International Journal of Pure and Applied Mathematics
Volume 50 No. 1 2009, 1-12

DECOMPOSITIONS OF CERTAIN SYMMETRIC FUNCTIONS OF POWERS OF COSINE AND SINE FUNCTIONS

Roman Wituła[1], Damian Słota[2 §]
[1,2]Institute of Mathematics
Silesian University of Technology
23, Kaszubska, Gliwice, 44-100, POLAND
[2]e-mail: d.slota@polsl.pl

Abstract: Various decompositions of certain basic symmetric functions of n-th power of cosine and sine functions are discussed in the paper. Some of these decompositions are then applied for the derivation of many new trigonometric identities.

AMS Subject Classification: 26C05, 11B37, 11B83
Key Words: Chebyshev polynomials, powers of cosine and sine functions

1. Introduction

The aim of this paper is to present two fundamental decompositions (and many additional ones) of the following four symmetric functions of the n-th powers of cosine and sine functions:

$$C_n^+(x,\varphi) := \cos^n(x+\varphi) + \cos^n(x-\varphi), \tag{1.1}$$

$$C_n^-(x,\varphi) := \cos^n(x-\varphi) - \cos^n(x+\varphi), \tag{1.2}$$

$$S_n^+(x,\varphi) := \sin^n(x+\varphi) + \sin^n(x-\varphi), \tag{1.3}$$

$$S_n^-(x,\varphi) := \sin^n(x+\varphi) - \sin^n(x-\varphi). \tag{1.4}$$

A direct reason for the investigation of these decompositions were attempts at generalizing some trigonometric identities previously presented by Wituła et al [6]:

Received: February 11, 2008

§Correspondence author

$$\sin^{2n}(x)+\cos^{2n}\left(x-\tfrac{\pi}{6}\right)+\cos^{2n}\left(x+\tfrac{\pi}{6}\right)+\cos^{2n}(x)$$
$$+\cos^{2n}\left(x-\tfrac{\pi}{3}\right)+\cos^{2n}\left(x+\tfrac{\pi}{3}\right)\equiv\text{const}\ \Leftrightarrow\ n=1,2,\dots,5. \quad (1.5)$$

As it turned out, only the decompositions of functions (1.1)–(1.4) of (2.13)–(2.21) type are known; whereas decompositions of (2.22)–(2.23) type and identities (2.24)–(2.31) related to Chebyshev polynomials of the first kind and Horadam polynomials (see identities (2.32)–(2.36) and Wituła et al [6]) do not seem to be well recognized in literature.

The second group of decompositions of functions (1.1)–(1.4) derived in Section 3 and related to Chebyshev polynomials of the second kind is original. Likewise, some identities for the derivatives of Chebyshev polynomials of the second kind seem to be innovative. We note that appropriate generalizations of the identity (1.5) are discussed in the separate paper (see Wituła et al [7]).

2. The First Decomposition

Let us start with the following Lemma of a technical nature:

Lemma 2.1. *The following elementary identities hold:*

$$C^{+}_{2n-1}\big(x+\tfrac{\pi}{2},\varphi+\tfrac{\pi}{2}\big)=C^{-}_{2n-1}(x,\varphi), \quad (2.1)$$
$$C^{-}_{2n-1}\big(x+\tfrac{\pi}{2},\varphi+\tfrac{\pi}{2}\big)=C^{+}_{2n-1}(x,\varphi), \quad (2.2)$$
$$C^{+}_{2n}(x+k\,\pi,\varphi+l\,\pi)=C^{+}_{2n}(x,\varphi), \quad (2.3)$$
$$C^{-}_{2n}(x+k\,\pi,\varphi+l\,\pi)=C^{-}_{2n}(x,\varphi), \quad (2.4)$$
$$C^{+}_{n}(x,\varphi)=C^{+}_{n}(\varphi,x), \quad (2.5)$$
$$C^{-}_{n}(x,\varphi)=C^{-}_{n}(\varphi,x), \quad (2.6)$$
$$S^{+}_{2n-1}(x,\varphi)=S^{-}_{2n-1}(\varphi,x), \quad (2.7)$$
$$S^{+}_{n}(x,\varphi)=C^{+}_{n}\big(x-\tfrac{\pi}{2},\varphi\big)=C^{+}_{n}\big(x,\varphi-\tfrac{\pi}{2}\big), \quad (2.8)$$
$$S^{+}_{2n}(x,\varphi)=C^{+}_{2n}\big(x+\tfrac{\pi}{2},\varphi\big)=C^{+}_{2n}\big(x,\varphi+\tfrac{\pi}{2}\big), \quad (2.9)$$
$$C^{+}_{2n}(x,\varphi)=S^{+}_{2n}\big(x+\tfrac{\pi}{2},\varphi\big)=S^{+}_{2n}\big(x,\varphi+\tfrac{\pi}{2}\big), \quad (2.10)$$
$$C^{-}_{n}\big(x+\tfrac{\pi}{2},\varphi\big)=(-1)^{n-1}S^{-}_{n}(x,\varphi) \quad (2.11)$$

and

$$C^{+}_{n+1}(x,\varphi)=C^{+}_{1}(x,\varphi)\,C^{+}_{n}(x,\varphi)-\cos(x+\varphi)\,\cos(x-\varphi)\,C^{+}_{n-1}(x,\varphi)$$
$$=2\,\cos\varphi\,\cos x\,C^{+}_{n}(x,\varphi)+\big(\sin^2\varphi-\cos^2x\big)\,C^{+}_{n-1}(x,\varphi)$$
$$=2\,\cos\varphi\,\cos x\,C^{+}_{n}(x,\varphi)-\frac{1}{2}\big(\cos(2\varphi)+\cos(2x)\big)\,C^{+}_{n-1}(x,\varphi). \quad (2.12)$$

Now, it will be presented the first of our two basic decompositions of functions (1.1)–(1.4).

Lemma 2.2. *The following two classical identities hold*

$$2^{n-2}\, C_n^+(x,\varphi) = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{k} \cos\big((n-2k)\varphi\big)\cos\big((n-2k)x\big) - \frac{1}{2}\left(\left\lfloor\frac{n}{2}\right\rfloor - \left\lfloor\frac{n-1}{2}\right\rfloor\right)\binom{n}{\lfloor\frac{n}{2}\rfloor} \tag{2.13}$$

and

$$2^{n-2}\, C_n^-(x,\varphi) = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \binom{n}{k} \sin\big((n-2k)\varphi\big)\sin\big((n-2k)x\big). \tag{2.14}$$

The proof of the above identities by induction follows and will be omitted here.

2.1. Corollaries

The sequence of the following identities can be deduced easily from identities (2.13) and (2.14):

$$\begin{aligned} 2^{2n-2}\, S_{2n}^+(x,\varphi) &= \sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{k}\cos\big(2(n-k)\varphi\big)\cos\big(2(n-k)x\big) - \frac{1}{2}\binom{2n}{n} \\ &= \sum_{k=0}^{n-1}(-1)^{n-k}\binom{2n}{k}\cos\big(2(n-k)\varphi\big)\cos\big(2(n-k)x\big) + \frac{1}{2}\binom{2n}{n}; \end{aligned} \tag{2.15}$$

and

$$2^{2n-3}\, S_{2n-1}^+(x,\varphi) = \sum_{k=0}^{n-1}(-1)^{n-k-1}\binom{2n-1}{k}\cos\big((2n-2k-1)\varphi\big)\sin\big((2n-2k-1)x\big); \tag{2.16}$$

$$2^{2n-3}\, S_{2n-1}^-(x,\varphi) = \sum_{k=0}^{n-1}(-1)^{n-k-1}\binom{2n-1}{k}\sin\big((2n-2k-1)\varphi\big)\cos\big((2n-2k-1)x\big); \tag{2.17}$$

$$2^{2n-2}\,S^-_{2n}(x,\varphi)=\sum_{k=0}^{n-1}(-1)^{n-k-1}\binom{2n}{k}\sin\big(2(n-k)\varphi\big)\sin\big(2(n-k)x\big);\quad (2.18)$$

$$2^{2n-2}\,\sin^{2n-1}(x)=\sum_{k=0}^{n-1}(-1)^{n-k-1}\binom{2n-1}{k}\sin\big((2n-2k-1)x\big);\quad (2.19)$$

$$\begin{aligned}2^{2n-1}\,\sin^{2n}(x)&=2^{2n-2}\,S^+_{2n}(x,0)\\&=\sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{k}\cos\big(2(n-k)x\big)-\frac{1}{2}\binom{2n}{n};\end{aligned}\quad (2.20)$$

$$\begin{aligned}2^{n-1}\,\cos^{n}(x)&=2^{n-2}\,C^+_{n}(x,0)\\&=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\cos\big((n-2k)x\big)-\frac{1}{2}\left(\left\lfloor\frac{n}{2}\right\rfloor-\left\lfloor\frac{n-1}{2}\right\rfloor\right)\binom{n}{\lfloor n/2\rfloor};\end{aligned}\quad (2.21)$$

$$\begin{aligned}2^{2n-2}\,C^+_{2n}\big(x,\tfrac{\pi}{4}\big)&=\sum_{k=0}^{n}\binom{2n}{k}\cos\big((n-k)\tfrac{\pi}{2}\big)\cos\big(2(n-k)x\big)-\frac{1}{2}\binom{2n}{n}\\&=\frac{1}{2}\binom{2n}{n}-\binom{2n}{n-2}\cos(4x)+\binom{2n}{n-4}\cos(8x)\\&\qquad-\binom{2n}{n-6}\cos(12x)+\ldots;\end{aligned}\quad (2.22)$$

$$\begin{aligned}2^{2n-2}\,C^+_{2n}\big(x+\tfrac{\pi}{4},\tfrac{\pi}{4}\big)&=\sin^{2n}(x)+\cos^{2n}(x)\\&=\sum_{k=0}^{n}\binom{2n}{k}\cos^2\big((n-k)\tfrac{\pi}{2}\big)\cos\big(2(n-k)x\big)-\frac{1}{2}\binom{2n}{n}\\&=\frac{1}{2}\binom{2n}{n}+\binom{2n}{n-2}\cos(4x)+\binom{2n}{n-4}\cos(8x)\\&\qquad+\binom{2n}{n-6}\cos(12x)+\ldots;\end{aligned}\quad (2.23)$$

Proof. (ad (2.15) and (2.16)) We have (by (2.13)):

$$\begin{aligned}2^{2n-2}\,S^+_{2n}(x,\varphi)&=2^{2n-2}\,C^+_{2n}(x,\varphi+\tfrac{\pi}{2})\\&=\sum_{k=0}^{n}\binom{2n}{k}\cos\big(2(n-k)\varphi+(n-k)\pi\big)\cos\big(2(n-k)x\big)-\frac{1}{2}\binom{2n}{n}\\&=\sum_{k=0}^{n}\binom{2n}{k}\big(-1\big)^{n-k}\cos\big(2(n-k)\varphi\big)\cos\big(2(n-k)x\big)-\frac{1}{2}\binom{2n}{n}.\end{aligned}$$

n	$C_n^+(x,\varphi)$	$C_n^-(x,\varphi)$
1	$2\cos(\varphi)\cos(x)$	$2\sin(\varphi)\sin(x)$
2	$1+\cos(2\varphi)\cos(2x)$	$\sin(2\varphi)\sin(2x)$
3	$\frac{1}{2}\big(3\cos(\varphi)\cos(x)+\cos(3\varphi)\cos(3x)\big)$	$\frac{1}{2}\big(3\sin(\varphi)\sin(x)+\sin(3\varphi)\sin(3x)\big)$
4	$\frac{1}{4}\big(3+4\cos(2\varphi)\cos(2x)+\cos(4\varphi)\cos(4x)\big)$	$\frac{1}{4}\big(4\sin(2\varphi)\sin(2x)+\sin(4\varphi)\sin(4x)\big)$
5	$\frac{1}{8}\big(10\cos(\varphi)\cos(x)+5\cos(3\varphi)\cos(3x)$	$\frac{1}{8}\big(10\sin(\varphi)\sin(x)+5\sin(3\varphi)\sin(3x)$
	$+\cos(5\varphi)\cos(5x)\big)$	$+\sin(5\varphi)\sin(5x)\big)$
6	$\frac{1}{16}\big(10+15\cos(2\varphi)\cos(2x)$	$\frac{1}{16}\big(15\sin(2\varphi)\sin(2x)$
	$+6\cos(4\varphi)\cos(4x)+\cos(6\varphi)\cos(6x)\big)$	$+6\sin(4\varphi)\sin(4x)+\sin(6\varphi)\sin(6x)\big)$
7	$\frac{1}{32}\big(35\cos(\varphi)\cos(x)+21\cos(3\varphi)\cos(3x)$	$\frac{1}{32}\big(35\sin(\varphi)\sin(x)+21\sin(3\varphi)\sin(3x)$
	$+7\cos(5\varphi)\cos(5x)+\cos(7\varphi)\cos(7x)\big)$	$+7\sin(5\varphi)\sin(5x)+\sin(7\varphi)\sin(7x)\big)$

Table 1:

On the other hand (also by (2.13)), we get:

$$2^{2n-3}\,S_{2n-1}^{+}(x,\varphi)=2^{2n-3}\,C_{2n-1}^{+}(x-\tfrac{\pi}{2},\varphi)$$
$$=\sum_{k=0}^{n-1}\binom{2n-1}{k}\cos\big((2n-2k-1)\varphi\big)\cos\big((2n-2k-1)x-(n-k)\pi+\tfrac{\pi}{2}\big)$$
$$=\sum_{k=0}^{n-1}\binom{2n-1}{k}\big(-1\big)^{n-k-1}\cos\big((2n-2k-1)\varphi\big)\sin\big((2n-2k-1)x\big).$$

(ad (2.17)) We have:

$$2^{2n-3}\,S_{2n-1}^{-}(x,\varphi)=-2^{2n-3}\,C_{2n-1}^{+}(x,\varphi+\tfrac{\pi}{2})$$
$$=-\sum_{k=0}^{n-1}\binom{2n-1}{k}\cos\big((2n-2k-1)\varphi+(n-k)\pi-\tfrac{\pi}{2}\big)\cos\big((2n-2k-1)x\big)$$
$$=\sum_{k=0}^{n-1}\binom{2n-1}{k}\big(-1\big)^{n-k-1}\sin\big((2n-2k-1)\varphi\big)\cos\big((2n-2k-1)x\big).$$

In Table 1 and Table 2 the first seven decompositions of the type (2.13)–(2.18) of every function C_n^+, C_n^-, S_n^+ and S_n^- are presented.

Remark 2.3. We note that by identities (Paszkowski [3] and Rivlin [5]):

$$T_n\big(\tfrac{1}{2}(y+y^{-1})\big)=\frac{1}{2}\big(y^n+y^{-n}\big),\qquad y\neq 0,$$

n	$S_n^+(x,\varphi)$	$S_n^-(x,\varphi)$
1	$2\cos(\varphi)\sin(x)$	$2\sin(\varphi)\cos(x)$
2	$1-\cos(2\varphi)\cos(2x)$	$\sin(2\varphi)\sin(2x)$
3	$\frac{1}{2}\big(3\cos(\varphi)\sin(x)-\cos(3\varphi)\sin(3x)\big)$	$\frac{1}{2}\big(3\sin(\varphi)\cos(x)-\sin(3\varphi)\cos(3x)\big)$
4	$\frac{1}{4}\big(3-4\cos(2\varphi)\cos(2x)+\cos(4\varphi)\cos(4x)\big)$	$\frac{1}{4}\big(4\sin(2\varphi)\sin(2x)-\sin(4\varphi)\sin(4x)\big)$
5	$\frac{1}{8}\big(10\cos(\varphi)\sin(x)-5\cos(3\varphi)\sin(3x)$	$\frac{1}{8}\big(10\sin(\varphi)\cos(x)-5\sin(3\varphi)\cos(3x)$
	$+\cos(5\varphi)\sin(5x)\big)$	$+\sin(5\varphi)\cos(5x)\big)$
6	$\frac{1}{16}\big(10-15\cos(2\varphi)\cos(2x)$	$\frac{1}{16}\big(15\sin(2\varphi)\sin(2x)-$
	$+6\cos(4\varphi)\cos(4x)-\cos(6\varphi)\cos(6x)\big)$	$-6\sin(4\varphi)\sin(4x)+\sin(6\varphi)\sin(6x)\big)$
7	$\frac{1}{32}\big(35\cos(\varphi)\sin(x)-21\cos(3\varphi)\sin(3x)$	$\frac{1}{32}\big(35\sin(\varphi)\cos(x)-21\sin(3\varphi)\cos(3x)+$
	$+7\cos(5\varphi)\sin(5x)-\cos(7\varphi)\sin(7x)\big)$	$+7\sin(5\varphi)\cos(5x)-\sin(7\varphi)\cos(7x)\big)$

Table 2:

and

$$T_n(x) = 2^{-n}\sum_{k=0}^{n}\binom{2n}{2k}(x+1)^{n-k}(x-1)^k = \frac{n}{2}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k}{n-k}\binom{n-k}{k}(2\,x)^{n-2k},$$

where $T_n(x)$ denotes the n-th Chebyshev polynomial of the first kind, we get (for $y=\tan x$) the following decompositions (see also (2.23)):

$$\sin^{2n}(x)+\cos^{2n}(x) = \frac{\sin^n(2x)}{2^{n-1}}\,T_n\big(\sin^{-1}(2x)\big)$$

$$= \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\,\frac{n}{n-k}\binom{n-k}{k}\frac{1}{2^{2k}}\,\sin^{2k}(2x) \tag{2.24}$$

$$= \frac{\sin^{2n}(x+\frac{\pi}{4})}{2^{n-1}}\sum_{k=0}^{n}\binom{2n}{2k}\left(\frac{\sin(x-\frac{\pi}{4})}{\sin(x+\frac{\pi}{4})}\right)^{2k}. \tag{2.25}$$

Remark 2.4. We have the so called "Kummer identities" (see Grzymkowski et al [1], Ribenboim [4] and Ma [2]):

$$x^n+y^n = \sum_{r=0}^{\lfloor n/2\rfloor}(-1)^r\,\frac{n}{n-r}\binom{n-r}{r}p^{n-2r}\,q^r \tag{2.26}$$

and

$$x^{n+1}-y^{n-1} = (x-y)\sum_{r=0}^{\lfloor n/2\rfloor}(-1)^r\binom{n-r}{r}p^{n-2r}\,q^r, \tag{2.27}$$

where $p = x + y$, $q = xy$. Hence, for example, we obtain four basic identities (for $x = \cos\varphi$, $y = i\sin\varphi$):

$$\cos^{4n+1}\varphi + i\sin^{4n+1}\varphi = \sum_{r=0}^{2n}(-1)^r \frac{4n+1}{4n+1-r}\binom{4n+1-r}{r} e^{i\,(4n+1-2r)\,\varphi}\left(\frac{i}{2}\sin(2\varphi)\right)^r, \quad (2.28)$$

$$\cos^{4n+3}\varphi - i\sin^{4n+3}\varphi = \sum_{r=0}^{2n+1}(-1)^r \frac{(4n+3}{4n+3-r}\binom{4n+3-r}{r} e^{i\,(4n+3-2r)\,\varphi}\left(\frac{i}{2}\sin(2\varphi)\right)^r, \quad (2.29)$$

$$\cos^{4n+1}\varphi - i\sin^{4n+1}\varphi = e^{-i\,\varphi}\sum_{r=0}^{2n}(-1)^r \binom{4n-r}{r} e^{i\,(4n-2r)\,\varphi}\left(\frac{i}{2}\sin(2\varphi)\right)^r, \quad (2.30)$$

$$\cos^{4n+3}\varphi + i\sin^{4n+3}\varphi = e^{i\,\varphi}\sum_{r=0}^{2n+1}(-1)^r \binom{4n+2-r}{r} e^{i\,(4n-2r)\,\varphi}\left(\frac{i}{2}\sin(2\varphi)\right)^r. \quad (2.31)$$

For example we have

$$\cos^7\varphi - i\sin^7\varphi = e^{i\,7\,\varphi} - i\,\frac{7}{2}\,\sin(2\varphi)\,e^{i\,5\,\varphi} - \frac{7}{2}\,\sin^2(2\varphi)\,e^{i\,3\,\varphi} + i\,\frac{7}{8}\,\sin^3(2\varphi)\,e^{i\,\varphi}$$

or equivalently:

$$8\cos^7\varphi = 8\cos(7\varphi) + 28\sin(2\varphi)\,\sin(5\varphi) - 28\sin^2(2\varphi)\,\cos(3\varphi) - 7\sin^3(2\varphi)\,\sin(\varphi)$$

and

$$8\sin^7\varphi = -8\sin(7\varphi) + 28\sin(2\varphi)\,\cos(5\varphi) + 28\sin^2(2\varphi)\,\sin(3\varphi) - 7\sin^3(2\varphi)\,\cos(\varphi).$$

It should be noted that decomposition (2.26) is related to the so-called Horadam polynomials $c_k(x)$ and $C_k(x)$ (see Wituła et al [6] and Grzymkowski et al [1]) in the following way:

$$x^{2k+1} + y^{2k+1} = p\,(-q)^k\,c_k\left(-\tfrac{p^2}{q}\right), \quad (2.32)$$

$$x^{2k} + y^{2k} = (-q)^k\,C_k\left(-\tfrac{p^2}{q}\right). \quad (2.33)$$

Assuming that $\Omega_n(x) := 2T_n(x/2)$, $n \in \mathbb{N}$, we obtain also the following identities:

$$\Omega_{2n+1}(x) = (-1)^n x\, c_n(-x^2), \tag{2.34}$$

$$\Omega_{2n}(x) = (-1)^n C_n(-x^2), \tag{2.35}$$

$$\Omega_n(x+2) = C_n(x), \tag{2.36}$$

for every $n \in \mathbb{N}$. So, Horadam polynomials are modified versions of Chebyshev polynomials of the first kind.

3. The Second Decomposition

The purpose of this section is to generate some new decompositions of all four discussed here functions C_n^+, C_n^-, S_n^+ and S_n^- (identities (3.5), (3.8), (3.9), (3.11), and (3.12) below). We start with the presentation of auxiliary identities related to the derivatives of Chebyshev polynomials of the second kind $U_n(x)$, $n \in \mathbb{N}$ (identities 2°–5° seem to be innovative).

Lemma 3.1. *The following identities hold for $n, k \in \mathbb{N}$:*

1° $U_n^{(k)}(x) = 2\,x\,U_{n-1}^{(k)}(x) + 2\,k\,U_{n-1}^{(k-1)}(x) - U_{n-2}^{(k)}(x)$;

2° $x\,U_n'(x) - U_{n-1}'(x) = n\,U_n(x)$;

3° $x\,U_n^{(k)}(x) - U_{n-1}^{(k)}(x) = (n-k+1)\,U_n^{(k-1)}(x)$;

4° $U_n^{(k)}(x) = 2\,n\,U_{n-1}^{(k-1)}(x) + U_{n-2}^{(k)}(x)$;

5° $(n+1)U_{n-k}^{(k-1)}(x) = 2nxU_{n-k-1}^{(k-1)}(x) + 2k(n-1)U_{n-k-1}^{(k-2)}(x) - (n-1)U_{n-k-2}^{(k-1)}(x)$.

Proof. Ad 1° It follows by k-times differentiation of the basic recurrence identity:

$$U_n(x) = 2\,x\,U_{n-1}(x) - U_{n-2}(x).$$

Ad 2° It is sufficient to prove the identity for the following representation of $U_n(x)$:

$$U_n(x) = \frac{\sin\big((n+1)\arccos x\big)}{\sin(\arccos x)} = \frac{\sin\big((n+1)\arccos x\big)}{\sqrt{1-x^2}}$$

which hold for $|x| < 1$. We have ($y := \arccos x$):

$$x\,U_n'(x) - U_{n-1}'(x) = \frac{x}{1-x^2}\bigg(- (n+1)\,\cos\big((n+1)\,y\big)$$

$$+\frac{x}{\sqrt{1-x^2}}\sin\big((n+1)\,y\big)\Big)-\frac{1}{1-x^2}\Big(-n\,\cos(n\,y)+\frac{x}{\sqrt{1-x^2}}\sin(n\,y)\Big)$$
$$=\sin^{-3}(y)\Big(-(n+1)\,\cos(y)\sin(y)\cos\big((n+1)\,y\big)+n\,\sin(y)\cos(n\,y)$$
$$+\cos^2(y)\sin\big((n+1)\,y\big)-\cos(y)\sin(n\,y)\Big)$$
$$=\sin^{-3}(y)\bigg(n\,\sin(y)\Big(\cos(n\,y)-\cos(y)\cos\big((n+1)\,y\big)\Big)$$
$$+\cos(y)\Big(\cos(y)\sin\big((n+1)\,y\big)-\sin(y)\cos\big((n+1)\,y\big)-\sin(n\,y)\Big)\bigg)$$
$$=n\,\sin^{-2}(y)\sin(y)\sin\big((n+1)\,y\big)=n\,U_n(x).$$

Ad 3° By $(k-1)$-times differentiating of the identity 2° we get 3°.

Ad 4° By 3° we have:

$$2\,x\,U_{n-1}^{(k)}(x)=2\,U_{n-2}^{(k)}(x)+2\,(n-k)\,U_{n-1}^{(k-1)}(x). \tag{3.1}$$

On the other hand, by 1°, we get:

$$2\,x\,U_{n-1}^{(k)}(x)=U_n^{(k)}(x)-2\,k\,U_{n-1}^{(k-1)}(x)+U_{n-2}^{(k)}(x). \tag{3.2}$$

From (3.1) and (3.2) we deduce desired identity 4°.

Ad 5° By replacing in 1° n by $n-k-1$, and next k by $k-1$ and multiplying the derived equation by n we obtain:

$$n\,U_{n-k}^{(k-1)}(x)=2\,n\,x\,U_{n-k-1}^{(k-1)}(x)+2\,(k-1)\,n\,U_{n-k-1}^{(k-2)}(x)-n\,U_{n-k-2}^{(k-1)}(x). \tag{3.3}$$

Next, from 4°, we obtain:

$$U_{n-k}^{(k-1)}(x)=2\,(n-k)\,U_{n-k-1}^{(k-2)}(x)+U_{n-k-2}^{(k-1)}(x). \tag{3.4}$$

We note that (3.3) + (3.4) yields 5°.

Now we are ready to present our second fundamental decompositions of C_n^+, C_n^-, S_n^+ and S_n^-.

Lemma 3.2. *The following identity hold:*

$$C_n^+(x,\varphi)=\sum_{k=0}^{\lfloor n/2\rfloor}A_k^{(n)}(\cos\varphi)\,(\cos x)^{n-2k}, \tag{3.5}$$

where:

$$A_0^{(n)}(y):=2\,T_n(y), \tag{3.6}$$
$$A_k^{(n)}(y):=\frac{2n}{(2k)!!}\,(1-y^2)^k\,U_{n-k-1}^{(k-1)}(y) \tag{3.7}$$

for any $k, n \in \mathbb{N}$, $k \le n$. For example, we have:

$$A_1^{(n)}(x) := n\,(1-x^2)\,U_{n-2}(x),$$
$$A_2^{(n)}(x) := \frac{n}{4}\,(1-x^2)^2\,U'_{n-3}(x),$$
$$A_3^{(n)}(x) := \frac{n}{24}\,(1-x^2)^3\,U''_{n-4}(x), \ldots$$

and:

n	$C_n^+(x,\varphi)$
1	$2\cos(\varphi)\cos(x)$
2	$2\big(2\cos^2(\varphi)-1\big)\cos^2(x)+2\sin^2(\varphi)$
	$=2\cos(2\varphi)\cos^2(x)+2\sin^2(\varphi)$
3	$2\,\cos(\varphi)\,\big(4\cos^2(\varphi)-3\big)\cos^3(x)+6\sin^2(\varphi)\cos(\varphi)\cos(x)$
	$=2\cos(3\varphi)\cos^3(x)+3\sin(\varphi)\sin(2\varphi)\cos(x)$
4	$2\cos(4\varphi)\cos^4(x)+4\,\sin^2(\varphi)\,\big(4\cos^2(\varphi)-1\big)\cos^2(x)+2\sin^4(\varphi)$
	$=2\cos(4\varphi)\cos^4(x)+4\sin(\varphi)\sin(3\varphi)\cos^2(x)+2\sin^4(\varphi)$

Proof. By Lemma 2.1 the following identity can be verified:

$$\sum_{k=0}^{\lfloor (n+1)/2\rfloor} A_k^{(n+1)}(\cos\varphi)\,(\cos x)^{n+1-2k}$$
$$= \sum_{k=0}^{\lfloor n/2\rfloor} 2\,\cos(\varphi)\,A_k^{(n)}(\cos\varphi)\,(\cos x)^{n+1-2k}$$
$$+ \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \sin^2(\varphi)\,A_k^{(n-1)}(\cos\varphi)\,(\cos x)^{n-1-2k}$$
$$- \sum_{k=0}^{\lfloor (n-1)/2\rfloor} A_k^{(n-1)}(\cos\varphi)\,(\cos x)^{n+1-2k},$$

hence, for $k = 1, 2, \ldots, \lfloor (n-1)/2\rfloor$:

$$A_k^{(n+1)}(x) = 2\,\cos(\varphi)\,A_k^{(n)}(x) + \sin^2(\varphi)\,A_{k-1}^{(n-1)}(x) - A_k^{(n-1)}(x)$$

which, after some manipulations, implies identity 5° of Lemma 3.1.

Corollary 3.3. *By (2.8) we get the following identities*

$$S_n^+(x) = \sum_{k=0}^{\lfloor n/2\rfloor} A_k^{(n)}\big(\cos\varphi\big)\,\big(\sin x\big)^{n-2k}$$
$$= \sum_{k=0}^{\lfloor n/2\rfloor} A_k^{(n)}\big(\sin\varphi\big)\,\big(\cos x\big)^{n-2k}. \qquad (3.8)$$

Remark 3.4. The following identity may be proved in a similar way as identity (3.5) be derived:

$$C_n^-(x,\varphi) = C_1^-(x,\varphi) \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} B_k^{(n)}(\cos\varphi)\,(\cos x)^{n-2k-1}$$

$$= 2\sin(\varphi)\sin(x) \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} B_k^{(n)}(\cos\varphi)\,(\cos x)^{n-2k-1}, \tag{3.9}$$

where:

$$B_k^{(n)}(x) := \frac{1}{(2k)!!}\,(1-x^2)^k\,U_{n-k-1}^{(k)}(x) \tag{3.10}$$

for any $k, n \in \mathbb{N}$. For example, we have:

$$B_0^{(n)}(x) = U_{n-1}(x),$$
$$B_1^{(n)}(x) = \frac{1}{2}\,(1-x^2)\,U'_{n-2}(x),$$
$$B_2^{(n)}(x) = \frac{1}{8}\,(1-x^2)^2\,U''_{n-3}(x)$$

and:

n	$C_n^-(x,\varphi)$
2	$C_1^-(x,\varphi)\cos(\varphi)\cos(x)$
3	$C_1^-(x,\varphi)\big(U_2(\cos(\varphi))\cos^2(x)+\sin^2(\varphi)\big)$
4	$C_1^-(x,\varphi)\big(U_3(\cos(\varphi))\cos^3(x)+4\sin^2(\varphi)\cos(\varphi)\cos(x)\big)$
5	$C_1^-(x,\varphi)\big(U_4(\cos(\varphi))\cos^4(x)+\frac{1}{2}\sin^2(\varphi)U'_3(\cos(\varphi))\cos^2(x)+\sin^4(\varphi)\big)$
6	$C_1^-(x,\varphi)\big(U_5(\cos(\varphi))\cos^5(x)+\frac{1}{2}\sin^2(\varphi)U'_4(\cos(\varphi))\cos^3(x)$ $+6\sin^2(\varphi)\cos(\varphi)\cos(x)\big)$

Moreover, from equation (2.11) we also get:

$$S_n^-(x,\varphi) = (-1)^{n-1}\,C_n^-(x+\tfrac{\pi}{2},\varphi) =$$

$$= C_1^-(x+\tfrac{\pi}{2},\varphi) \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} B_k^{(n)}(\cos\varphi)\,(\sin x)^{n-2k-1}$$

$$= 2\sin(\varphi)\cos(x) \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} B_k^{(n)}(\cos\varphi)\,(\sin x)^{n-2k-1}. \tag{3.11}$$

Hence, by (2.7) we obtain another interesting decomposition:

$$S_{2n-1}^+(x,\varphi) = 2\sin(x)\cos(\varphi) \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} B_k^{(n)}(\cos x)\,(\sin\varphi)^{n-2k-1}. \tag{3.12}$$

References

[1] R. Grzymkowski, R. Wituła, *Computational Methods in Algebra*, Part I, WPKJS, Gliwice (2000), In Polish.

[2] X. Ma, A generalization of the Kummer identity and its application to Fibonacci-Lucas sequences, *Fib. Quarterly*, **36** (1998), 339-347.

[3] S. Paszkowski, *Numerical Applications of Chebyshev Polynomials and Series*, PWN, Warsaw (1975), In Polish.

[4] P. Ribenboim, *Fermat's Last Theorem for Amateurs*, Springer, New York (1999).

[5] T. Rivlin, *Chebyshev Polynomials from Approximation Theory to Algebra and Number Theory*, Wiley, New York (1990).

[6] R. Wituła, D. Słota, On modified Chebyshev polynomials, *J. Math. Anal. Appl.*, **324** (2006), 321-343.

[7] R. Wituła, D. Słota, Some trigonometric identities related to powers of cosine and sine functions, *Folia Math.*, To Appear.