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EMBEDDINGS OF STABLE CURVES IN $\mathbb{P}^n$, $n \geq 4$, WITH GOOD POSTULATION

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Abstract: Let X be a stable curve of genus g. Here we prove the existence of high degree embeddings of X in $\mathbb{P}^n$ with good postulation (maximal rank). We may also fix the degrees of all components, provided that all these degrees are sufficiently large.

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1. Introduction

Let X be a connected and projective curve with only nodes as singularities. Set $g := p_a(X)$. We will always assume $g \geq 2$. For the case $g = 1$ see Remark 7. Let $S(X)$ be the set of all smooth and rational irreducible components of X intersecting at most two other irreducible components of X. Set $s(X) := \sharp(S(X))$.

Let $T \subset \mathbb{P}^n$ be any scheme. For every integer $t \geq 0$ let $\rho_{T,t} : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(T, \mathcal{O}_T(t))$ denote the restriction map. The scheme T has *maximal rank* if for every integer $t \geq 0$ the linear map $\rho_{T,t}$ has maximal rank, i.e. it is injective or surjective. Now assume $\dim(T) \leq 1$ and that $h^1(T, \mathcal{O}_T(x)) = 0$ for every $x \geq 1$. Let d be the degree of the one-dimensional part of T. Obviously, $\rho_{T,0}$ has maximal rank. Fix an integer $t \geq 1$. Since $h^0(T, \mathcal{O}_T(t)) = td + \chi(\mathcal{O}_T)$, $\rho_{T,t}$ has

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maximal rank if and only if either $h^1(\mathbb{P}^n, I_T(t)) = 0$ (case $td + \chi(\mathcal{O}_T) \le \binom{n+t}{n}$) or $h^0(\mathbb{P}^n, I_T(t)) = 0$ (case $td + \chi(\mathcal{O}_T) \ge \binom{n+t}{n}$). For any reduced projective curve A let $\mathcal{B}(A)$ denote the set of the irreducible components of A. Here we prove the following results.

Theorem 1. *Fix integers $n \ge 4$, $g \ge 2$ and $s \ge 0$. There is an integer $\alpha(n, g, s) > 0$ with the following property. Let X be a nodal and connected projective curve such that $p_a(X) = g$ and $s(X) = s$. Fix integers d_T, $T \in \mathcal{B}(X)$, with the following properties:*

(i) $d_T > 0$ if $T \in S(X)$;

(ii) $d_T \ge 2p_a(T) + 2 + \sharp(T \cap \overline{X \backslash T})$ for all $T \in \mathcal{B}(X) \backslash S(X)$;

(iii) $\sum_{T \in \mathcal{B}(X)} d_T \ge \alpha(n, g, s)$.

Then there exists an embedding $j : X \hookrightarrow \mathbb{P}^n$ such that $\deg(j(T)) = d_T$ for all $T \in \mathcal{B}(X)$ and $j(X)$ has maximal rank.

If X is a stable curve, then $s(X) = 0$. Hence as an immediate corollary of Theorem 1 we get the following result.

Corollary 1. *Fix integers $n \ge 4$, and $g \ge 2$. There is an integer $\alpha(n, g) > 0$ with the following property. Let X be a stable curve of genus g. Fix integers d_T, $T \in \mathcal{B}(X)$ such that $d_T \ge 2p_a(T) + 2 + \sharp(T \cap \overline{X \backslash T})$ for all T and $\sum_{T \in \mathcal{B}(X)} d_T \ge \alpha(n, g)$. Then there exists an embedding $j : X \hookrightarrow \mathbb{P}^n$ such that $\deg(j(T)) = d_T$ for all $T \in \mathcal{B}(X)$ and $j(X)$ has maximal rank.*

Notice that we allow $d_T = 1$ in Theorem 1 only if $T \cong \mathbb{P}^1$ and $\sharp(T \cap \overline{X \backslash T}) \le 2$.

If X is quasi-stable, the case $d_T = 1$ for all $T \in S(X)$ of Theorem 1 has an important geometric interpretation in terms or relative Picard scheme on $\overline{\mathcal{M}}_g$ (see [3]).

We will use the so-called Horace method introduced by A. Hirschowitz.

2. The Proofs

For all integers q, t, n, such that $n \ge 3$ and $t \ge 1$ define the integers $a(n, q, t)$ and $b(n, q, t)$, by the following relation:

$$t \cdot a(n, q, t) + 1 - q + b(n, t, q) = \binom{n+t}{n}, \; 0 \le b(n, t, q) \le t - 1\,. \qquad (1)$$

Obviously, $a(n, q, t) \ge a(n, 0, t)$ for all $q \ge 0$. If $t \ge q \ge 0$, then either $b(n, 0, t) \le n - 1 - q$, $a(n, q, t) = a(n, 0, t)$ or $b(n, q, t) \ge n - g$, $a(n, q, t) = a(n, 0, t) + 1$ and $b(n, q, t) = b(n, 0, t) + g - n$. For every integer $d \ge |q| + n$

let $c_{n,q,d}$ be the minimal integer t such that $d \le a(n,q,t)$. The integer $c_{n,q,d}$ is called the *critical value* of the triple (n,q,d).

Remark 1. Let $C \subset \mathbb{P}^n$ be a reduced and connected curve such that $p_a(C) = g$, $\deg(C) = d$ and $h^1(C, \mathcal{O}_C(1)) = 0$. Hence $h^1(C, \mathcal{O}_C(t)) = 0$ for every $t \ge 1$. The curve C has maximal rank if and only $h^1(\mathbb{P}^n, \mathcal{I}_C(c_{n,g,d}-1)) = 0$ and $h^0(\mathbb{P}^n, \mathcal{I}_C(c_{n,g,d})) = 0$.

Remark 2. Let M be a projective scheme and D an effective Cartier divisor of M. For any closed subscheme $Y \subseteq M$ let $\mathrm{Res}_D(Y)$ denote the residual scheme of Y with respect to D, i.e. the closed subscheme of M with $\mathcal{I}_{D,M} : \mathcal{I}_{Y,M}$ as its ideal sheaf. For any $L \in \mathrm{Pic}(M)$ we have an exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Y),M} \otimes L(-D) \to \mathcal{I}_{Y,M} \otimes L \to \mathcal{I}_{Y \cap D,D} \otimes (L|D) \to 0\,. \qquad (2)$$

From (2) we get

$$h^i(M, \mathcal{I}_{Y,M} \otimes L) \le h^i(M, \mathcal{I}_{\mathrm{Res}_D(Y),M} \otimes L(-D)) + h^i(D, \mathcal{I}_{Y \cap D,D} \otimes (L|D)) \quad (3)$$

for all integers $i \ge 0$.

Remark 3. Fix $Q \in \mathbb{P}^n$, $n \ge 3$, and any 3-dimensional linear subspace M of $\mathbb{P}^n$ such that $P \in M$. Let $\chi_M(Q)$ denote the first infinitesimal neighborhood of Q in M, i.e. the closed subscheme of M with $\mathcal{I}^2_{Q,M}$ as its ideal sheaf. We have $\mathrm{length}(\chi_M(Q)) = 4$. Now assume $n \ge 4$ and take any hyperplane $H \subset \mathbb{P}^n$ containing Q, but not containing M. Then $\mathrm{Res}_H(\chi_M(Q)) = \{Q\}$.

Remark 4. Let $C \subset \mathbb{P}^m$, $m \ge 3$, be a nodal integral curve. We do not assume that C is non-degenerate. Assume $h^1(C, \mathcal{O}_C(1)) = 0$. Fix positive integers k, d_i, $1 \le i \le k$, and k distinct points of C_{reg}. Set $L := \mathcal{O}_C(1)(d_1P_1 + \cdots + d_kP_k)$. Notice that $h^1(C, L) = 0$, $\deg(L) = \deg(C) + d_1 + \cdots + d_k$ and $h^0(C, L) = h^0(C, \mathcal{O}_C(1)) + d_1 + \cdots = d_k$.

Remark 5. Let $Y \subset \mathbb{P}^m$, $m \ge 2$, be a connected nodal curve such that $p_a(Y) = 0$. It is classical (see [7]) that Y is a limit in the Hilbert scheme of $\mathbb{P}^m$ of a flat family of smooth rational curves with the same degrees.

Lemma 1. *Fix an integral nodal curve $C \subset \mathbb{P}^m$, $m \ge 3$, such that C spans $\mathbb{P}^m$, and $h^1(C, \mathcal{O}_C(1)) = 0$. Fix $P \in C_{reg}$, and an integer $d > 0$. Let $D \subset \mathbb{P}^m$ be a general smooth rational curve such that $P \in D$ and $\deg(D) = d$. The curve $C \cup D$ is connected, nodal, and $p_a(C \cup D) = p_a(C)$. There are an integral curve E, $o \in E$, and a flat family $\{Y_t\}_{t \in E}$ of curves in $\mathbb{P}^m$ such that $Y_o = C \cup D$, while $Y_t \cong C$ for all $t \in T \backslash \{o\}$.*

Proof. First assume that C is linearly normal, i.e. assume $h^0(C, \mathcal{O}_C(1)) = m+1$. The case $m = 3$ and C smooth was proved by A. Tannenbaum (see [7], Theorem 1.1). The same proof (working in a product $C \times \mathbb{P}^1$) works if $m > 3$ and C is sigular, because $P \in C_{reg}$. Now assume $x := h^0(C, \mathcal{O}_C(1)) - 1 \geq m+1$. Consider the linearly normal embedding $\phi : C \hookrightarrow \mathbb{P}^x$. Notice that there is a smooth rational degree d curve $D' \subset \mathbb{P}^x$ such that $D' \cap \phi(C) = \{\phi(P)\}$, $\phi(C) \cup D'$ is nodal and $C \cup D$ is a linear projection of $\phi(C) \cup D'$. Apply the linearly normal case to $\phi(C) \cup D'$ and then apply the given linear projection to all nearby fibers of the flat deformation of $\phi(C) \cup D'$. □

Lemma 2. *Fix integers $m \geq 3$, $x \geq 0$ and $c > 0$. There exists an integer $k(m, x, c)$ with the following property. Fix any $S \subset \mathbb{P}^m$ such that $\sharp(S) = x$, any integer $k \geq k(m, x, c)$ and integers $d_1, \dots, d_c$ such that $x + c + d_1 + \cdots + d_c \geq \binom{m+k-1}{m}$. Let $Y \subset \mathbb{P}^m$ be a general union of S and c smooth rational curves of degrees $d_1, \dots, d_s$. Then Y has maximal rank.*

Proof. The case $x = 0$ is a very weak form of [1] (case $m = 3$) or [2], case $m \geq 4$. The general case may be proved in a similar way either inserting one point of S at each inductive step (from critical value $t-2$ to critical value t when $m = 3$ with the new point of S in a smooth quadric surface, from critical value $t-1$ to critical value t with the new point of S is a hyperplane) or taking a general $S' \subset \mathbb{P}^m$ such that $\sharp(S') = \binom{m+s}{m} - x$ and $h^i(\mathbb{P}^m, \mathcal{I}_{S \cup S'}(x)) = 0$, $i = 0, 1$, and then copying the proof of Theorem 1 below with S instead of W and adding only rational curves disjoint from S. □

Lemma 3. *Fix an integer $m \geq 3$, an integral non-degenerate curve $C \subset \mathbb{P}^m$ and a finite set $S \subset C_{reg}$. Assume $h^1(C, \mathcal{O}_C(1)(-S)) = 0$ and $h^0(C, \mathcal{O}_C(1)(-S)) \geq m+1$. Fix a flat family $\{S_t\}_{t \in E}$ of subsets of $\mathbb{P}^m$ such that E is integral and $S_o = S$ for some $o \in E$. Then there exist a quasi-finite covering $u : E' \to E$ such that $o \in u(E')$, and an open neighborhood U of $u^{-1}(o)$, a flat family $\{C_t\}_{t \in U}$ of curves in $\mathbb{P}^m$ and a flat family $\{f_t\}_{t \in U}$ of isomorphism $f_t : C \to C_t$ such that $C_{o'} = C$ and $f_{o'} = Id_C$ for all $o' \in u^{-1}(o)$, and $S_t = f_t(S)$ for all $t \in U$.*

Proof. Set $x := \sharp(S)$. Since $h^1(C, \mathcal{O}_C(1)(-S)) = 0$, we have $h^1(C, \mathcal{O}_C(1)) = 0$ and $h^0(C, \mathcal{O}_C(1)) = h^0(C, \mathcal{O}_C(1)(-S)) + x$. Since $\mathcal{O}_C(1)$ is very ample, its complete linear system induces an embedding $\psi : C \hookrightarrow \mathbb{P}^r$, $r := h^0(C, \mathcal{O}_C(1)) - 1$. Let M be the $(m-r-1)$-dimensional linear subspace of $\mathbb{P}^r$ such that the linear projection of $\psi(C)$ from M induces the given embedding $C \hookrightarrow \mathbb{P}^m$. By assumption $M \cap \psi(M) = \emptyset$. Since $h^0(C, \mathcal{O}_C(1)) = h^0(C, \mathcal{O}_C(1)(-S)) + x$, the set

$\psi(S)$ is a set of $\sharp(S)$ points in linearly general projection and $\sharp(S) \le \dim(M)-1$. If $x = 1$, then we take $E' = E$ and $u = \mathrm{Id}_E$. If $x \ge 2$ then we take a normalization E' of a certain iterated fiber product to get the existence on E' of x morphisms $v_P : E' \to \mathbb{P}^m$, $P \in S$ such that $S_t = \{v_P(t)\}_{P\in S}$ for all $t \in E'$. We see $\mathbb{P}^m$ as a linear subspace of $\mathbb{P}^r$. For any $P \in S$ and any $t \in E'$ set $W(P,t) : \langle W \cup \{v_P(t)\}\rangle$. Restricting if necessary E' to a neighborhood of $u^{-1}(o)$ we may assume $\dim(W(P,t)) = r - m$ for all t. It is sufficient to prove the existence (restricting if necessary to an open neighborhood of $u^{-1}(o)$ and then taking if a finite covering of this neighborhood) of a flat family $\{W_t\}_{t\in E'}$ of $(r-m-1)$-dimensional linear subspaces of $\mathbb{P}^r$ such that $W_{o'} = W$ for all $o' \in u^{-1}(o)$ and $W_t \cap W(P,t)$ a hyperplane of $W(P,t)$ for all $P \in S$ and t in a neighborhood of $u^{-1}(o)$. This is possible (after a finite covering), because $\dim(W) \ge s-1$. □

Remark 6. Fix integers $d, t > 0$ and $n \ge 3$. Fix a hyperplane $H \subset \mathbb{P}^n$ and $P \in \mathbb{P}^n \backslash H$. Let $C \subset \mathbb{P}^n$ be a general smooth rational curve such that $\deg(C) = d$ and $P \in C$. Let $D \subset H$ be a general degree t smooth rational curve. Let N_C denote the normal bundle of C in $\mathbb{P}^n$ and $N_{D,H}$ the normal bundle of D in H. Thus N_C is a rank $n-1$ vector bundle on C with degree $(n+1)d-2$, while $N_{D,H}$ is a rank $n-2$ vector bundle on D with degree $tn-2$. First assume $d \le n-1$. Let M be the linear span of C. The generality of C gives $\dim(M) = d$. Hence C is a rational normal curve of M. Hence N_C is a direct sum of $d-1$ line bundles of degree $d+2$ and $n-d$ line bundles of degree d. Now assume $d \ge n$. The generality of C gives that N_C is rigid (see [6]), i.e. each rank 1 indecomposable factor of N_C has either degree $\lfloor((n+1)d-2)/(n-1)\rfloor$ or degree $\lceil((n+1)d-2)/(n-1)\rceil$. In both cases we have $h^1(C, N_C(-1)) = 0$. Hence for general C we may assume that $C \cap H$ is formed by d general points of H (see [5], Theorem 1.5). Now assume $t \le n-2$ and call V the linear span of D. The generality of D gives $\dim(V) = t$ and that D is a rational normal curve of V. Varying V and then D inside V we see that for a $S \subset H$ such that $\sharp(S) = t+1$ there is D as above such that $S \subset D$. We may also fix a codimension 2 closed subset T of H and then find (S, D) with the additional condition $D \cap T = \emptyset$. Now assume $t \ge n-1$. Since every indecomposable rank 1 factor of $N_{D,H}$ has degree at least $\lfloor(nt-2)/(n-2)\rfloor$, we may find D as above containing a general $S' \subset H$ such that $\sharp(S') = \lfloor(nt-2)/(n-2)\rfloor + 1$ and $D \cap T = \emptyset$.

Lemma 4. *We may take k_0 so large (depending only from the fixed integers n, g, s) such that the following inequalities are satisfied for all integers $t > k_0$:*

Lemma 5. *For large k_0 (depending only from n, g and s) we have*

$$a(n,g,t) - a(n,g,t-1) \geq 3t + 2n \tag{4}$$

for all $t > k_0$.

Proof. For fixed n and g the function $a(n,g,t)$ increases as $t^{n-1}/n!$ (use (1) and that $0 \leq b(n,g,t) \leq t-1$). Hence $a(n,g,t) - a(n,g,t-1)$ increases as $(n-1)t^{n-2}/n!$. □

Proof of Theorem 1. By [2] we may assume that X is singular. Fix a non-degenerate embedding $W \hookrightarrow \mathbb{P}^n$ of X such that the following properties hold:

(i) $h^1(W, \mathcal{O}_W(1)) = 0$;

(ii) for any $T \in \mathcal{B}(W)$ set $w_T := \deg(T)$; assume $w_T = 1$ if $T \in S(X)$ and $w_T = 2p_a(T) + 2 + 2 \cdot \sharp(\mathrm{Sing}(X) \cap T)$ for all $T \in \mathcal{B}(X) \backslash S(X)$.

Thus there is an upper bound for the integer $x := \deg(W)$ depending only on n, g, s. Hence there is an integer k_0 depending only on n, g and s such that $h^1(\mathbb{P}^n, \mathcal{I}_W(t)) = 0$ for all $t \geq k_0$. Increasing if necessary k_0 we assume that Lemma 2 holds for $m := n-1$ and $x := \deg(W)$ and any positive integer $c \leq 4g + 4s$, i.e. that $k_0(m,x,c) \leq k_0$ for all such c. Fix a general $A \subset \mathbb{P}^m$ such that $\sharp(A) = \binom{n+k_0}{n} - k_0 \cdot \deg(W) + 1 - g$. The generality of A gives $h^i(\mathbb{P}^n, \mathcal{I}_{W \cup A}(k_0)) = 0$, $i = 0, 1$. For all integers $t \geq k_0$ set $a_t := \max\{0, \sharp(A) - t + k_0\}$. For every integer $t \geq k_0 - 1$ we fix a subset $A_t \subseteq A$ such that $\sharp(A_t) = A + k_0 - 2 - t$ if $t \leq k_0 - 1 + \sharp(A)$ and $A_t = \emptyset$ if $t \geq k_0 + \sharp(A)$. Set $A' := A_{k_0}$ and $\{P\} := A \backslash A'$. The generality of A' gives $h^0(\mathbb{P}^n, \mathcal{I}_{W \cup A'}(k_0 - 1)) = 0$. We fix a general hyperplane H of $\mathbb{P}^m$. Thus $W \cap H$ is formed by $\deg(W)$ distinct points. Since $h^0(\mathbb{P}^n, \mathcal{I}_{W \cup A' \cup H}(k_0) = 0$, we get $h^i(\mathbb{P}^m, \mathcal{I}_{W \cup A' \cup \{Q\}}(k_0)) = 0$, $i = 0, 1$, where A'' is a union of A' and a general point of H. For all integer $t \geq k_0$ set $\epsilon(t) = 1$ if $t \leq k_0 + \sharp(A) + 1$ and $\epsilon(t) = 0$ if $t \geq k_0 + \sharp(A) + 2$. For all non-negative integers c, c' such that $c' \leq \deg(W)$ set $u(k_0, c, c') = 0$ and $v(k_0, c, c') = 0$. For all integers $t > k_0$ and all non-negative integers c, c' such that $c' \leq \deg(W)$ and $c + c' > 0$ define the integers $u(t, c, c')$ and $v(t, c, c')$ (depending also from n, g and s) by the relations

$$x - c' + c_{t,c,c'} + \epsilon(t) + t \cdot u(t,c,c') + u(t-1,c,c') = \binom{n+t-1}{n-1},$$
$$0 \leq v(t,c,c') \leq n-1\,. \tag{5}$$

Let $H \subset \mathbb{P}^n$ be a general hyperplane. Thus $W \cap H$ is a union of $x := \deg(W)$ distinct points. We also fix $E_t \subset H \backslash H \cap W$ such that $\sharp(E_t) = \epsilon(t)$. Thus E_t

is a point if $k_0 < t \le k_0(n,g,s) + \sharp(A) + 1$ and $E_t = \emptyset$ if $t \ge k_0 + \sharp(A) + 2$. If $E_t \ne \emptyset$ we fix a 3-dimensional linear subspace M_t of $\mathbb{P}^n$ containing E_t, but not contained in H. Recall that $\mathrm{Res}_H(\chi_{M_t}(E_t)) = \{E_t\}$ (Remark 3).

(a) After fixing the integers g, s we found W. Hence we may assume that $x := \deg(W)$ is a fixed integer. We define the following assertion $A(t,c,c')$ for all integers $t > k_0$ and all integers c, c' such that $c \ge 0$, $0 \le c' \le x$, $0 < c + c' \le 2x$:

$A(t,c,c')$: There exist a finite set $A_t \subset H$, a disjoint union $Y_t \subset H$ of $c + c' + v(t,c,c')$ smooth rational curves such that exactly c' of these curves intersects $W \cap H$, each of them not at a point of a component in B, such that $\sharp(A_t) = u(t-1,c,c')$, $\deg(Y_t) = u(t,c,c')$, $A_t \cap W = A_t \cap E_t = Y_t \cap A_t = Y_t \cap E_t = \emptyset$, and $h^i(H, \mathcal{I}_{W\cap H\cup E_t\cup A_t\cup Y_t}(t)) = 0$, $t = 0,1$.

The equality in (5) shows that in the assertion $A(t,c,c')$ the h^0-vanishing is equivalent to the h^1-vanishing. Since for fixed W, E_t, Y_t we may take A_t general, the assertion $A(t,c,c')$ is equivalent to the existence of Y_t as above such that $h^1(H, \mathcal{I}_{W\cap H\cup E_t\cup Y_t}(t)) = 0$.

We will also use the following assertion $B(t,c,c')$ defined for all integers $t > k_0$ and all integers c, c' such that $c \ge 0$, $0 \le c' \le \deg(W)$, $0 < c + c' \le 2 \cdot \deg(W)$:

$B(t,c,c')$: Fix any integer a such that $1 \le u(t,c,c') - 1 - c_t - c' - v(t,c,c')$. There exists a disjoint union $Y_t \subset H$ of $c_t + c' + v(t,c,c')$ curves such only that one of them is not a line, exactly c' of these curves intersect $W \cap H$, $\deg(Y_t) = u(t,c,c')$, $Y_t \cap E_t = \emptyset$, and $h^1(H, \mathcal{I}_{W\cap H\cup E_t\cup A_t\cup Y_t}(t)) = 0$; the degree > 1 connected component C of Y_t is nodal, with a unique singular point and with two irreducible components, both rational, one of them having degree a.

Notice that the curve C appearing in $B(t,c,c')$ has arithmetic genus zero and that its irreducible components are smooth. We also state the following assertion $D(c,t)$ and $D(c,t)'$ defined for all integers t, c such that $1 \le c \le \sharp(\mathcal{B}(X))$ and all integers $t > k_0 + \sharp(A)$.

$D(c,t)$ Fix integers $a_1, \dots, a_c$ such that $a_i > 0$ for all i and $x + c + t(a_1 + \cdots + a_c) \le \binom{n+t-1}{n-1}$. Then there exists a disjoint union $Y_t \subset H$ of c smooth rational curves of degrees $a_1, \dots, a_c$ such that $Y_t \cap W = \emptyset$ and $h^1(H, \mathcal{I}_{Y_t\cup W\cap H}(t)) = 0$.

$D(c,t)'$ Fix integers $a_1, \dots, a_c$ such that $a_i > 0$ for all i and $x + c + t(a_1 + \cdots + a_c) \le \binom{n+t-1}{n-1}$. Then there exists a disjoint union $Y_t \subset H$ of c smooth rational curves of degrees $a_1, \dots, a_c$ such that $Y_t \cap W = \emptyset$ and $h^1(H, \mathcal{I}_{Y_t\cup W\cap H}(t)) = 0$.

$D(c,t)''$ Fix integers $a_1, \dots, a_c$ such that $a_i > 0$ for all i and $x + c + t(a_1 + \cdots + a_c) \le \binom{n+t-1}{n-1}$. Fix an integer $j \in \{1, \dots, c\}$ such that $a_j \ge n+1$. We asume that j exists. Then there exists a disjoint union $Y_t \subset H$ of c of c curves $U_1, \dots, U_c$, such that:

(i) if $i \ne j$, then U_i is a smooth rational curve of degree a_i;

(ii) U_j is connected and nodal; U_j is the union of a smooth degree $a_j - 2$ rational curve V_j and a reducible conic D_j such that each line of D_j intersects V_j at a unique point;

(iii) $Y_t \cap W = \emptyset$;

(iiv) $h^1(H, \mathcal{I}_{Y_t \cup W \cap H}(t)) = 0$.

(b) Assertions $A(t,c,c')$ and $B(t,c,c')$ easily follow from a very small part of the proofs in [1] (case $n-1 = 3$) or [2] (case $n-1 \geq 4$). To get $D(t,c)$ one also need through the proofs and use that k_0 is large with respect to x. Obviously, $D(t,c)'$ is a particular (and very weak form) of $D(t,c)$. We will not use $D(t,c)$, but only $D(t,c)'$. As in [1] and [2] their proofs use induction on D and in the last inductive step ($t-1 \mapsto t$ if $n-1 \geq 4$, $t-2 \mapsto t$ if $n-1 = 3$ we may also obtain the reducible conic needed to prove $D(c,t)''$. A nodal curve $T \subset \mathbb{P}^m$, $m \geq 3$, is called a *bamboo* of degree δ if its irreducible components are lines and we there is an ordering $T_1, \dots, T_\delta$ of them such that $T_i \cap T_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Any bamboo has arithmetic genus zero. There are solutions for the statement of $A(t,c,c)$, $D(t,c)$, $D(t,c)'$ and $D(t,c)''$ in which we take a bamboo instead of smooth rational curves of the same degree (see [2]), Lemma 5, for the case $n-1 = 3$, while the case $n-1 \geq 4$ is far easier, as explained in [2].

(c) Fix an integer $t > k_0$ and assume the existence of a disjoint union B_{t-1} of smooth rational curves, each of them intersecting W at most at one point and with $W \cup B_t$ nodal such that $h^i(\mathbb{P}^n, \mathcal{I}_{A_{t-1} \cup B_{t-1} \cup W}(t-1)) = 0$, $i = 0,1$. First assume $t \leq k_0 + \sharp(A) + 1$. Fix integers z_i, $1 \leq i \leq c$, such that $z_i \geq 0$ for all i, $z_i \leq \max\{a_i + 1, 1 + \lfloor n a_i/(n-2) \rfloor\}$ for all $i \neq j$, and $z_j \leq \max\{a_j - 1, 1 + \lfloor n(a_j - 2)/(n-2) \rfloor\}$ and $z_1 + \cdots + z_c = z$. ++By semicontinuity and Remark 3 we may assume that, after fixing $W \cap H$, $B_{t-1} \cap H$ is a general subset of $\deg(B_t)$ points of H. Since $h^0(\mathbb{P}^n, \mathcal{I}_{A_{t-2} \cup B_{t-1} \cup W}(t-2)) = 0$, we also see that $h^i(\mathbb{P}^n, \mathcal{I}_{A_{t-2} \cup B_{t-1} \cup W \cup \{Q\}}(t-1)) = 0$, $i = 0,1$, for a general $Q \in H$. Let Y_t be a general curve satisfying $D(t,c)''$. Call $Y[i]$, $1 \leq i \leq c$, the connected components of Y_t with $\deg(Y[i]) = a_i$ for all i. We assume $\sharp(Y[i] \cap B_{t-1}) = z_i$ for all i and that the reducible conic of $Y[j]$ is disjoint from B_{t-1}. For general Y_t and Q we may also assume that Q is the singular point of the reducible conic of Y_t. Fix a 3-dimensional linear subspace M_Q of $\mathbb{P}^n$ containing the reducible conic of Y_t, but not contained in H. We impose that $W \cup B_{t-1} \cup Y_t$ is nodal, and that each connected component of $B_{t-1} \cup Y_t$ has arithmetic genus zero and it intersects W at at most at one point. Hence the connected component of $W \cup B_{t-1} \cup Y_t$ containing W is a specialization of a nodal union of W and some smooth rational curves, each of them intersecting W at a unique point. Define the non-negative integer z by the following equality

$$x + \epsilon(t) + c + t(a_1 + \cdots + a_c) + \deg(B_{t-1}) - z = \binom{n+t-1}{n-1}. \qquad (6)$$

We choose $a_1, \dots, a_c$ maximal so that $0 \le z \le t-1$. We get positive integers a_i and with $a_1 + \cdots + a_c$ large by Lemma 5. Set $U[t] := W \cup A_{t-1} \cup B_{t-1} \cup Y_t \cup \chi_{M_Q}(Q)$. Notice that $\mathrm{Res}_H(U[t]) = W \cup A_{t-1} \cup B_{t-1} \cup \{Q\}$ (Remark 3). Hence $h^i(\mathbb{P}^n, \mathcal{I}_{\mathrm{Res}_H(U[t])}(t-1)) = 0$, $i = 0, 1$. $U[t] \cap H$ is the disjoint union of Y_t, $W \cap H$, and $B_{t-1} \cap H$. Since M_Q contains the reducible conic of Y_t with Q as its singular point, but it is not contained in H, $(Y_t \cup \chi_{M_Q}(Q)) \cap H = Y_t$ (scheme-theoretic intersection). By (6) and $\sharp(B_{t-1} \cap (H \backslash Y_t)) = z$, we have $h^0(U[t] \cap H, \mathcal{O}_{U[t]\cap H}(t)) = \binom{n+t-1}{n-1}$. Hence $h^0(H, \mathcal{I}_{U[t]\cap H}(t)) = 0$ if and only if $h^1(H, \mathcal{I}_{U[t]\cap H}(t)) = 0$. Since (after fixing $W \cap H$ and Y_t we may assume that $B_{t-1} \cap (H \backslash Y_t)$ is general in H and ++, $h^1(H, \mathcal{I}_{U[t]\cap H}(t)) = 0$ if and only if $h^1(H, \mathcal{I}_{W\cap H \cup Y_t}(t)) = 0$. The latter vanishing holds by $D(t,c)''$. Hence Remark 2 gives $h^i(\mathbb{P}^n, \mathcal{I}_{U[t]}(t)) = 0$, $i = 0, 1$. Then we continue in the same way, from postulation in degree t to postulation in degree $t+1$; if $t \le k_0 + \sharp(A)$ we first use $A_t \cup \{Q'\}$, Q' general in H, instead of A_t. If $t \ge k_0 + \sharp(A) + 2$, we make a similar proof, without the point Q and using $D(t,c)'$ instead of $D(t,c)''$. At each step we have a different pair (c, z), but the difference $c - z$ is uniquely determined by other data (see (6). Varying the integers z_i and a_i we get the number of connected components we want and the control the degrees of the rational connected components of $\overline{U[t] \backslash W}$ intersecting each $T \in \mathcal{B}(X)$.

(c) Take integers d_T, $T \in \mathcal{B}(X)$ as in the statement of Theorem 1 and set $\beta := \sum_{T \in \mathcal{B}(X)} d_T$. Let y be the critical value for the triple (n, β, g), i.e. the first integer k such that $\beta \le a(n, g, y)$. In step (2) we got some curve $Y[y-1]$ containing W such that $h^i(\mathbb{P}^n, \mathcal{I}_{U[y-1]}(y-1)) = 0$, $i = 0, 1$. In the step $y-1 \mapsto y$ we wolud obtain a curve $U[y]$ such that $U[y-1] \subset U[y]$. We modify the last step to obtained a connectected curve K of arithmetic genus g, $h^1(K, \mathcal{O}_K(1)) = 0$ and with degree β and with $h^1(\mathbb{P}^n, \mathcal{I}_K(y)) = 0$. Since $U[y-1] \subset K$,we have $h^0(\mathbb{P}^n, \mathcal{I}_K(y-1)) = 0$. Thus K has maximal rank.

(d) Fix an irreducible component T of W such that $w_T \ge 2$. We found Y with good postulation and such that the union Y_T of T and the rational subcurces of Y intersecting T has degree d_t. Remark 5 and Lemma 1 gives that we may smooth Y_T inside $\mathbb{P}^n$ to a degree d_T curve X_T isomorphic to T. By Lemma 3 we may move all other components of W so that they intersects the new curve isomorphic to T at the points of X_T corresponding to the images of $T \cap \overline{W \backslash T}$ by these isomorphisms. Notice that the union Z_T of X_T and the deformations of the other moved components is a curve isomorphic to X. Then we continue in the same way for the other irreducible components of X. Notice that there is no problem for the components C such that $d_C = 1$, because any two points of $\mathbb{P}^n$ are contained in a line. □

$$0 \leq u(t, c, c') \leq (n + t - 1)^{n-1}/n! \tag{7}$$

$$u(t, c, c') \geq (n + t - 1)^{n-2}/((n - 1)!) - (n + t - 2)^{n-2}/n! - 2x - t\,. \tag{8}$$

Proof. Recall that $u(k_0, c, c') = 0$. Fix an integer $t \geq k_0 + 1$. We assume the inequalities in (7) for the integer $t' := t - 1$. Now we use (5). Since $u(t - 1, c, c') \geq 0$, we get the last inequality in (7). Since $u(t - 1, c, c') \leq (n + t - 2)^{n-1}/n!$, we get (8. For $k_0 \gg x$ (8) implies the first inequality in (7). □

Remark 7. Take a connected nodal curve X such that $p_a(X) = 1$. Either X is irreducible or $X = X_1 \cup X_2$ with $\sharp(X_1 \cap X_2) = 2$ or there is a unique $C \in \mathcal{B}(X)$ and an ordering $T_1, \ldots, T_k$ of the other irreducible components of X such that $p_a(C) = 1$, $T_i \cong \mathbb{P}^1$ for all i, $\sharp(C \cap T_1) = 1$ and $\sharp(T_{i+1} \cap (C \cup T_1 \cup \cdots \cup T_i)) = 1$ for all $i \in \{1, \ldots, k - 1\}$. It is trivial to do the proof of Theorem 1 in each of these cases.

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References

[1] E. Ballico, On the postulation of disjoint rational curves in a projective space, *Rend. Sem. Mat. Univers. Politec. Torino*, **44**, No. 2 (1986), 207-249.

[2] E. Ballico, Ph. Ellia, On the postulation of many disjoint rational curves in $\mathbb{P}_N$, $N \geq 4$, *Boll. Un. Mat. Ital. B*, **6**, No. 2 (1985), 585-599.

[3] L. Caporaso, A compactification of the universal Picard variety over the moduli space of stable curves, *J. Amer. Math. Soc.*, **7**, No. 3 (1994), 589-660.

[4] A. Hirschowitz, R. Hartshorne, Droites en position générale dans l'espace projectif, In: *Algebraic Geometry, Proceedings La Rabida* (1981), 169-189; *Lect. Notes in Math.*, **961**, Springer, Berlin (1982).

[5] D. Perrin, Courbes passant par m points g'enéraux de $\mathbb{P}^3$, *Bull. Soc. Math. France*, Mémoire, **28/29** (1987).

[6] G. Sacchiero, Normal bundles of rational curves in projective space, *Ann. Univ. Ferrara Sez.*, **VII**, No. 26 (1980), 33-40.

[7] A. Tannembaum, Deformations of space curves, *Arch. Math.*, Basel, **34** (1980), 37-42.