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ON A TIME-DEPENDENT EXTRA SPATIAL DIMENSION

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Abstract: In the usual brane-world scenario matter fields are confined to the four-dimensional spacetime, called a 3-brane, embedded in a higher-dimensional space, usually referred to as the bulk spacetime. In this paper we assume that the 3-brane is a de Sitter space; there is only one extra spatial dimension, assumed to be time dependent. By using the form of the brane-world energy-momentum tensor suggested by Shiromizu et al in the five-dimensional Einstein equations, it is proposed that the cosmological expansion of the 3-brane may provide a possible explanation for the collapse of the extra dimension. More precisely, whenever the bulk cosmological constant Λ is negative, the extra spatial dimension rapidly shrinks during the inflation of the brane. When Λ is positive, on the other hand, the extra spatial dimension either completely follows the cosmological expansion of the brane or completely ignores it, thereby shrinking relative to the expanding space. This behavior resembles the all-or-nothing behavior of ordinary systems in an expanding universe, as recently demonstrated by R.H. Price.

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1. Introduction

The notion that our world may contain more than three spatial dimensions can be traced to the pioneer work of Kaluza and Klein starting in 1919. The development of string/M-theory has resulted in a revival of this idea. At this stage

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in the evolution of the Universe the extra spatial dimensions are hidden from us four-dimensional observers. It has been conjectured that the extra dimensions had suddenly compactified to become unobservable, but the mechanisms for this dimension breaking has remained somewhat of a mystery [1]. It is proposed in this paper that cosmic inflation may provide a possible explanation, provided that certain conditions are met.

We are going to confine ourselves to a single extra spatial dimension with a scale factor that is necessarily time dependent to allow the size to vary. Accordingly, our starting point is the spacetime topology $M \times S^1$, where M refers to a de Sitter space and S^1 to an extra-dimensional 1-sphere. So our metric is given by

$$ds^2 = -dt^2 + [R(t)]^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] + [\rho(t)]^2 d\chi^2, \tag{1}$$

where χ is the coordinate in the fifth dimension (note the time-dependent scale factor $\rho(t)$). In the usual brane-world picture matter fields are confined to the four-dimensional spacetime (or 3-brane), while gravity acts in five dimensions (the bulk). Our *basic assumption* is that when dealing with the Early Universe we may apply the five-dimensional Einstein field equations $G_{\mu\nu} = k_5^2\, T_{\mu\nu}$ using a particular form of the energy-momentum tensor: following [2], $(M, q_{\mu\nu})$ denotes our 3-brane in a five-dimensional spacetime $(V, g_{\mu\nu})$ and

$$T_{\mu\nu} = -\Lambda g_{\mu\nu} + \delta(\chi)(-\lambda q_{\mu\nu} + \tau_{\mu\nu}). \tag{2}$$

Here Λ is the cosmological constant of the bulk spacetime, while the hypersurface $\chi = 0$ corresponds to our 3-brane (the δ-function expresses the confinement of matter in the brane). Also, λ and $\tau_{\mu\nu}$ are the vacuum energy and the energy-momentum tensor, respectively, of the 3-brane.

Remark. Concerning our basic assumption, it must be kept in mind that the coefficients in equation (1) are independent of χ, an assumption more in line with the Kaluza-Klein model than the brane-world model. While our model must therefore be viewed as a special case, it is not unreasonable to assume that during inflation the enormous rate of expansion is so dominant that outside influences, including the existence of an extra spatial dimension, are negligible.

Returning to equation (1), if $(M, q_{\mu\nu})$ is to be a de Sitter space, we need to let $R(t) = e^{Ht}$, where $H = \sqrt{\Lambda_4/3}$ and Λ_4 is the cosmological constant of the 3-brane. Universes with exponential expansion are usually called *inflationary.*

Since the derivations in [2] do not depend on the sign of Λ, we are justified in considering the cases $\Lambda > 0$ and $\Lambda < 0$ separately. In the latter case, discussed in Section 3, the extra spatial dimension rapidly shrinks during the inflation of the brane. The former case, discussed next, is the more interesting of the two: the extra spatial dimension either completely follows the cosmological expansion of the brane or completely ignores it, thereby shrinking relative to the expanding space.

2. The Case $\Lambda > 0$

Our first step is to calculate the nonzero components of the Einstein tensor in the orthonormal frame. These are given next:

$$G_{\hat{t}\hat{t}} = 3\frac{[R'(t)]^2}{[R(t)]^2} + 3\frac{R'(t)\rho'(t)}{R(t)\rho(t)}, \tag{3}$$

$$G_{\hat{r}\hat{r}} = -2\frac{R''(t)}{R(t)} - \frac{[R'(t)]^2}{[R(t)]^2} - 2\frac{R'(t)\rho'(t)}{R(t)\rho(t)} - \frac{\rho''(t)}{\rho(t)}, \tag{4}$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = -2\frac{R''(t)}{R(t)} - \frac{[R'(t)]^2}{[R(t)]^2} - \frac{\rho''(t)}{\rho(t)} - 2\frac{R'(t)\rho'(t)}{R(t)\rho(t)}, \tag{5}$$

$$G_{\hat{\chi}\hat{\chi}} = -3\frac{R''(t)}{R(t)} - 3\frac{[R'(t)]^2}{[R(t)]^2}. \tag{6}$$

2.1. Solutions

From equation (3), we have

$$3\frac{[He^{Ht}]^2}{[e^{Ht}]^2} + 3\frac{He^{Ht}\rho'(t)}{e^{Ht}\rho(t)} = k_5^2\, T_{\hat{t}\hat{t}},$$

which reduces to

$$\frac{\rho'(t)}{\rho(t)} = -H + \frac{k_5^2\, T_{\hat{t}\hat{t}}}{3H}. \tag{7}$$

Let A_{in} be the initial value of $\rho(t)$ (at the onset of inflation), i.e., $\rho(0) = A_{\text{in}}$. Then the solution is

$$\rho(t) = A_{\text{in}} e^{-Ht} e^{k_5^2\, T_{\hat{t}\hat{t}} t/3H}.$$

By equation (2), since $\delta(\chi) = 0$ in the bulk,

$$\rho(t) = A_{\text{in}} e^{-Ht} e^{k_5^2\, \Lambda t/3H}. \tag{8}$$

Similarly, from both equations (4) and (5), we get

$$\rho''(t) + 2H\rho'(t) + (3H^2 - k_5^2\, \Lambda)\rho(t) = 0 \tag{9}$$

and

$$\rho(t) = e^{-Ht}\left(c_1\, e^{\sqrt{-2H^2+k_5^2\,\Lambda}\,t} + c_2\, e^{-\sqrt{-2H^2+k_5^2\,\Lambda}\,t}\right). \tag{10}$$

Since c_1 and c_2 are arbitrary constants, we are free to choose $c_2 = 0$. So

$$\rho(t) = A_{\text{in}} e^{-Ht} e^{\sqrt{-2H^2+k_5^2\,\Lambda}\,t}. \tag{11}$$

To get agreement between these two solutions, we need to let

$$\frac{k_5^2\,\Lambda}{3H} = \sqrt{-2H^2 + k_5^2\,\Lambda} \tag{12}$$

with $H = \sqrt{\Lambda_4/3}$. The only solutions are $\Lambda = \Lambda_4/k_5^2$ and $\Lambda = \Lambda_4/(\frac{1}{2}k_5^2)$.

It remains to show that these solutions are consistent with $G_{\hat{\chi}\hat{\chi}}$. Observe that this component is completely independent of $\rho(t)$, suggesting that $\chi = 0$ in equation (2). Retaining the notation $\delta(\chi)$, equation (2) becomes

$$T_{\hat{\chi}\hat{\chi}} = -\Lambda g_{\hat{\chi}\hat{\chi}} + \delta(\chi)(-\lambda q_{\hat{\chi}\hat{\chi}} + \tau_{\hat{\chi}\hat{\chi}}), \tag{13}$$

and, since $R(t) = e^{Ht}$, with $H = \sqrt{\Lambda_4/3}$,

$$G_{\hat{\chi}\hat{\chi}} = -3\frac{R''(t)}{R(t)} - 3\frac{[R'(t)]^2}{[R(t)]^2} = -\Lambda_4 - \Lambda_4. \tag{14}$$

In equation (13), $q_{\hat{\chi}\hat{\chi}} = 0$ and $g_{\hat{\chi}\hat{\chi}} = 1$, so that

$$-\Lambda_4 - \Lambda_4 = k_5^2[-\Lambda + \delta(\chi)\tau_{\hat{\chi}\hat{\chi}}]. \tag{15}$$

It now follows that $\delta(\chi)\tau_{\hat{\chi}\hat{\chi}} = 0$ if, and only if, $\Lambda = \Lambda_4/(\frac{1}{2}k_5^2)$. So if $\delta(\chi)\tau_{\hat{\chi}\hat{\chi}} \neq 0$, then $\Lambda = \Lambda_4/k_5^2$, the other solution. For this case, equation (15) implies that

$$-2\Lambda_4 = k_5^2[-\Lambda_4/k_5^2 + \delta(\chi)\tau_{\hat{\chi}\hat{\chi}}]. \tag{16}$$

One of the properties of the δ-function is that $\delta(y)f(y) = \delta(y)f(0)$ for any continuous function f. Applying this property to $\tau_{\hat{\chi}\hat{\chi}}$, we have

$$\delta(\chi)\tau_{\hat{\chi}\hat{\chi}} = \delta(\chi)\left(\tau_{\hat{\chi}\hat{\chi}}\,|_{\chi=0}\right).$$

So equation (16) becomes

$$\delta(\chi)\left(\tau_{\hat{\chi}\hat{\chi}}\,|_{\chi=0}\right) = -\frac{\Lambda_4}{k_5^2},$$

emphasizing the confinement to the brane. This form is similar to the classical de Sitter forms

$$T_{\hat{r}\hat{r}} = T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = -\frac{\Lambda_4}{8\pi}$$

obtainable from equations (4) and (5) by letting $k_5^2 = 8\pi$ and $\rho(t) \equiv 0$.

2.2. Analysis

We now examine each solution in turn:

(1) Suppose $\Lambda = \Lambda_4/k_5^2$. Substituting in equations (8) and (11) and recalling that $H = \sqrt{\Lambda_4/3}$, we get in both cases,

$$\rho(t) = A_{\text{in}} e^{-Ht} e^{Ht} = A_{\text{in}}. \tag{17}$$

(2) For the other solution, $\Lambda = \Lambda_4/(\frac{1}{2}k_5^2)$, we obtain from equations (8) and (11),

$$\rho(t) = A_{\text{in}} e^{-Ht} e^{2Ht} = A_{\text{in}} e^{Ht}. \tag{18}$$

So the "radius" of the extra dimension expands by the factor e^{Ht}, the same as for any other distance in the brane.

To emphasize the rescaling of the r coordinate on each $t =$ constant slice, one could write

$$\frac{\ell(t)}{e^{Ht}} = r_B - r_A.$$

In the first case, equation (17), the corresponding equation for a fixed $\ell_1(t) = r_B - r_A$ can be written

$$\frac{\ell_1(t)}{e^{Ht}} = e^{-Ht}(r_B - r_A),$$

that is, if $\rho(t)$ remains fixed, then the original size has the appearance of having shrunk by a factor of e^{-HT} relative to the expanded space at the end of inflation.

3. The Case $\Lambda < 0$

If $\Lambda < 0$, the solution given by equation (8) retains the form

$$\rho(t) = A_{\text{in}} e^{-Ht} e^{k_5^2 \Lambda t/3H}, \tag{19}$$

but equation (10) becomes

$$\rho(t) = e^{-Ht}\left(c_1 \cos\sqrt{2H^2 - k_5^2\Lambda}\, t + c_2 \sin\sqrt{2H^2 - k_5^2\Lambda}\, t\right). \tag{20}$$

Once again letting $c_2 = 0$, we get

$$\rho(t) = A_{\text{in}}\, e^{-Ht} \cos\sqrt{2H^2 - k_5^2\Lambda}\, t. \tag{21}$$

As before, we want the solutions to agree, but the relationship between Λ and Λ_4 is not so apparent. So let us recall that $H = \sqrt{\Lambda_4/3}$ and let

$$\rho_1(\Lambda) = e^{k_5^2\,\Lambda/\sqrt{3\Lambda_4}} = \cos\sqrt{\frac{2}{3}\Lambda_4 - k_5^2\,\Lambda} = \rho_2(\Lambda) \tag{22}$$

for some Λ. Then $\rho_1(0) = 1 > \rho_2(0)$. To the left of the origin, $\rho_1(\Lambda)$ is a decaying exponential, that is, $\rho_1(\Lambda) \to 0$ as $\Lambda \to -\infty$. So $\rho_2(\Lambda)$, being sinusoidal, will intersect $\rho_1(\Lambda)$ for some $\Lambda < 0$, thereby yielding a solution to equation (22). Returning to equations (19) and (20), the now identical solutions are dominated by the decaying exponential function e^{-Ht}, implying that for $\Lambda < 0$, $\rho(t)$ will have shrunk significantly.

4. Summary

It is proposed in this paper that in the case of a de Sitter 3-brane world, cosmic inflation may provide an explanation for the collapse of the extra spatial dimension. More precisely, whenever the cosmological constant Λ is negative, $\rho(t)$ shrinks rapidly during the inflation of the brane. When Λ is positive, the extra spatial dimension either completely follows the cosmological expansion of the brane or completely ignores it. In the former case the extra dimension expands, rather than shrinks. In the latter case, $\rho(t)$ shrinks relative to the expanding space.

For $\Lambda > 0$, the conclusion resembles the interesting all-or-nothing behavior demonstated in [3]: a system will either completely follow the cosmological expansion of the universe or completely ignore it.

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