International Journal of Pure and Applied Mathematics
Volume 49 No. 4 2008, 539-545

APPROXIMATE APPROXIMATIONS AND THE POISSON EQUATION IN THE WHOLE SPACE

Werner Varnhorn
Faculty of Mathematics
University of Kassel
Kassel, 34109, GERMANY
e-mail: varnhorn@mathematik.uni-kassel.de

Abstract: The method of approximate approximations is based on generating functions representing an approximate partition of the unity, only. In the present paper this method is used for the numerical solution of the Poisson equation in $\mathbb{R}^n$ $(n = 2, 3)$. The corresponding approximate volume potentials will be computed explicitly in these cases, containing a one-dimensional integral, only. Numerical simulations show the efficiency of the method and confirm the expected convergence of essentially second order, depending on the smoothness of the data.

AMS Subject Classification: 31B10, 35J05, 41A30, 65N12, 76D07
Key Words: approximate approximations, volume potentials, Poisson equation

1. Introduction

In 1991 V. Maz‘ya introduced an approximation method, called the method of approximate approximations [1]. Here a given function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is approximated by a linear combination f_h $(h > 0)$ of radial smooth exponentially decreasing basic functions. In contrast to a linear combination of splines this system of basic functions leads only to an approximate partition of the unity. Hence the approximation procedure does not converge as $h \to 0$. For practical

Received: August 14, 2008

computations, however, this lack of convergence does not play an important role, since the error between f and its approximation f_h can be controlled via a certain parameter and hence chosen to be of the same magnitude as the computer accuracy. Furthermore, the method of approximate approximations has great advantages for the numerical solution of Cauchy problems of the form $Du = f$, where D is a suitable linear partial differential operator in $\mathbb{R}^n$. In some cases explicit formulas for the approximate volume potentials can be developed if the right hand side f is approximated by f_h. In these formulas, instead of a multi-dimensional integration, often there remains a one-dimensional integral only, for instance an expression containing the error function (see [2]). Recently, the method of approximate approximations has also been applied successfully for the numerical treatment of boundary value problems (see [3, 4]).

In the present paper the method of approximate approximations is carried out explicitly for the Poisson-Cauchy problem in $\mathbb{R}^n$ $(n = 2, 3)$. In Section 2 the method is motivated and introduced for the approximation of functions given on the real line. In Section 3 the method is applied to the Poisson equation. Here explicit expressions for the corresponding approximate volume potentials in two and three dimensions are given. In Section 4 numerical simulations for the Poisson equation are carried out. In all cases the numerical simulations show essential convergence of second order, as expected from the error estimates.

2. Approximate Approximations on the Real Line

We consider the Gaussian probability function $\varphi_{\mu,\sigma}$ of the normal distribution with mean μ and variance σ^2, defined by

$$\varphi_{\mu,\sigma}(x) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\mu - x)^2}{2\sigma^2}\right). \tag{1}$$

Since $\varphi_{\mu,\sigma}$ is a probability density on the real line we have

$$\int_{-\infty}^{+\infty} \varphi_{\mu,\sigma}(x)dx = 1. \tag{2}$$

Replacing integration by a simple quadrature rule we obtain

$$\sum_{k\in\mathbb{Z}} \varphi_{\mu,\sigma}(k) \approx 1.$$

Let us consider the left–hand side as a function of μ, i.e.

$$\mu \mapsto \Phi_\sigma(\mu) := \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{(\mu-k)^2}{2\sigma^2}\right). \tag{3}$$

We investigate the difference between Φ_σ and the constant 1. Since Φ_σ is an even function having the period $p=1$ we obtain the Fourier series expansion

$$\Phi_\sigma(\mu) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(2m\pi\mu), \quad |\mu| < \tfrac{1}{2}.$$

An easy calculation leads to the Fourier coefficients

$$a_m = 2\exp\left(-2\sigma^2 m^2 \pi^2\right), \quad m \in \mathbb{N}_0.$$

It follows

$$\Phi_\sigma(\mu) - 1 = 2\sum_{k=1}^{\infty} \exp\left(-2\sigma^2 k^2 \pi^2\right) \cos(2k\pi\mu),$$

and this implies

$$|\Phi_\sigma(\mu) - 1| \quad \le \quad 2\sum_{k=1}^{\infty} \exp\left(-2\sigma^2 k^2 \pi^2\right) \approx \begin{cases} 10^{-2}\ , & \sigma = \frac{1}{2}, \\ 10^{-9}\ , & \sigma = 1, \\ 10^{-34}, & \sigma = 2. \end{cases} \tag{4}$$

Analogously, for the derivatives

$$\begin{aligned} \Phi'_\sigma(\mu) \quad &:= \quad -4\pi \sum_{k=1}^{\infty} k \exp\left(-2\sigma^2 k^2 \pi^2\right) \sin(2k\pi\mu), \\ \Phi''_\sigma(\mu) \quad &:= \quad -8\pi^2 \sum_{k=1}^{\infty} k^2 \exp\left(-2\sigma^2 k^2 \pi^2\right) \cos(2k\pi\mu) \end{aligned}$$

we find

$$|\Phi'_\sigma(\mu)| \approx \begin{cases} 10^{-1}\ , & \sigma = \frac{1}{2}, \\ 10^{-6}\ , & \sigma = 1, \\ 10^{-34}, & \sigma = 2, \end{cases} \qquad \text{and} \qquad |\Phi''_\sigma(\mu)| \approx \begin{cases} 10^{-1}\ , & \sigma = \frac{1}{2}, \\ 10^{-7}\ , & \sigma = 1, \\ 10^{-33}, & \sigma = 2. \end{cases}$$

In the following, let us assume $\sigma := 1$. In contrast to splines, the function

$$\Phi(\mu) := \Phi_1(\mu) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{(\mu-k)^2}{2}\right) \tag{5}$$

generates only an approximate partition of the unity. Now let us use the function (5) for the approximation of a given function $f : \mathbb{R} \to \mathbb{R}$. For this purpose we choose $h > 0$ and define

$$f_h(x) := \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{1}{2}\left(\frac{x-hk}{h}\right)^2\right) f(hk). \tag{6}$$

Since we are using an approximate partition of the unity, only, we cannot expect convergence of the resulting sequence if h tends to zero. Anyhow, let us study the error

$$\varepsilon_h(x) := f_h(x) - f(x)$$

for $h \to 0$ assuming a certain regularity on f. To do so we need the space $C_b^m(\mathbb{R})$ of functions having bounded continuous derivatives on $\mathbb{R}$ up to the order $m \in \mathbb{N}$.

Lemma 1. *Let $f \in C_b^2(\mathbb{R})$, $h > 0$, and f_h defined by (6). Then the error $\varepsilon_h(x)$ satisfies in $x \in \mathbb{R}$ the following estimate:*

$$|\varepsilon_h(x)| \le \frac{h^2}{2}\|f''\|_\infty\left(\left|\Phi\left(\tfrac{x}{h}\right)\right| + \left|\Phi''\left(\tfrac{x}{h}\right)\right|\right) + h\,|f'(x)|\left|\Phi'\left(\tfrac{x}{h}\right)\right| + |f(x)|\left|\Phi\left(\tfrac{x}{h}\right) - 1\right|.$$

Here Φ is defined by (5) and $\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)|$ is the norm in $L^\infty(\mathbb{R})$.

The estimate of Lemma 1 shows that we are using an approximation essentially of second order, since in practise only the term

$$\frac{h^2}{2}\|f''\|_\infty\left|\Phi\left(\tfrac{x}{h}\right)\right|$$

has to be taken into account, all other factors are neglectably small. Therefore the expression approximate approximation seems to be reasonable (compare [1]).

The method caries over immediately to the n–dimensional case, where a given function $f : \mathbb{R}^n \to \mathbb{R}$ can be approximated by

$$f_h(x) := \frac{1}{\sqrt{(2\pi)^n}} \sum_{k\in\mathbb{Z}^n} \exp\left(-\frac{1}{2}\left|\frac{x-hk}{h}\right|^2\right) f(hk). \tag{7}$$

All the above statements hold true in this case, too.

3. Application to the Poisson Equation

To use this approximation method for the numerical solution of the Poisson equation

$$-\triangle v = f \quad \text{in } \mathbb{R}^n \qquad (n = 2, 3) \tag{8}$$

we proceed as follows: It is well–known that a solution of (8) is given by the volume potential

$$Vf(x) := \int_{\mathbb{R}^n} e(x-y)f(y)\,dy \qquad (n=2,3). \tag{9}$$

Here

$$e(x) = \begin{cases} \frac{1}{2\pi}\ln\frac{1}{|x|}, & n=2, \\ \frac{1}{4\pi}\frac{1}{|x|} \quad , & n=3 \end{cases} \tag{10}$$

denotes the fundamential solution of the Laplacian in $\mathbb{R}^n$. To approximate the volume potential Vf we replace the given function f by the approximation f_h, defined by (7). This leads to an approximate solution of (8) in the form

$$\begin{aligned} v_h(x) &:= Vf_h = \int_{\mathbb{R}^n} e(x-y)\cdot\frac{1}{\sqrt{(2\pi)^n}}\sum_{m\in\mathbb{Z}^n}\exp\left(-\tfrac{1}{2}\left|\tfrac{y}{h}-m\right|^2\right) f(hm)\,dy \\ &= \sum_{m\in\mathbb{Z}^n} S_{m,h}(x)f(hm) \end{aligned} \tag{11}$$

with

$$S_{m,h}(x) := \begin{cases} \frac{1}{4\pi^2}\int\limits_{\mathbb{R}^2}\ln\frac{1}{|x-y|}\cdot\exp\left(-\frac{1}{2}\left|\frac{y}{h}-m\right|^2\right) dy \quad , & n=2, \\ \frac{1}{4\pi\sqrt{(2\pi)^3}}\int\limits_{\mathbb{R}^3}\frac{1}{|x-y|}\cdot\exp\left(-\frac{1}{2}\left|\frac{y}{h}-m\right|^2\right) dy, & n=3. \end{cases}$$

The weights $S_{m,h}(x)$ can be determined analytically:

Theorem 2. *Let $\xi := x/h - m$. Then for we have*

$$S_{m,h}(x) = -\frac{h^2}{4\pi}\left\{\ln(2h^2) - C + \operatorname{exint}\left(\tfrac{1}{2}|\xi|^2\right)\right\}, \qquad n=2,$$

$$S_{m,h}(x) = \frac{h^2}{\sqrt{(2\pi)^3}}\frac{1}{|\xi|}\int\limits_0^{|\xi|}\exp\left(-\frac{t^2}{2}\right) dt, \qquad n=3.$$

Here $C = 0,577216\ldots$ is Euler's constant and the function exint is defined by

$$\operatorname{exint}(x) := \int\limits_0^x \frac{1-\exp(-t)}{t}\,dt.$$

h	$\beta = 3$	$\beta = 4$	$\beta = 5$	$\beta = 6$
0,1	1,41e-01	2,49e-01	2,97e-01	3,34e-01
0,05	4,23e-02	7,49e-02	9,22e-02	1,08e-01
0,025	1,10e-02	1,96e-02	2,44e-02	2,92e-02
0,0125	2,80e-03	4,97e-03	6,21e-03	7,45e-03
0,00625	7,02e-04	1,24e-03	1,56e-03	1,87e-03
0,003125	1,75e-04	3,12e-04	3,90e-04	4,68e-04
0,0015625	4,39e-05	7,81e-05	9,76e-05	1,17e-04

Table 1. Maximal error expansion

h	$\beta = 3$	$\beta = 4$	$\beta = 5$	$\beta = 6$
0,05	1,73938	1,73616	1,69007	1,62291
0,025	1,93466	1,92975	1,91317	1,89169
0,0125	1,98366	1,98216	1,97767	1,97194
0,00625	1,99591	1,99552	1,99438	1,99292
0,003125	1,99898	1,99888	1,99860	1,99823
0,0015625	1,99974	1,99972	1,99965	1,99956

Table 2. Order of convergence

4. Numerical Simulations

In the following we present the results of some numerical simulations using the above formulas for the two-dimensional Poisson equation. Let us choose $3 \leq \beta \in \mathbb{N}$ and define the test function

$$v(x_1, x_2) := \begin{cases} 16^\beta \left(\frac{1}{4} - x_1^2\right)^\beta \left(\frac{1}{4} - x_2^2\right)^\beta & \text{in} \quad Q, \\ 0 & \text{in} \quad \mathbb{R}^2 \setminus Q, \end{cases} \tag{12}$$

where

$$Q := \left\{ (x_1, x_2) \in \mathbb{R}^2 \,\middle|\, |x_1| < \tfrac{1}{2},\ |x_2| < \tfrac{1}{2} \right\}$$

denotes the open two-dimensional unit square. For $f := -\triangle v$ we obtain continuity in thr whole $\mathbb{R}^2$ if $\beta \geq 3$.

The exponential integral function $\operatorname{exint}(x)$ in Theorem 2 has been evaluated with help of the *NAG Fortran Library* (see www.nag.co.uk).

The error $\varepsilon_h := \max |v(x) - v_h(x)|$ for different values of the smoothness parameter β is shown in Table 1.

The corresponding order $\alpha_h := \log_2 \frac{\varepsilon_{2h}}{\varepsilon_h}$ of convergence is presented in Table 2 and confirms an approximate approximation of second order.

References

[1] V. Maz'ya, A new approximation method and its applications to the calculation of volume potentials. Boundary point method. In: 3-th Kolloquium des DFG-Forschungsschwerpunktes Randelementmethoden (1991).

[2] V. Maz'ya, G. Schmidt, Approximate approximations and the cubature of potentials, *Rend. Lincei, Mat. Appl.*, **9**, No. 6 (1995), 161-184.

[3] F. Müller, W. Varnhorn, An approximation method using approximate approximations, *Applic. Anal.*, **85** (2006), 669-680.

[4] F. Müller, W. Varnhorn, Error estimates for approximate approximations on compact intervals, *J.A.T.*, **145** (2007), 171-181.