{\Omega}_2, \dots, \overline{\Omega}_n$ respectively and we have:*

$$\max_{1 \le i \le n} \left\{ \max_{(x, y) \in \overline{\Omega}_i} |v_{ik}(x, y)| \right\} \le C Q^{k-1}, \quad C = const.$$

Put on the subdomains Ω_i the following problems

$$u_{ik} = f(x, y, u_{ik}), \quad (x, y) \in \Omega_i, \tag{10}$$

$$i = 1, 2, \dots, n; \quad k = 1, 2, \dots,$$

$$u_{ik}(a_i, y) = u_{i-1,k}(a_i, y), \quad y \in [0, 1], \tag{11}$$

$$i = 2, 3, \dots, n; \quad k = 1, 2, \dots,$$

$$u_{ik}(b_i, y) = u_{i+1,k-1}(b_i, y), \quad y \in [0, 1], \tag{12}$$

$$i = 1, 2, \dots, n-1; \quad k = 2, 3, \dots,$$

$$u_{1,\,k}\,(a_1,\,y)=\varphi(y),\;\; u_{n,\,k}\,(b_n,\,y)=u_{i_\xi,\,k}\,(\xi,\,y)\,,\;\; y\in[0,\,1], \tag{13}$$
$$k=1,\,2,\,...,$$
$$u_{ik}\,(x,0\,)=\psi_0(x),\;\; u_{ik}\,(x,1)=\psi_1(x),\;\; x\in[a_i,\,b_i]\,, \tag{14}$$
$$i=1,\,2,\,...\,,\,n;\;\; k=1,\,2,\,...,$$
$$u_{i,1}\,(b_i,\,y)=l_i\,(y)\,,\;\; y\in[0,\,1],\;\; i=1,\,2,\,...\,,\,n-1\,, \tag{15}$$

where l_1, l_2, ..., l_{n-1} are arbitrarily taken continuous functions, consistent with ψ_0 and ψ_1.

It is true the following

Theorem 4. *Let functions f, φ, ψ_0, ψ_1 be continuous and let f have continuous nonnegative partial derivative on argument u. If the properties of function f provide existence of the solution u of the problem (1)-(3) and existence of the solutions of the following boundary value problems to (1) equation:*

$$w_i=f(x,\,y,\,w_i),\;\; (x,\,y)\in\Omega_i,\;\; i=1,\,2,\,...\,,\,n,$$
$$w_i\,(a_i,y\,)=g_i\,(y)\,,\;\; w_i\,(b_i,\,y)=h_i\,(y)\,,\;\; y\in[0,\,1],\;\; i=1,\,2,\,...\,,\,n,$$
$$w_i\,(x,0\,)=\psi_0(x),\;\; w_i\,(x,1)=\psi_1(x),\;\; x\in[a_i,\,b_i]\,,\;\; i=1,\,2,\,...\,,\,n,$$

for $g_1\equiv\phi$ and any continuous functions $g_2, g_3, \ldots, g_n$ and $h_1, h_2, \ldots, h_n$ consistent with ψ_0 and ψ_1, then exist the solutions of problems (4)-(9) and sequences $(u_{1k})_{k=1}^{\infty}$, $(u_{2k})_{k=1}^{\infty}$, ..., $(u_{n\,k})_{k=1}^{\infty}$ of the solutions of problems (4)-(9) uniformly converge to the solution of the problem (1)-(3) in domains $\overline{\Omega}_1$, $\overline{\Omega}_2$, ..., $\overline{\Omega}_n$ respectively and we have:

$$\max_{1\le i\le n}\left\{\max_{(x,\,y)\in\bar{\Omega}_i}|u_{i\,k}\,(x,\,y)-u\,(x,\,y)|\right\}\le CQ^{k-1},\;\; C=const.$$

Proof. Using the hypothesis of the theorem it is easy to show that the solutions of problems (4)-(9) exist.

Subtract from the equations (10) the equation (1). We obtain:

$$(u_{i\,k}-u)=f(x,\,y,\,u_{i\,k})-f(x,\,y,\,u),\;\; (x,\,y)\in\Omega_i, \tag{16}$$
$$i=1,\,2,\,...\,,\,n;\;\; k=1,\,2,\,....$$

Transform the right side of equations (16). We have:

$$f(x,\,y,\,u_{i\,k})-f(x,\,y,\,u)$$
$$=\frac{\partial f}{\partial u}\,(x,\,y,\,\theta_{i\,k}\,(x,\,y)\,u_{i\,k}+(1-\theta_{i\,k}\,(x,\,y))\,u)\,[u_{i\,k}-u]\,,$$

where $0\le\theta_{i\,k}\,(x,\,y)\le 1$.

Enter the following notations:

$$q_{ik} = \frac{\partial f}{\partial u}\left(x,\, y,\, \theta_{i\,k}\left(x,\, y\right) u_{i\,k} + \left(1 - \theta_{i\,k}\left(x,\, y\right)\right) u\right),$$

$$i = 1,\, 2,\, ...,\, n;\;\; k = 1,\, 2,\, \ldots$$

$$v_{ik} = u_{ik} - u,\;\; i = 1,\, 2,\, ...,\, n,\;\; k = 1,\, 2,\, \ldots.$$

Then (16) can be written as follows

$$v_{ik} = q_{ik}(x,\, y) v_{ik},\;\; (x,\, y) \in \Omega_i,\;\; i = 1,\, 2,\, ...,\, n;\;\; k = 1,\, 2,\, ...$$

According the hypothesis of the theorem functions $q_{ik}(x,\, y)$ are continuous and nonnegative. As for functions v_{ik}, we have:

$$v_{ik}\left(a_i, y\right) = u_{ik}\left(a_i, y\right) - u\left(a_i, y\right) = u_{i-1,k}\left(a_i,\, y\right) - u\left(a_i,\, y\right)$$

$$= v_{i-1,k}\left(a_i,\, y\right),\;\; y \in [0,\, 1],\;\; i = 2,\, 3,\, ...,\, n-1;\;\; k = 1,\, 2,\, ...,$$

$$v_{ik}\left(b_i,\, y\right) = u_{ik}\left(b_i,\, y\right) - u\left(b_i,\, y\right) = u_{i+1,k-1}\left(b_i,\, y\right) - u\left(b_i,\, y\right)$$

$$= v_{i+1,k-1}\left(b_i,\, y\right),\;\; y \in [0,\, 1],\;\; i = 2,\, 3,\, ...,\, n-1;\;\; k = 2,\, 3,\, ...,$$

$$v_{1,\,k}\left(a_1,\, y\right) = u_{1,\,k}\left(a_1,\, y\right) - u\left(a_1,\, y\right) = 0,\;\; y \in [0,\, 1],\;\; k = 1,\, 2,\, ...,$$

$$v_{n\,k}\left(b_n,\, y\right) = u_{n\,k}\left(b_n,\, y\right) - u\left(b_n,\, y\right) = u_{i_\xi,\,k}\left(\xi,\, y\right) - u\left(\xi,\, y\right)$$

$$= v_{i_\xi,\,k}\left(\xi,\, y\right),\;\; y \in [0,\, 1],\;\; k = 1,\, 2,\, ...,$$

$$v_{ik}\left(x, 0\right) = u_{ik}\left(x, 0\right) - u\left(x, 0\right) = 0,\;\; x \in \left[a_i,\, b_i\right],$$

$$i = 1,\, 2,\, ...,\, n;\;\; k = 1,\, 2,\, ...$$

$$v_{ik}\left(x, 1\right) = u_{ik}\left(x, 1\right) - u\left(x, 1\right) = 0,\;\; x \in \left[a_i,\, b_i\right],$$

$$i = 1,\, 2,\, ...,\, n;\;\; k = 1,\, 2,\, ...,$$

$$v_{i1}\left(b_i,\, y\right) = u_{i1}\left(b_i,\, y\right) - u\left(b_i,\, y\right) = l_i\left(y\right) - u\left(b_i,\, y\right) \equiv r_i\left(x\right),$$

$$i = 1,\, 2,\, ...,\, n-1.$$

It is clear, that functions v_{ik} are the solutions of the problems (4)-(9) and satisfy the requirements of Theorem 3, therefore we can apply Theorem 3 and we have:

$$\max_{1 \le i \le n}\left\{\max_{(x,\,y)\in\bar{\Omega}_i}\left|u_{i\,k}\left(x,\, y\right) - u\left(x,\, y\right)\right|\right\}$$

$$= \max_{1 \le i \le n}\left\{\max_{(x,\,y)\in\overline{\Omega}_i}\left|v_{i\,k}\left(x,\, y\right)\right|\right\} \le CQ^{k-1},\;\; C = \text{const}.$$

Theorem 4 is proved. □

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