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A NEW WAY TO OBTAIN, WITHOUT USING EXPANSIONS, A 14 MOMENTS MODEL IN EXTENDED THERMODYNAMICS

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Abstract: In this paper we consider a 14 moments model for Extended Thermodynamics in the case where the flux appearing in a balance equations is not an independent variable of the subsequent balance equation, except for the conservation law of mass, whose subsequent one is the conservation law of momentum; moreover the symmetry of the fluxes is requested only for the stress tensor, in order to obtain conservation of angular momentum. Solutions of the restrictions imposed by the entropy principle and that of Galilean relativity for such a model have been until now obtained in literature only in an approximate manner up to a certain order with respect to thermodynamic equilibrium; for more restrictive models they have been obtained up to whatever order, but by using Taylor expansions around equilibrium and without proving convergence. Here we have found an exact solution without using expansions.

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1. Introduction

The idea, which has allowed to find the above mentioned results, has been to write firstly a relativistic model, for which it is easy to impose the Einsteinian relativity principle, and then taking its non relativistic limit. The symmetry of the relativistic energy-stress tensor will yield the above mentioned conditions.

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In particular, the balance equations are

$$\partial_t F + \partial_k F_k = 0\,, \qquad \partial_t F_i + \partial_k G_{ki} = 0\,, \quad \partial_t F_{ij} + \partial_k G_{kij} = P_{<ij>}\,, \tag{1}$$
$$\partial_t F_{ill} + \partial_k G_{kill} = P_{ill}\,, \quad \partial_t F_{iill} + \partial_k G_{kiill} = P_{iill}\,.$$

Here F_{ij} and the flux G_{ki} are symmetric, while the fluxes G_{kij}, G_{kill} are symmetric over all indexes, except for k.

It is well known [1] that the entropy principle for the system (1) amounts in assuming the existence of the potential functions h' and ϕ'_k depending on the Lagrange multipliers λ, λ_i, λ_{ij}, λ_{ill}, λ_{iill} such that

$$\begin{aligned}
F &= \frac{\partial h'}{\partial \lambda}, \quad F_i = \frac{\partial h'}{\partial \lambda_i}, \quad F_{il} = \frac{\partial h'}{\partial \lambda_{il}},\\
F_{ill} &= \frac{\partial h'}{\partial \lambda_{ill}}, \quad F_{iill} = \frac{\partial h'}{\partial \lambda_{iill}}\,,\\
F_k &= \frac{\partial \phi'_k}{\partial \lambda}, \quad G_{ki} = \frac{\partial \phi'_k}{\partial \lambda_i}, \quad G_{kil} = \frac{\partial \phi'_k}{\partial \lambda_{il}},\\
G_{kill} &= \frac{\partial \phi'_k}{\partial \lambda_{ill}}, \quad G_{kiill} = \frac{\partial \phi'_k}{\partial \lambda_{iill}}\,.
\end{aligned} \tag{2}$$

More precisely, equations $(2)_{1-5}$ define the Lagrange multipliers in terms of the moments in their left hand sides; once these are substituted in equations $(2)_{7-10}$, these give the fluxes in terms of the moments. But compatibility of $(2)_2$ and $(2)_6$, jointly with the symmetry of equation $(2)_7$, impose on h' and ϕ'_k the following conditions

$$\frac{\partial \phi'_k}{\partial \lambda} = \frac{\partial h'}{\partial \lambda_k} \quad , \quad \frac{\partial \phi'_{[k}}{\partial \lambda_{i]}} = 0\,. \tag{3}$$

Moreover, the Galilean relativity principle is equivalent to the following two other conditions

$$\begin{aligned}
0 &= \frac{\partial h'}{\partial \lambda}\lambda_i + 2\lambda_{ij}\frac{\partial h'}{\partial \lambda_j} + \lambda_{jpp}\left(\frac{\partial h'}{\partial \lambda_{rs}}\delta_{rs}\delta_{ij} + 2\frac{\partial h'}{\partial \lambda_{ij}}\right) + 4\lambda_{ppqq}\frac{\partial h'}{\partial \lambda_{ill}} \\
0 &= \frac{\partial \phi'_k}{\partial \lambda}\lambda_i + 2\lambda_{ij}\frac{\partial \phi'_k}{\partial \lambda_j} + \lambda_{jpp}\left(\frac{\partial \phi'_k}{\partial \lambda_{rs}}\delta_{rs}\delta_{ij} + 2\frac{\partial \phi'_k}{\partial \lambda_{ij}}\right) + 4\lambda_{ppqq}\frac{\partial \phi'_k}{\partial \lambda_{ill}} + h'\delta_{ik}\,.
\end{aligned} \tag{4}$$

The result of the present work is that a scalar function φ exists such that

$$h' = \frac{\partial \varphi}{\partial \lambda} \quad , \quad \phi'_k = \frac{\partial \varphi}{\partial \lambda_k}\,. \tag{5}$$

These define the potentials h' and ϕ'_k except for the scalar function φ. For this functions we can use the results of [2], i.e., it is an arbitrary function of the scalars $X_1 - X_8$ whose expressions we cannot here report for the sake of brevity, but that can be read in [2].

By direct calculations it can be verified that this result satisfies identically the conditions (3) and (4). But let us now see how it has been obtained.

2. An Idea from a Relativistic Approach

The idea, from which we obtained the above result, has been to consider firstly a relativistic model

$$\partial_\alpha T^{\alpha\beta} = 0 \quad , \quad \partial_\alpha A^{\alpha\beta\gamma} = I^{\beta\gamma} \, , \tag{6}$$

where $T^{\alpha\beta}$ is symmetric, while $A^{\alpha\beta\gamma}$ and $I^{\beta\gamma}$ are symmetric only with respect to the indexes $\beta\gamma$. The first of these is the conservation law of momentum-energy, while the trace of the second one is the conservation law of mass, so that $I^{\beta\gamma}$ is traceless. With the usual passages, we have that the entropy principle is equivalent to assume the existence of Lagrange multipliers λ_β, $\Lambda_{\beta\gamma}$ and of the 4-potential h'^α such that

$$T^{\alpha\beta} = \frac{\partial h'^\alpha}{\partial \Lambda_\beta} \quad , \quad A^{\alpha\beta\gamma} = \frac{\partial h'^\alpha}{\partial \Lambda_{\beta\gamma}} \, , \tag{7}$$

where Λ_β, $\Lambda_{\beta\gamma}$ have been taken as independent variables. It follows that from the knowledge of h'^α we obtain $T^{\alpha\beta}$ and $A^{\alpha\beta\gamma}$; we have only 2 conditions on h'^α:

1) It must be such that $(7)_1$ implies symmetry of $T^{\alpha\beta}$ and

2) It has to satisfy the Einsteinian relativity principle.

By using (7), the system (6) becomes

$$\partial_\alpha \left(\frac{\partial h'^\alpha}{\partial \Lambda_A} \right) = P^A \quad , \text{with } \Lambda_A = (\Lambda_\beta \, , \, \Lambda_{\beta\gamma}) \tag{8}$$

and this system maintains its form under a linear and invertible change of independent variables $\Lambda_A = \Lambda_A{}^B \mu_B$ (obviously, $\Lambda_A{}^B$ is a constant matrix); this can be seen by the following relations

$$\Lambda_A{}^B P^A = \Lambda_A{}^B \partial_\alpha \left(\frac{\partial h'^\alpha}{\partial \Lambda_A} \right) = \partial_\alpha \left(\Lambda_A{}^B \frac{\partial h'^\alpha}{\partial \Lambda_A} \right) = \partial_\alpha \left(\frac{\partial h'^\alpha}{\partial \mu_B} \right) . \tag{9}$$

We see now how our system changes under the following linear and invertible change of independent variables, where c is the light speed and m_0 the particle mass

$$\begin{aligned} \Lambda_\beta &= \frac{c^3}{m_0} \left[\begin{pmatrix} -8\lambda_{pll} \\ 0_i \end{pmatrix} + \frac{1}{c} \begin{pmatrix} 0 \\ -2\lambda_{ill} \end{pmatrix} + \frac{1}{c^3} \begin{pmatrix} 0 \\ \lambda_i \end{pmatrix} + \frac{1}{c^4} \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \right] , \qquad (10) \\ \Lambda_{\beta\gamma} &= \frac{c^2}{m_0^2} \left[\begin{pmatrix} 8\lambda_{pll} & 0_j \\ 0_i & -4\lambda_{pll}\delta_{ij} \end{pmatrix} + \frac{1}{c} \begin{pmatrix} 0 & \lambda_{jll} \\ \lambda_{ill} & 0_{ij} \end{pmatrix} + \frac{1}{c^2} \begin{pmatrix} 0 & 0_j \\ 0_i & \lambda_{ij} \end{pmatrix} \right] . \end{aligned}$$

By using these expressions and (7) we obtain

$$\frac{\partial h'^0}{\partial \lambda} = \frac{\partial h'^0}{\partial \Lambda_0}\frac{1}{m_0 c} = \frac{T^{00}}{m_0 c} = (m_0)^3 F_2\,, \tag{11}$$

$$\frac{\partial h'^0}{\partial \lambda_i} = \frac{\partial h'^0}{\partial \Lambda_i}\frac{1}{m_0} = \frac{T^{0i}}{m_0} = (m_0)^3 F_2^i\,,$$

$$\frac{\partial h'^0}{\partial \lambda_{ij}} = \frac{\partial h'^0}{\partial \Lambda_{ij}}\frac{1}{(m_0)^2} = \frac{A^{0ij}}{(m_0)^2} = (m_0)^3 F_3^{ij}\,,$$

$$\frac{\partial h'^0}{\partial \lambda_{ill}} = \frac{\partial h'^0}{\partial \Lambda_i}\frac{-2c^2}{m_0} + 2\frac{\partial h'^0}{\partial \Lambda_{i0}}\frac{c}{(m_0)^2}$$

$$= T^{0i}\frac{-2c^2}{m_0} + 2A^{0i0}\frac{c}{(m_0)^2} = -2(m_0)^3 c^2 F_2^i + 2(m_0)^3 c^2 F_3^i\,,$$

$$\frac{\partial h'^0}{\partial \lambda_{aabb}} = \frac{\partial h'^0}{\partial \Lambda_0}\frac{-8c^3}{m_0} + \frac{\partial h'^0}{\partial \Lambda_{00}}\frac{8c^2}{(m_0)^2} + \frac{\partial h'^0}{\partial \Lambda_{ij}}\frac{-4c^2}{(m_0)^2}\delta_{ij} = T^{00}\frac{-8c^3}{m_0}$$

$$+A^{000}\frac{8c^2}{(m_0)^2} + A^{0ij}\frac{-4c^2}{(m_0)^2}\delta_{ij} = -8(m_0)^3 c^4 F_2 + 8(m_0)^3 c^4 F_3 - 4(m_0)^3 c^2 F_3^{ll}\,,$$

where the last side of each equation is the definition of F_2, F_2^i, F_3^{ij}, F_3^i, F_3 respectively; they are important because, at least for a kinetic approach, have finite and independent limits for $c \to \infty$, as it will be seen in the next section; moreover, this limits, divided by $(m_0)^3$, are exactly our variables F, F^i, F^{ij}, F^{ill}, F^{aabb}. In other words, by defining $h' = \frac{1}{(m_0)^3}h'^0$, equations (11) become equations $(2)_{1-5}$.

Moreover, the symmetry of $T^{\alpha\beta}$ implies that

$$\frac{\partial h'^k}{\partial \lambda} = \frac{\partial h'^k}{\partial \Lambda_0}\frac{1}{m_0 c} = \frac{T^{k0}}{m_0 c} = \frac{1}{c}\frac{\partial h'^0}{\partial \lambda_k}\,,\quad \frac{\partial h'^k}{\partial \lambda_i} = \frac{\partial h'^k}{\partial \Lambda_i}\frac{1}{m_0} = \frac{T^{ki}}{m_0} \tag{12}$$

the first of these implies $(3)_1$ if we define $\phi'^k = \frac{c}{(m_0)^3}h'^k$, while the second one implies $(3)_2$, as we desired.

It is useful now to take the inverse relations of (10); they are

$$\lambda_{aabb} = \frac{(m_0)^2}{8c^2}\Lambda_{00}\,,\ \lambda_{ill} = \frac{(m_0)^2}{c}\Lambda_{i0}\,,\ \lambda_{ij} = (m_0)^2\left(\Lambda_{ij} + \frac{1}{2}\Lambda_{00}\delta_{ij}\right)\,,$$

$$\lambda_i = m_0\Lambda_i + 2(m_0)^2 c\Lambda_{i0}\,,\ \lambda = m_0 c\Lambda_0 + (m_0)^2 c^2 \Lambda_{00}\,. \tag{13}$$

These relations allow to see how the symmetry of $T^{\alpha\beta}$ implies equation (3). To this end, we firstly see that the symmetry of $T^{\alpha\beta}$, by using $(7)_1$, amounts in assuming the existence of ψ such that $h'^\alpha = \frac{\partial \psi}{\partial \Lambda_\alpha}$. Consequently, we have

$$h' = \frac{1}{(m_0)^3}h'^0 = \frac{1}{(m_0)^3}\frac{\partial \psi}{\partial \Lambda_0} = \frac{c}{(m_0)^2}\frac{\partial \psi}{\partial \lambda} = \frac{\partial \varphi}{\partial \lambda}\,, \tag{14}$$

$$\phi'^k = \frac{c}{(m_0)^3} h'^k = \frac{c}{(m_0)^3} \frac{\partial \psi}{\partial \Lambda_k} = \frac{c}{(m_0)^2} \frac{\partial \psi}{\partial \lambda_k} = \frac{\partial \varphi}{\partial \lambda_k} \,, \tag{15}$$

with $\varphi = \frac{c}{(m_0)^2}\psi$. In other words, we have found equation (5).

We conclude this section nothing that, from equation (5) it follows

$$\begin{aligned} h' &= 8H_0X_1 + H_1X_2 + H_2X_3 + H_3X_4 \,, \\ \phi'^k &= H_0V_0^k + H_1V_1^k + H_2V_2^k + H_3V_3^k h' \,, \end{aligned} \tag{16}$$

with

$$H_0 = 2\frac{\partial \varphi}{\partial X_5}, \quad H_1 = 2\frac{\partial \varphi}{\partial X_6}, \quad H_2 = 2\frac{\partial \varphi}{\partial X_7}, \quad H_3 = 2\frac{\partial \varphi}{\partial X_8}, \tag{17}$$

because $X_1 - X_4$ do not depend on λ, λ_i and for

$$\begin{pmatrix} \frac{\partial X_5}{\partial \lambda} \\ \frac{\partial X_5}{\partial \lambda_k} \end{pmatrix} = 2 \begin{pmatrix} 8X_1 \\ V_0^k \end{pmatrix}; \quad \begin{pmatrix} \frac{\partial X_6}{\partial \lambda} \\ \frac{\partial X_6}{\partial \lambda_k} \end{pmatrix} = 2 \begin{pmatrix} X_2 \\ V_1^k \end{pmatrix};$$

$$\begin{pmatrix} \frac{\partial X_7}{\partial \lambda} \\ \frac{\partial X_7}{\partial \lambda_k} \end{pmatrix} = 2 \begin{pmatrix} X_3 \\ V_2^k \end{pmatrix}; \quad \begin{pmatrix} \frac{\partial X_8}{\partial \lambda} \\ \frac{\partial X_8}{\partial \lambda_k} \end{pmatrix} = 2 \begin{pmatrix} X_4 \\ V_3^k \end{pmatrix};$$

proof of these last relations can be obtained from the expressions $X_1 - X_8$ in [2], together with the expressions of V_i^k.

So we have seen how condition 1), on the 3-th page of the present paper, is satisfied. In order to satisfy the condition 2) we have to impose only that ψ and φ are arbitrary functions of the scalars

$$Q_1 = \Lambda^\beta_\beta \quad , \quad Q_2 = \overset{2}{\Lambda}{}^{\alpha\gamma} g_{\alpha\gamma} \quad , \quad Q_3 = \overset{3}{\Lambda}{}^{\alpha\gamma} g_{\alpha\gamma} \quad , \quad Q_4 = \overset{4}{\Lambda}{}^{\alpha\gamma} g_{\alpha\gamma} \quad ,$$

$$P_0 = \Lambda_\beta \Lambda^\beta \,, \; P_1 = \Lambda_\beta \Lambda_\gamma \Lambda^{\beta\gamma} \,, \; P_2 = \Lambda_\beta \Lambda_\gamma \overset{2}{\Lambda}{}^{\beta\gamma} \,, \; P_3 = \Lambda_\beta \Lambda_\gamma \overset{3}{\Lambda}{}^{\beta\gamma} \,,$$

where the following definitions have been used

$$\overset{2}{\Lambda}{}^{\alpha\gamma} = \Lambda^{\alpha\beta} \Lambda_{\beta\delta} g^{\delta\gamma} \quad , \quad \overset{3}{\Lambda}{}^{\alpha\gamma} = \Lambda^{\alpha\beta} \Lambda_{\beta\delta} \Lambda^{\delta\gamma} \,.$$

But in [2] we have seen that these scalars can be expressed as invertible function of other scalars, whose non relativistic limits are $X_1 - X_8$. Consequently, φ is an arbitrary function of $X_1 - X_8$ and no other condition has to be imposed.

3. The Functions F_2, F_2^i, F_3^{ij}, F_3^i, F_3 in the Kinetic Approach

In the kinetic approach we have that

$$T^{\alpha\beta} = \int f p^\alpha p^\beta \frac{dp^1 dp^2 dp^3}{p^0} \,, \quad A^{\alpha\beta\gamma} = \int f p^\alpha p^\beta p^\gamma \frac{dp^1 dp^2 dp^3}{p^0} \,, \tag{18}$$

where $p^0 = m_0\gamma c$, $p^i = m_0\gamma u^i$ and γ is the Lorentz factor $(1-\frac{u^2}{c^2})^{-\frac{1}{2}}$.

With the change of integration variables (from p^i to u^i), the Jacobian is $J = det[m_0\gamma(\delta^{ij} + \frac{\gamma^2}{c^2}u^i u^j)] = (m_0)^3\gamma^5$. It follows, from equations (11), that

$$(m_0)^3 F_2 = (m_0)^3 \int f\gamma^6 d\mathbf{u} \quad , \qquad (m_0)^3 F_2^i = (m_0)^3 \int f\gamma^6 u^i d\mathbf{u}\,,$$

$$(m_0)^3 F_3^{ij} = (m_0)^3 \int f\gamma^7 u^i u^j d\mathbf{u}\,,$$

$$-2(m_0)^3 c^2 F_2^i + 2(m_0)^3 c^2 F_3^i = -2(m_0)^3 c^2 \int f\gamma^6 u^i d\mathbf{u} +$$

$$+2(m_0)^3 c^2 \int f\gamma^7 u^i d\mathbf{u} = 2(m_0)^3 c^2 \int \left(1 - \frac{1}{\gamma}\right) f\gamma^7 u^i d\mathbf{u}\,,$$

$$-8(m_0)^3 c^4 F_2 + 8(m_0)^3 c^4 F_3 - 4(m_0)^3 c^2 F_3^{ll} = -8(m_0)^3 c^4 \int f\gamma^6 d\mathbf{u} +$$

$$+8(m_0)^3 c^4 \int f\gamma^7 d\mathbf{u} - 4(m_0)^3 c^2 \int f\gamma^7 u^2 d\mathbf{u} =$$

$$= 8(m_0)^3 c^4 \int \left(1 - \frac{1}{\gamma} - \frac{1}{2}\frac{u^2}{c^2}\right) f\gamma^7 d\mathbf{u}\,.$$

It is interesting to see that these expressions, in the non relativistic limit and divided by $(m_0)^3$, give respectively

$$\int \tilde{f} d\mathbf{u}\,, \quad \int \tilde{f} u^i d\mathbf{u}\,, \int \tilde{f} u^i u^j d\mathbf{u}\,, \int \tilde{f} u^i u^2 d\mathbf{u}\,, \quad \int \tilde{f} u^4 d\mathbf{u}\,. \qquad (19)$$

This relation, between the 4-dimensional and 3-dimensional moments, implies another one for the Lagrange multipliers, that is equation (10).

References

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