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ON FACTORIZATION OF q–DIFFERENCE EQUATION FOR CONTINUOUS q–ULTRASPHERICAL POLYNOMIALS

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Abstract: We prove that a customary Sturm-Liouville form of second-order q-difference equation for the continuous q-ultraspherical polynomials $C_n(x;\beta|\,q)$ of Rogers can be written in a factorized form in terms of some explicitly defined q-difference operator $\mathcal{D}_x^{\beta,\,q}$. This reveals the fact that the continuous q-ultraspherical polynomials $C_n(x;\beta|\,q)$ are actually governed by the q-difference equation $\mathcal{D}_x^{\beta,\,q}\,C_n(x;\beta|\,q) = \left(q^{-n/2} + \beta\,q^{n/2}\right)\,C_n(x;\beta|\,q)$, which can be regarded as a square root of the equation, obtained from its original form.

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1. Introduction

It is well known that for many purposes it proves practical, as in the case of linear second-order ordinary differential equations, to represent the difference

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equation of hypergeometric type for classical orthogonal polynomials in Sturm-Liouville (or self-adjoint) form [13]

$$\frac{\Delta}{\Delta x(s-1/2)}\left[\sigma(s)\,\rho(s)\,\frac{\nabla f(s)}{\nabla x(s)}\right] + \lambda\,\rho(s)\,f(s) = 0\,, \tag{1.1}$$

where $\Delta\, y(s) := y(s+1) - y(s)$ and $\nabla\, y(s) := y(s) - y(s-1)$ (we employ standard notations of the theory of special functions, see, for example, [7] or [2]).

The important feature of this form (1.1) is that it requires the introduction of a function $\rho(s)$ through the Pearson-type difference equation

$$\frac{\Delta}{\Delta x(s-1/2)}\left[\,\sigma(s)\,\rho(s)\,\right] = \tau(s)\,\rho(s)\,, \tag{1.2}$$

with polynomials $\sigma(s)$ and $\tau(s)$ of respective degrees at most two and one, which characterize an original form of the difference equation (1.1). The full importance of the self-adjoint form (1.1) becomes apparent when one takes into account that the same function $\rho(s)$ enables one to formulate the orthogonality property of solutions of equation (1.1). Moreover, one can construct explicit representation ([13], p. 66)

$$f_n(s) := \frac{B_n}{\rho(s)}\,\frac{\nabla}{\nabla\, x_1(s)}\,\cdots\,\frac{\nabla}{\nabla\, x_{n-1}(s)}\,\frac{\nabla}{\nabla\, x_n(s)}\,\left[\rho(s+n)\,\prod_{k=1}^{n}\sigma(s+k)\right],$$

in terms of the function $\rho(s)$ for the polynomial solutions $f_n(s)$ of equation (1.1), which correspond to the values $\lambda_n := -n\,\tau' - n(n-1)\,\sigma''/2$ of the parameter λ (for a more detailed discussion of this topic, see [13]).

An example to illustrate this point is provided by the continuous q-Hermite polynomials of Rogers,

$$H_n(x|\,q) := \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q e^{\mathrm{i}(n-2k)\,\theta}\,, \qquad 0 < q < 1\,, \tag{1.3}$$

which are orthogonal on the finite interval $-1 \le x := \cos\theta \le 1$ with respect to the weight function

$$\widetilde{w}(x|\,q) := \frac{1}{\sin\theta}\,\left(e^{2\mathrm{i}\theta},\, e^{-2\mathrm{i}\theta}; q\right)_\infty\,. \tag{1.4}$$

These polynomials $H_n(x|\,q)$ satisfy the following q-difference equation

$$D_q\,\left[\widetilde{w}(x|\,q)\,D_q\,H_n(x|\,q)\right] = \frac{4\,q\,(1-q^{-n})}{(1-q)^2}\,H_n(x|\,q)\,\widetilde{w}(x|\,q)\,, \tag{1.5}$$

written in self-adjoint form (1.1) (see [10], p. 115). The symbol D_q in (1.5) is the conventional notation for the Askey-Wilson divided-difference operator (see, for example, [2], p. 529), defined as

$$D_q\, f(x) := \frac{\delta_q\, f(x)}{\delta_q\, x}\,, \quad \delta_q\, g(e^{\mathrm{i}\,\theta}) := g(q^{1/2}\, e^{\mathrm{i}\,\theta}) - g(q^{-1/2}\, e^{\mathrm{i}\,\theta})\,, \quad x = \cos\theta\,. \tag{1.6}$$

As was observed in [4], one may eliminate the weight function $\widetilde{w}(x|\,q)$ from (1.5) by utilizing its readily verified property that

$$\exp\left(\pm\,\mathrm{i}\ln q^{1/2}\,\partial_\theta\right)\widetilde{w}(x|\,q) = -\frac{e^{\pm 2\mathrm{i}\theta}}{\sqrt{q}}\,\widetilde{w}(x|\,q)\,. \tag{1.7}$$

It should be noted that following [4] we find it more convenient to write (1.7) (and subsequent q-difference equations) in terms of the shift operators (or the operators of the finite displacement, [12]) $e^{\pm a\,\partial_\theta}\, g(\theta) := g(\theta \pm a)$ with respect to the variable θ.

This elimination of the weight function $\widetilde{w}(x|\,q)$ from (1.5) yields the following q-difference equation

$$\frac{1}{2\mathrm{i}\sin\theta}\left[\frac{e^{\mathrm{i}\theta}}{1-q\,e^{-2\mathrm{i}\theta}}\left(e^{\mathrm{i}\ln q\,\partial_\theta}-1\right)+\frac{e^{-\mathrm{i}\theta}}{1-q\,e^{2\mathrm{i}\theta}}\left(1-e^{-\mathrm{i}\ln q\,\partial_\theta}\right)\right]H_n(x|\,q)$$
$$= \left(q^{-n}-1\right)\,H_n(x|\,q) \tag{1.8}$$

for the continuous q-Hermite polynomials $H_n(x|\,q)$. The resultant q-difference equation (1.8) then admits factorization of the form

$$(\mathcal{D}_x^q)^2\, H_n(x|\,q) = q^{-n}\, H_n(x|\,q)\,, \tag{1.9}$$

where the q-difference operator $\mathcal{D}_x^q$ is equal to

$$\mathcal{D}_x^q := \frac{1}{1-e^{-2\mathrm{i}\theta}}\;e^{\mathrm{i}\,\ln q^{1/2}\,\partial_\theta} + \frac{1}{1-e^{2\mathrm{i}\theta}}\;e^{-\mathrm{i}\,\ln q^{1/2}\,\partial_\theta}$$
$$\equiv \frac{1}{2\mathrm{i}\sin\theta}\left(e^{\mathrm{i}\theta}\,e^{\mathrm{i}\,\ln q^{1/2}\,\partial_\theta} - e^{-\mathrm{i}\theta}\,e^{-\mathrm{i}\,\ln q^{1/2}\,\partial_\theta}\right)\,, \quad x=\cos\theta\,. \tag{1.10}$$

This means that the continuous q-Hermite polynomials are in fact governed by a simpler q-difference equation,

$$\mathcal{D}_x^q H_n(x|\,q) = q^{-n/2}\, H_n(x|\,q)\,, \tag{1.11}$$

which represents a "square root" of (1.8) or (1.9).

This curious interrelation between two q-difference equations (1.5) and (1.11), studied in detail in [4], leads to the natural question whether the continuous q-Hermite polynomials $H_n(x|\,q)$ represent the exceptional case or there exist other instances of orthogonal polynomials from the Askey q-scheme [10], which admit the same type of factorization in corresponding q-difference equations for them.

The present paper is aimed at proving that the continuous q-ultraspherical

(Rogers) polynomials $C_n(x;\beta|\,q)$ exhibit the same property of factorization as the continuous q-Hermite polynomials $H_n(x|\,q)$. The next section collects those known facts about the q-ultraspherical polynomials $C_n(x;\beta|\,q)$ and their $q\to 1$ limit counterpart, the Gegenbauer (ultraspherical) polynomials $C_n^{(\gamma)}(x)$, which are needed in Section 3 for proving that a q-difference equation for the $C_n(x;\beta|\,q)$, derived from its appropriate self-adjoint form like (1.1), does admit a factorization of the type (1.9). In the concluding Section 4 we briefly discuss some special and limit cases of the parameter β, which are related with other well-known families of q-polynomials.

2. Rogers and Gegenbauer Polynomials

To proceed further we need to recall in this section some standard facts about continuous q-ultraspherical (Rogers) polynomials and their $q\to 1$ limit counterpart, Gegenbauer (ultraspherical) polynomials. The continuous q-ultraspherical polynomials

$$C_n(x;\beta|\,q) := \sum_{k=0}^{n} \frac{(\beta;q)_k\,(\beta;q)_{n-k}}{(q;q)_k\,(q;q)_{n-k}}\, e^{\mathrm{i}(n-2k)\,\theta}\,, \qquad 0<q<1\,, \tag{2.1}$$

are known to be orthogonal on the finite interval $-1\le x:=\cos\theta\le 1$,

$$\frac{1}{2\pi}\int_{-1}^{1} C_m(x;\beta|\,q)\,C_n(x;\beta|\,q)\,\widetilde{w}(x;\beta|\,q)\,dx \;=\; d_n^{-1}(\beta;q)\,\delta_{mn}\,,$$

$$d_n(\beta;q) := \frac{(1-\beta q^n)}{(1-\beta)}\,\frac{(q;q)_n}{(\beta^2;q)_n}\,\frac{(\beta^2,q;q)_\infty}{(\beta,\beta q;q)_\infty}\,, \qquad |\beta|<1\,, \tag{2.2}$$

with respect to the weight function (see, for example, [10], p. 86)

$$\widetilde{w}(x;\beta|\,q) := \frac{1}{\sin\theta}\,\frac{\left(e^{2\mathrm{i}\theta},\,e^{-2\mathrm{i}\theta};q\right)_\infty}{\left(\beta\,e^{2\mathrm{i}\theta},\,\beta\,e^{-2\mathrm{i}\theta};q\right)_\infty}\,. \tag{2.3}$$

They satisfy the Sturm-Liouville type q-difference equation

$$D_q\,\left[\widetilde{w}(x;\beta q|\,q)\,D_q\,C_n(x;\beta|\,q)\right] = \lambda_n(\beta)\,C_n(x;\beta|\,q)\,\widetilde{w}(x;\beta|\,q) \tag{2.4}$$

with eigenvalues $\lambda_n(\beta) := 4\,q\,(1-q^{-n})\,(1-\beta^2q^n)/(1-q)^2$ (see, for example, [10], p. 86). Note that the D_q in (2.4) is the same Askey–Wilson divided-difference operator, defined above in (1.6), namely,

$$D_q = \frac{\sqrt{q}}{\mathrm{i}(1-q)}\,\frac{1}{\sin\theta}\left(e^{\,\mathrm{i}\ln q^{1/2}\,\partial_\theta} - e^{-\mathrm{i}\ln q^{1/2}\,\partial_\theta}\right), \qquad \partial_\theta \equiv \frac{d}{d\theta}\,. \tag{2.5}$$

Observe also that one readily derives from definition (2.3) the relation

$$\widetilde{w}(x;\beta q|\,q) = \left[(1+\beta)^2 - 4\beta x^2\right]\widetilde{w}(x;\beta|\,q) \tag{2.6}$$

between the weight functions $\widetilde{w}(x;\beta|\,q)$ with the two distinct parameters β and βq. Therefore a q-analogue of the factor $\sigma(s)$ from the self-adjoint equation (1.1) in the case of the q-difference equation (2.4) is just

$$\sigma_q(x;\beta) := (1+\beta)^2 - 4\beta x^2\,.$$

If one sets $\beta = q^\gamma$ in (2.1) and then evaluates its limit as $q \to 1$, this results in

$$\lim_{q\to 1} C_n(x;q^\gamma\,|\,q) = C_n^{(\gamma)}(x)\,,$$

where $C_n^{(\gamma)}(x)$ are the Gegenbauer polynomials:

$$C_n^{(\gamma)}(x) := \sum_{k=0}^{n} \frac{(\gamma)_k\,(\gamma)_{n-k}}{k!\,(n-k)!}\; e^{\mathrm{i}(n-2k)\,\theta}\,, \qquad x=\cos\theta\,. \tag{2.7}$$

The self-adjoint form of the second-order differential equation for the Gegenbauer polynomials (2.7) is known to be of the form

$$\frac{d}{dx}\left[(1-x^2)\,w(x)\,\frac{d\,C_n^{(\gamma)}(x)}{dx}\right] + n(n+2\gamma)\,w(x)\,C_n^{(\gamma)}(x) = 0, \tag{2.8}$$

where $w(x) := (1-x^2)^{\gamma-1/2}$, $\gamma > -1/2$, is the orthogonality weight function for the $C_n^{(\gamma)}(x)$ on the finite interval $-1 \le x \le 1$. After eliminating the weight function $w(x)$ from (2.8), one can rewrite it as

$$\left[(1-x^2)\,\frac{d^2}{dx^2} - (2\,\gamma+1)\,x\,\frac{d}{dx} + n(n+2\gamma)\right] C_n^{(\gamma)}(x) = 0\,. \tag{2.9}$$

In contrast to (2.8), this differential equation is evidently not self-adjoint; but to transform it into self-adjoint equation (2.8) one needs only to multiply it by $w(x)$ from the left and employ the readily verified identity

$$w(x)\left[(1-x^2)\,\frac{d^2}{dx^2} - (2\,\gamma+1)\,x\,\frac{d}{dx}\right] \;=\; \frac{d}{dx}\left[(1-x^2)\,w(x)\,\frac{d}{dx}\right]. \tag{2.10}$$

3. Factorization for Rogers Polynomials

To eliminate the weight function $\widetilde{w}(x;\beta|\,q)$ from q-difference equation (2.4), we employ first the relations

$$\exp\left(\pm\,\mathrm{i}\ln q^{1/2}\,\partial_\theta\right)\widetilde{w}(x;\beta q|\,q) = \frac{1}{\sqrt{q}}\,\left(1-\beta\,e^{\mp\,2\mathrm{i}\theta}\right)\,\left(\beta q - e^{\pm\,2\mathrm{i}\theta}\right)\,\widetilde{w}(x;\beta|\,q)\,, \tag{3.1}$$

which are straightforward to derive upon using the explicit expression (2.3) for $\widetilde{w}(x;\beta|\,q)$ and relation (2.6). Substituting (3.1) into (2.4), one obtains the q-difference equation

$$\frac{1}{\mathrm{i}\sin\theta}\left[e^{\mathrm{i}\theta}\,\frac{\left(1-\beta\,e^{-2\mathrm{i}\theta}\right)\,\left(1-\beta q\,e^{-2\mathrm{i}\theta}\right)}{1-q\,e^{-2\mathrm{i}\theta}}\,\left(e^{\mathrm{i}\ln q\,\partial_\theta}-1\right)\right.$$
$$\left.+\,e^{-\mathrm{i}\theta}\,\frac{\left(1-\beta\,e^{2\mathrm{i}\theta}\right)\,\left(1-\beta q\,e^{2\mathrm{i}\theta}\right)}{1-q\,e^{2\mathrm{i}\theta}}\,\left(1-e^{-\mathrm{i}\ln q\,\partial_\theta}\right)\right]\,C_n(x;\beta|\,q)$$
$$=2\,\left(q^{-n}-1\right)\,\left(1-\beta^2\,q^n\right)\,C_n(x;\beta|\,q) \tag{3.2}$$

for the q-ultraspherical polynomials $C_n(x;\beta|\,q)$, which does not contain the weight function $\widetilde{w}(x;\beta|\,q)$. This equation is a q-extension of the second-order differential equation (2.9) for the Gegenbauer polynomials $C_n^{(\gamma)}(x)$.

The next step is to use two simple trigonometric identities

$$\frac{e^{\pm\,\mathrm{i}\theta}}{\mathrm{i}\sin\theta}=\pm\,\frac{2}{1-e^{\mp\,2\mathrm{i}\theta}}$$

in order to write a q-difference operator on the left side of equation (3.2) as

$$2\left[\frac{\left(1-\beta\,e^{-2\mathrm{i}\theta}\right)\left(1-\beta q\,e^{-2\mathrm{i}\theta}\right)}{\left(1-\,e^{-2\mathrm{i}\theta}\right)\left(1-q\,e^{-2\mathrm{i}\theta}\right)}\,\left(e^{\mathrm{i}\ln q\,\partial_\theta}-1\right)\right.$$
$$\left.-\frac{\left(1-\beta e^{2\mathrm{i}\theta}\right)\left(1-\beta q e^{2\mathrm{i}\theta}\right)}{\left(1-e^{2\mathrm{i}\theta}\right)\left(1-q\,e^{2\mathrm{i}\theta}\right)}\,\left(1-e^{-\mathrm{i}\ln q\,\partial_\theta}\right)\right]$$
$$=2\left[\frac{\left(1-\beta\,e^{-2\mathrm{i}\theta}\right)\left(1-\beta\,q\,e^{-2\mathrm{i}\theta}\right)}{\left(1-\,e^{-2\mathrm{i}\theta}\right)\left(1-q\,e^{-2\mathrm{i}\theta}\right)}\,e^{\mathrm{i}\ln q\,\partial_\theta}+\frac{\left(1-\beta\,e^{2\mathrm{i}\theta}\right)\left(1-\beta\,q\,e^{2\mathrm{i}\theta}\right)}{\left(1-e^{2\mathrm{i}\theta}\right)\left(1-q\,e^{2\mathrm{i}\theta}\right)}\,e^{-\mathrm{i}\ln q\,\partial_\theta}\right.$$
$$\left.-\frac{\left(1-\beta\,e^{-2\mathrm{i}\theta}\right)\left(1-\beta\,q\,e^{-2\mathrm{i}\theta}\right)}{\left(1-\,e^{-2\mathrm{i}\theta}\right)\left(1-q\,e^{-2\mathrm{i}\theta}\right)}-\frac{\left(1-\beta\,e^{2\mathrm{i}\theta}\right)\left(1-\beta\,q\,e^{2\mathrm{i}\theta}\right)}{\left(1-e^{2\mathrm{i}\theta}\right)\left(1-q\,e^{2\mathrm{i}\theta}\right)}\right]. \tag{3.3}$$

The last important step is to employ a readily verified identity

$$\frac{1-\beta\,q\,e^{\mp\,2\mathrm{i}\theta}}{1-q\,e^{\mp\,2\mathrm{i}\theta}}\,e^{\pm\,\mathrm{i}\,\ln q^{1/2}\,\partial_\theta}=e^{\pm\,\mathrm{i}\,\ln q^{1/2}\,\partial_\theta}\,\frac{1-\beta\,e^{\mp\,2\mathrm{i}\theta}}{1-e^{\mp\,2\mathrm{i}\theta}} \tag{3.4}$$

for the shift operators $\exp\left(\pm\,\mathrm{i}\,\ln q^{1/2}\,\partial_\theta\right)$, which enter into first two terms in (3.3). With the aid of (3.4) one can thus cast (3.3) into the form

$$2\left[\frac{1-\beta\,e^{-2\mathrm{i}\theta}}{1-\,e^{-2\mathrm{i}\theta}}\,e^{\mathrm{i}\ln q^{1/2}\,\partial_\theta}\,\frac{1-\beta\,e^{-2\mathrm{i}\theta}}{1-\,e^{-2\mathrm{i}\theta}}\,e^{\mathrm{i}\ln q^{1/2}\,\partial_\theta}\right.$$

$$+\,\frac{1-\beta\,e^{2\mathrm{i}\theta}}{1-\,e^{2\mathrm{i}\theta}}\,e^{-\mathrm{i}\ln q^{1/2}\,\partial_\theta}\,\frac{1-\beta\,e^{2\mathrm{i}\theta}}{1-\,e^{2\mathrm{i}\theta}}\,e^{-\mathrm{i}\ln q^{1/2}\,\partial_\theta}$$

$$-\,\frac{\left(1-\beta\,e^{-2\mathrm{i}\theta}\right)\left(1-\beta q\,e^{-2\mathrm{i}\theta}\right)}{\left(1-\,e^{-2\mathrm{i}\theta}\right)\left(1-q\,e^{-2\mathrm{i}\theta}\right)}\,-\,\frac{\left(1-\beta\,e^{2\mathrm{i}\theta}\right)\left(1-\beta q\,e^{2\mathrm{i}\theta}\right)}{\left(1-\,e^{2\mathrm{i}\theta}\right)\left(1-q\,e^{2\mathrm{i}\theta}\right)}\Bigg]$$

$$=2\left[\frac{1-\beta\,e^{-2\mathrm{i}\theta}}{1-\,e^{-2\mathrm{i}\theta}}\,e^{\mathrm{i}\ln q^{1/2}\,\partial_\theta}\,\frac{1-\beta\,e^{-2\mathrm{i}\theta}}{1-\,e^{-2\mathrm{i}\theta}}\,e^{\mathrm{i}\ln q^{1/2}\,\partial_\theta}\,-\,\frac{(1+q)(1-\beta)(\beta-q)}{(1+q)^2-4qx^2}\right.$$

$$\left.+\,\frac{1-\beta\,e^{2\mathrm{i}\theta}}{1-\,e^{2\mathrm{i}\theta}}\,e^{-\mathrm{i}\ln q^{1/2}\,\partial_\theta}\,\frac{1-\beta\,e^{2\mathrm{i}\theta}}{1-\,e^{2\mathrm{i}\theta}}\,e^{-\mathrm{i}\ln q^{1/2}\,\partial_\theta}\,-1-\beta^2\right]. \qquad (3.5)$$

It is not hard to verify now that the above expression represents a product of two q-difference operators,

$$2\left[\left(\mathcal{D}_x^{\beta,\,q}\right)^2-(1+\beta)^2\right]=2\,\left(\mathcal{D}_x^{\beta,\,q}+1+\beta\right)\,\left(\mathcal{D}_x^{\beta,\,q}-1-\beta\right)\,,$$

where $\mathcal{D}_x^{\beta,\,q}$ is equal to (cf. (1.10))

$$\mathcal{D}_x^{\beta,\,q}:=\frac{1-\beta\,e^{-2\mathrm{i}\theta}}{1-\,e^{-2\mathrm{i}\theta}}\,e^{\mathrm{i}\ln q^{1/2}\,\partial_\theta}+\frac{1-\beta\,e^{2\mathrm{i}\theta}}{1-\,e^{2\mathrm{i}\theta}}\,e^{-\mathrm{i}\ln q^{1/2}\,\partial_\theta}\equiv\mathcal{D}_x^{q}\,+\,\beta\,\mathcal{D}_x^{1/q}$$

$$\equiv\frac{1}{2\mathrm{i}\sin\theta}\,\left[\left(e^{\mathrm{i}\theta}-\beta\,e^{-\mathrm{i}\theta}\right)\,e^{\mathrm{i}\ln q^{1/2}\,\partial_\theta}-\left(e^{-\mathrm{i}\theta}-\beta\,e^{\mathrm{i}\theta}\right)\,e^{-\mathrm{i}\ln q^{1/2}\,\partial_\theta}\right]. \qquad (3.6)$$

Finally, taking into account that the factor $(q^{-n}-1)\left(1-\beta^2\,q^n\right)$ on the right side of (3.2) can be written as $\left(q^{-n/2}+\beta\,q^{n/2}\right)^2-(1+\beta)^2$, one arrives at the following factorized form of equation (3.2):

$$\left(\mathcal{D}_x^{\beta,\,q}\right)^2 C_n(x;\beta|\,q)=\left(q^{-n/2}+\beta\,q^{n/2}\right)^2 C_n(x;\beta|\,q)\,. \qquad (3.7)$$

Note that the operator $(\mathcal{D}_x^{\beta,\,q})^2$ represents, as equation (3.7) implies, an unbounded operator on the Hilbert space $L^2(S^1)$ with the scalar product

$$\langle g_1,g_2\rangle=\frac{1}{2\pi}\int_{-1}^{1} g_1(x)\,\overline{g_2(x)}\,\widetilde{w}(x;\beta|\,q)\,dx\,, \qquad (3.8)$$

where the weight function $\widetilde{w}(x;\beta|\,q)$ is defined by (2.3). In view of (2.2) the polynomials $p_n(x):=d_n^{1/2}(\beta;q)\,C_n(x;\beta|\,q)$, $n=0,1,2,\cdots$, constitute an orthonormal basis in this space such that $(\mathcal{D}_x^{q})^2\,p_n(x)=\left(q^{-n/2}+\beta\,q^{n/2}\right)^2 p_n(x)$. In particular, the operator $(\mathcal{D}_x^{\beta,\,q})^2$ is defined on the linear span $\mathcal{H}$ of the basis functions $p_n(x)$, which is everywhere dense in $L^2(S^1)$. We close $\left(\mathcal{D}_x^{\beta,\,q}\right)^2$ with respect to the scalar product (3.8). Since $\left(\mathcal{D}_x^{\beta,\,q}\right)^2$ is diagonal with respect to the orthonormal basis $p_n(x)$, $n = 0, 1, 2, \cdots$, its closure $\overline{\left(\mathcal{D}_x^{\beta,\,q}\right)^2}$ is a self-adjoint operator, which coincides on $\mathcal{H}$ with $\left(\mathcal{D}_x^{\beta,\,q}\right)^2$. According to the theory of self-adjoint operators (see [1], Chapter 6), we can take a square root of the operator $\overline{\left(\mathcal{D}_x^{\beta,\,q}\right)^2}$. This square root is a self-adjoint operator too and has the same eigenfunctions as the operator $\overline{\left(\mathcal{D}_x^{\beta,\,q}\right)^2}$ does. We denote this operator by $\overline{\mathcal{D}_x^{\beta,\,q}}$. It is evident that on the subspace $\mathcal{H}$ the operator $\overline{\mathcal{D}_x^{\beta,\,q}}$ coincides with the $\mathcal{D}_x^{\beta,\,q}$. That is, the $\mathcal{D}_x^{\beta,\,q}$ is a well-defined operator on the Hilbert space $L^2(S_1)$ with everywhere dense subspace of definition. Moreover, according to the definition of a function of a self-adjoint operator (see [1], Chapter 6), we have $\overline{\mathcal{D}_x^{\beta,\,q}}\, p_n(x) = \left(q^{-n/2} + \beta\, q^{n/2}\right)\, p_n(x)$. This means that the continuous q-ultraspherical polynomials $C_n(x;\beta|\,q)$ are in fact governed by a simpler q-difference equation,

$$\mathcal{D}_x^{\beta,\,q}\, C_n(x;\beta|\,q) = \left(q^{-n/2} + \beta\, q^{n/2}\right)\, C_n(x;\beta|\,q)\,, \tag{3.9}$$

which can be regarded as a "square root" of (3.7).

Observe that the q-difference operator $\mathcal{D}_x^{\beta,\,q}$ in (3.9) may be expressed in terms of the Askey-Wilson divided-difference operator D_q, defined in (1.6), as

$$\mathcal{D}_x^{\beta,\,q} = (1+\beta)\,\mathcal{A}_q + \frac{1-q}{2\sqrt{q}}\,(1-\beta)\,x\,D_q\,, \tag{3.10}$$

where the $\mathcal{A}_q$ is so-called *averaging difference operator*, that is (see, for example [8]),

$$\left(\mathcal{A}_q f\right)(x) = \frac{1}{2}\left(e^{\mathrm{i}\ln q^{1/2}\partial_\theta} + e^{-\mathrm{i}\ln q^{1/2}\partial_\theta}\right) f(x) \equiv \cos\left(\ln q^{1/2}\partial_\theta\right) f(x). \tag{3.11}$$

We emphasize that q-difference equation (3.8) is consistent with the generating function

$$\sum_{n=0}^{\infty} t^n\, C_n(x;\beta|\,q) = \frac{\left(\beta t\, e^{\mathrm{i}\theta}, \beta t\, e^{-\mathrm{i}\theta}; q\right)_\infty}{\left(t\, e^{\mathrm{i}\theta}, t\, e^{-\mathrm{i}\theta}; q\right)_\infty} \tag{3.12}$$

for the continuous q-ultraspherical polynomials $C_n(x;\beta|\,q)$ (see [7], p. 169). Indeed, apply the q-difference operator $\mathcal{D}_x^{\beta,\,q}$ to both sides of (3.12) to verify that

$$\sum_{n=0}^{\infty} t^n\, \mathcal{D}_x^{\beta,\,q}\, C_n(x;\beta|\,q) = \mathcal{D}_x^{\beta,\,q}\, \frac{\left(\beta t\, e^{\mathrm{i}\theta}, \beta t\, e^{-\mathrm{i}\theta}; q\right)_\infty}{\left(t\, e^{\mathrm{i}\theta}, t\, e^{-\mathrm{i}\theta}; q\right)_\infty}$$

$$= \frac{\left(q^{-1/2}\,\beta t\,e^{\mathrm{i}\theta},\, q^{-1/2}\,\beta t\,e^{-\mathrm{i}\theta}; q\right)_\infty}{\left(q^{-1/2}\,t\,e^{\mathrm{i}\theta},\, q^{-1/2}\,t\,e^{-\mathrm{i}\theta}; q\right)_\infty} + \beta\,\frac{\left(q^{1/2}\,\beta t\,e^{\mathrm{i}\theta},\, q^{1/2}\,\beta t\,e^{-\mathrm{i}\theta}; q\right)_\infty}{\left(q^{1/2}\,t\,e^{\mathrm{i}\theta},\, q^{1/2}\,t\,e^{-\mathrm{i}\theta}; q\right)_\infty}$$

$$= \sum_{n=0}^{\infty} \left(q^{-n/2} + \beta\,q^{n/2}\right) t^n\,C_n(x;\beta|\,q)\,. \tag{3.13}$$

Equating coefficients of like powers of t on the extremal sides of (3.13), one completes the another proof of equation (3.9).

As it was recalled in Section 2, if $\beta = q^\gamma$, then the q-ultraspherical polynomials $C_n(x;q^\gamma|\,q)$ reduce to the Gegenbauer polynomials $C_n^{(\gamma)}(x)$ in the limit as $q \to 1$. This fact can be also expressed as the following limit property of the q-difference operator $\mathcal{D}_x^{\beta,\,q}$ in (3.6):

$$\lim_{q\to 1}\left\{\frac{1}{(\ln q)^2}\,\left[(1+q^\gamma)I - \mathcal{D}_x^{q^\gamma,\,q}\right]\right\} = \frac{1}{4}\,\left[(1-x^2)\frac{d^2}{dx^2} - (2\gamma+1)x\frac{d}{dx}\right],$$

where I is the identity operator.

Observe also that the q-ultraspherical polynomials $C_n(x;\beta|\,q)$ are known to possess the simple transformation property

$$C_n(x;\beta|\,q^{-1}) = (\beta q)^n\,C_n(x;\beta^{-1}|\,q) \tag{3.14}$$

with respect to the changes $q \to q^{-1}$ and $\beta \to \beta^{-1}$ (see [10], p. 88). It is not hard to check that q-difference equation (3.9) agrees with this property (3.14) since by definition (3.6)

$$\mathcal{D}_x^{\beta,\,q} \equiv \beta\,\mathcal{D}_x^{\beta^{-1},\,q^{-1}}\,.$$

We close this section with the following remark about equation (3.9). Koornwinder has recently examined raising and lowering relations for the Askey–Wilson polynomials $p_n(x;a,b,c,d|\,q)$ [11], which are known to reduce to the continuous q-ultraspherical polynomials $C_n(x;\beta|\,q)$, when one specializes the parameters a,b,c,d as $a = -\,c = \sqrt{\beta}$ and $b = -\,d = \sqrt{q\,\beta}$. So equation (3.9) coincides with "the second order q-difference formula" (6.10) in Koornwinder's paper [11], upon taking into account that variables z and t in (6.10) are equal to $e^{\mathrm{i}\theta}$ and β, respectively, in our notations. But [11] does not contain any hint about the interrelation between equation (3.9) and its commonly used counterpart (2.4), and that has been our main goal in the present work.

4. Special and Limit Cases of Parameter β

The q-difference equation (3.9) for the q-ultraspherical polynomials, derived in the previous section, does actually contain some special and limit cases of the parameter β, which correspond to other well-known families of q-polynomials. We recall (see, for example, [10], p. 88) that in the case when $\beta = q^{\alpha+1/2}$ the q-ultraspherical polynomials $C_n(x; q^{\alpha+1/2}|\, q)$ reduce to (up to a normalization factor) the continuous q-Jacobi polynomials $P_n^{(\alpha,\alpha)}(x|\, q)$; when $\beta = q^{1/2}$ the $C_n(x; q^{1/2}|\, q)$ are related to the continuous q-Legendre polynomials $P_n(x|\, q)$; and when $\beta = q$ the q-ultraspherical polynomials $C_n(x; q|\, q)$ embrace the Chebyshev polynomials of the second kind $U_n(x)$.

There is also the limit case $\beta \to 1$, which leads to the Chebyshev polynomials of the first kind $T_n(x)$ in the following way:

$$\lim_{\beta\to 1} \frac{1-q^n}{2(1-\beta)}\, C_n(x;\beta|\, q) = T_n(x) \equiv \cos n\theta\,, \quad n = 1, 2, 3, \ldots .$$

But the point is that q-difference equation (3.9) in this limit reduces to the difference equation

$$\left[e^{\mathrm{i}\ln q^{1/2}\partial_\theta} + e^{-\mathrm{i}\ln q^{1/2}\partial_\theta} \right] T_n(x) = \left(q^{n/2} + q^{-n/2} \right) T_n(x)\,, \tag{4.1}$$

although we all know well that the Chebyshev polynomials of the first kind $T_n(x)$ satisfy the second-order differential equation

$$\left[(1-x^2)\,\frac{d^2}{dx^2} - x\,\frac{d}{dx} + n^2 \right] T_n(x) = 0\,.$$

Nevertheless, there is no contradiction here since one readily verifies that the Chebyshev polynomials of the first kind $T_n(x) = \cos n\theta$, $n = 0, 1, 2, \ldots$, do satisfy difference equation (4.1) as well.

5. Concluding Remarks

To summarize, we have proved that the conventional q-difference equation (2.4) of Sturm-Liouville type for the continuous q-ultraspherical polynomials $C_n(x;\beta|\, q)$ of Rogers admits factorization of the form (3.9). The special case of the $C_n(x;\beta|\, q)$ with the vanishing parameter β is known to correspond to the continuous q-Hermite polynomials $H_n(x|\, q)$. The above-presented formulas in this case when $\beta = 0$ are in accord with that obtained by M. Atakishiyev and A. Klimyk in [4]. So it would be of considerable interest to explore now whether the situation here described obtains for other families of orthogonal polynomials on higher levels in the Askey q-scheme [10]. Work on clarifying this point is in progress.

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