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ON NONLINEAR MIXED QUASIVARIATIONAL INEQUALITIES

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Abstract: In this paper, we introduce and study a new class of nonlinear mixed quasivariational inequalities in Hilbert spaces. By applying the projection technique, we prove an existence and uniqueness theorem of solution for the nonlinear mixed quasivariational inequality, suggest and analyze an iterative method to compute the approximate solutions of the nonlinear mixed quasivariational inequality and establish the convergence criteria of the iterative method. The results presented in this paper improve, extend and unify some known results in this area.

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1. Introduction

Variational inequality theory has emerged an interesting branch of applicable mathematics. It has appeared as an effective and powerful tool to study and investigate a wide class of problems arising in pure and applied sciences including elasticity, optimization, economics, transportation and structural analysis. Using nonlinear analysis technique and method, many authors investigated various variational inequalities involving some classes of monotone mappings, established the existence and uniqueness of solution for these variational inequalities, and suggested iterative algorithms to compute approximation solutions, see [2]-[10] and the references therein.

Motivated and inspired by the above research work, in this paper, we introduce and study a new class of nonlinear mixed quasivariational inequalities, suggest a new iterative method with errors for finding the approximation solutions of the nonlinear mixed quasivariational inequality. Under certain conditions, we obtain an existence and uniqueness result of solution for the nonlinear mixed quasivariational inequality and discuss the convergence of the iterative sequence generated by the iterative method. The results presented in this paper improve, extend and unify several known results in this area.

2. Preliminaries

Throughout this paper, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let 2^H and $CC(H)$ stand for the families of all nonempty subsets and all nonempty closed convex subsets, respectively. For $X \in CC(H)$, we assume that P_X denote the projection of H onto X.

Given mappings $g, A, B, D, T : H \to H$, $N : H \times H \to H$ and $K : H \to CC(H)$, we consider the following nonlinear mixed quasivariational inequality:

Find $u \in H$ such that $gu \in K(Du)$ and

$$\langle N(Au, Bu) + Tu, v - gu \rangle \geq 0, \quad \forall v \in K(Du). \tag{2.1}$$

The nonlinear mixed quasivariational inequality (2.1) includes the variational inequalities in [9, 10] as special cases.

Definition 2.1. Let $A, B : H \to H$ and $N : H \times H \to H$ be mappings.

(1) A is said to be *strongly monotone* with respect to the first argument of the mapping N if there exists a constant $r > 0$ such that

$$\langle N(Au, w) - N(Av, w), u - v\rangle \geq r\|u - v\|^2, \quad \forall u, v, w \in H;$$

(2) B is said to be *relaxed monotone* with respect to the second argument of the mapping N if there exists a constant $r > 0$ such that

$$\langle N(w, Bu) - N(w, Bv), u - v\rangle \geq -r\|u - v\|^2, \quad \forall u, v, w \in H.$$

Definition 2.2. A mapping $N : H \times H \to H$ is said to be *Lipschitz continuous* in the first argument if there exists a constant $t > 0$ such that

$$\|N(x, u) - N(y, u)\| \leq t\|x - y\|, \quad \forall x, y, u \in H.$$

Definition 2.3. A mapping $g : H \to H$ is said to be *strongly monotone* and *Lipschitz continuous* if there exist constants $a, b > 0$ such that

$$\langle gu - gv, u - v\rangle \geq a\|u - v\|^2 \quad \text{and} \quad \|gu - gv\| \leq b\|u - v\|, \quad \forall u, v \in H.$$

Lemma 2.1. (see [1]) *Let $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ be three non-negative real sequences satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq 0,$$

where $\{t_n\}_{n\geq 0} \subset [0, 1]$, $\sum_{n=0}^{\infty} t_n = +\infty$, $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty} c_n < +\infty$. Then $\lim_{n\to\infty} a_n = 0$.

3. A Perturbed Ishikawa Iterative Algorithm with Errors

The following results play a crucial role in the proofs of our main results.

Lemma 3.1. (see [8]) *Let $K \in CC(H)$. Then, given $z \in H$, $u = P_K z$ if and only if $u \in K$ satisfies*

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in K.$$

Furthermore, the projection operator P_K is nonexpansive, that is, $\|P_K u - P_K v\| \leq \|u - v\|$, $\forall u, v \in H$.

Lemma 3.2. *$u \in H$ is a solution of the problem* (2.1) *if and only if there exists $u \in H$ such that*

$$u = u - g(u) + P_{K(Du)}(g(u) - \rho(N(Au, Bu) + Tu)), \tag{3.1}$$

where $\rho > 0$ is a constant.

Based on Lemma 3.2, we now suggest the following Ishikawa type iterative scheme with errors for solving the nonlinear mixed quasivariational inequality (2.1).

Algorithm 3.1. Let $g, A, B, D, T : H \to H$, $N : H \times H \to H$ and $K, K^n : H \to CC(H)$ for each $n \geq 0$. Given $u_0 \in H$, compute the sequence $\{u_n\}_{n\geq 0}$ by the iterative scheme

$$\begin{aligned} u_{n+1} &= (1-a_n)u_n + a_n[v_n - g(v_n) + P_{K^n(Dv_n)}(g(v_n) \\ &\quad - \rho N(Av_n, Bv_n) - \rho T v_n)] + a_n e_n, \\ v_n &= (1-b_n)u_n + b_n[u_n - g(u_n) + P_{K^n(Du_n)}(g(u_n) \\ &\quad - \rho N(Au_n, Bu_n) - \rho T u_n)] + b_n f_n \end{aligned} \tag{3.2}$$

for $n \geq 0$, where $\{e_n\}_{n\geq 0}$ and $\{f_n\}_{n\geq 0}$ are two sequences of the elements in H introduced to take into account possible inexact computation and the sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ satisfy the conditions

$$\Sigma_{n=0}^{\infty} a_n = +\infty \quad \text{and} \quad 0 \leq a_n, b_n \leq 1, \quad \forall n \geq 0; \tag{3.3}$$

$$\lim_{n\to\infty} \|e_n\| = 0, \quad \lim_{n\to\infty} \|f_n\| = 0. \tag{3.4}$$

4. Existence and Convergence

In this section, we establish the existence and uniqueness of solution for the nonlinear quasivariational inequality (2.1) and prove the convergence of the iterative sequence generated by Algorithm 3.1.

Theorem 4.1. *Let $g, A, B, D, T : H \to H$ be Lipschitz continuous with constants b, p, h, l, d, respectively, g and T be strongly monotone with constants a and c, respectively. Let $N : H \times H \to H$ be Lipschitz continuous with constants f and k in the first and second arguments, respectively. Assume that A is strongly monotone with constant e with respect to the first argument of N, B is relaxed monotone with constant i with respect to the second argument of N. Let $K : H \to CC(H)$ and $\{K^n\}_{n\geq 0}$ be a sequence of multivalued mappings from H into $CC(H)$ such that*

$$\|P_{K(x)}(z) - P_{K(y)}(z)\| \leq \xi \|x - y\|, \quad \forall x, y, z \in H, \tag{4.1}$$

$$\|P_{K^n(x)}(z) - P_{K^n(y)}(z)\| \leq \xi \|x - y\|, \quad \forall x, y, z \in H,\ n \geq 0, \tag{4.2}$$

$$\lim_{n\to\infty} P_{K^n(x)}(z) = P_{K(x)}(z), \quad \forall x, z \in H, \tag{4.3}$$

where $\xi > 0$ is a constant. If there exists a positive constant ρ satisfying

$$\begin{aligned} &2\sqrt{1-2a+b^2} + \sqrt{1-2\rho c+\rho^2 d^2} + \rho\sqrt{1-2e+f^2p^2} \\ &+ \rho\sqrt{1+2i+h^2k^2} + \xi l < 1, \end{aligned} \tag{4.4}$$

then the nonlinear quasivariational inequality (2.1) *has a unique solution $u^* \in H$ and the sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 3.1 converges strongly to u^*.*

Proof. First, we prove that there exists $u^* \in H$, which is a unique solution of the nonlinear quasivariational inequality (2.1). Define a mapping $F : H \to H$ by

$$F(u) = u - g(u) + P_{K(Du)}(g(u) - \rho(N(Au, Bu) + Tu)), \quad \forall u \in H.$$

By virtue (4.1) and Lemma 3.1, we infer that for any $u, v \in H$

$$\begin{aligned} \|F(u) - F(v)\| &\leq \|u - v - (g(u) - g(v))\| \\ &\quad + \|P_{K(Du)}(g(u) - \rho N(Au, Bu) - \rho Tu) \\ &\qquad - P_{K(Du)}(g(v) - \rho N(Av, Bv) - \rho Tv)\| \\ &\quad + \|P_{K(Du)}(g(v) - \rho N(Av, Bv) - \rho Tv) \\ &\qquad - P_{K(Dv)}(g(v) - \rho N(Av, Bv) - \rho Tv)\| \\ &\leq \|u - v - (g(u) - g(v))\| + \|g(u) - g(v) - \rho(N(Au, Bu) \\ &\qquad - N(Av, Bv)) - \rho(Tu - Tv)\| + \xi\|Du - Dv\| \\ &\leq 2\|u - v - (g(u) - g(v))\| + \|u - v - \rho(Tu - Tv)\| \\ &\quad + \rho\|N(Au, Bu) - N(Av, Bu) - (u - v)\| \\ &\quad + \rho\|(u - v) - (N(Av, Bv) - N(Av, Bu))\| + \xi\|Du - Dv\|. \end{aligned} \tag{4.5}$$

By the Lipschitz continuity of g, D and the strong monotonicity of g, we obtain that

$$\|u - v - (g(u) - g(v))\|^2 \leq (1 - 2a + p^2)\|u - v\|^2 \tag{4.6}$$

and

$$\|Du - Dv\| \leq l\|u - v\|. \tag{4.7}$$

Further, since A is strongly monotone with respect to the first argument of N and B is relaxed monotone with respect to the second argument of N, by the Lipschitz continuity of A, B, T and N in the first and the second arguments, respectively, and the strong monotonicity of g, we have

$$\|N(Au, Bu) - N(Av, Bu) - (u - v)\|^2 \leq (1 - 2e + f^2p^2)\|u - v\|^2, \tag{4.8}$$

$$\|N(Av, Bv) - N(Av, Bu) - (u - v)\|^2 \leq (1 + 2i + h^2k^2)\|u - v\|^2 \tag{4.9}$$

and

$$\|u - v - \rho(Tu - Tv)\|^2 \le (1 - 2\rho c + \rho^2 d^2)\|u - v\|^2. \tag{4.10}$$

Using (4.5)-(4.10), we obtain that

$$\|F(u) - F(v)\| \le q\|u - v\|, \tag{4.11}$$

where

$$q = 2\sqrt{1 - 2a + b^2} + \sqrt{1 - 2\rho c + \rho^2 d^2} + \rho\sqrt{1 - 2e + f^2 p^2} \\ + \rho\sqrt{1 + 2i + h^2 k^2} + \xi l.$$

It follows from (4.4) and (4.11) that F is a contraction mapping. Therefore F has a unique fixed point $u^* \in H$. Lemma 3.2 ensures that the nonlinear quasivariational inequality (2.1) has a unique solution u^*.

Next, we prove that the iterative sequence $\{u_n\}_{n\ge 0}$ defined by Algorithm 3.1 converges strongly to u^*. It follows from (3.2) that

$$\begin{aligned} &\|u_{n+1} - u^*\| \\ &\le (1 - a_n)\|u_n - u^*\| + a_n\|v_n - u^* - (g(v_n) - g(u^*))\| \\ &\quad + a_n\|P_{K^n(Dv_n)}(g(v_n) - \rho N(Av_n, Bv_n) - \rho Tv_n) \\ &\qquad - P_{K^n(Dv_n)}(g(u^*) - \rho N(Au^*, Bu^*) - \rho Tu^*)\| \\ &\quad + a_n\|P_{K^n(Dv_n)}(g(u^*) - \rho N(Au^*, Bu^*) - \rho Tu^*) \\ &\qquad - P_{K^n(Du^*)}(g(u^*) - \rho N(Au^*, Bu^*) - \rho Tu^*)\| \\ &\quad + a_n\|P_{K^n(Du^*)}(g(u^*) - \rho N(Au^*, Bu^*) - \rho Tu^*) \\ &\qquad - P_{K(Du^*)}(g(u^*) - \rho N(Au^*, Bu^*) - \rho Tu^*)\| + a_n\|e_n\| \\ &\le (1 - a_n)\|u_n - u^*\| + qa_n\|v_n - u^*\| + a_n c_n + a_n\|e_n\|, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} c_n = \|&P_{K^n(Du^*)}(g(u^*) - \rho N(Au^*, Bu^*) - \rho Tu^*) \\ &- P_{K(Du^*)}(g(u^*) - \rho N(Au^*, Bu^*) - \rho Tu^*)\|. \end{aligned} \tag{4.13}$$

Similarly, we get

$$\|v_n - u^*\| \le (1 - b_n)\|u_n - u^*\| + qb_n\|u_n - u^*\| + b_n c_n + b_n\|f_n\|. \tag{4.14}$$

It follows from (4.12), (4.13) and (4.14) that

$$\begin{aligned} \|u_{n+1} - u^*\| &\le (1 - a_n(1 - q))\|u_n - u^*\| \\ &\quad + a_n((q + 1)c_n + q\|f_n\| + \|e_n\|), \quad \forall n \ge 0. \end{aligned} \tag{4.15}$$

Using (3.3), (3.4), (4.15) and Lemma 2.1, we infer that $u_n \to u^*$ as $n \to \infty$. This completes the proof. □

Remark 4.1. Theorem 4.1 extends and improve the corresponding results in [9], [10].

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