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A REMARK ON A SEMI-LINEAR ELLIPTIC PROBLEM WITH THE DE GENNES BOUNDARY CONDITION ASSOCIATED WITH SUPERCONDUCTIVITY

Junichi Aramaki

Department of Mathematical Sciences
Faculty of Science and Engineering
Tokyo Denki University
Hatoyama-machi, Hiki-Gun, Saitama, 350-0394, JAPAN
e-mail: aramaki@r.dendai.ac.jp

Abstract: We shall show that the results of Lu and Pan [8] also hold for the equation with more general non-linearity associated with the Ginzburg-Landau system. We impose the de Gennes boundary condition and show that some of the results hold in a domain in $\mathbb{R}^n$, where n may be general.

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1. Introduction

In the present paper, we study a semi-elliptic equation under the de Gennes boundary condition.

Lu and Pan [8] considered the following semi-linear elliptic system:

$$\begin{cases} -\Delta u = \frac{1}{\varepsilon^2}(1-u^2)u & \text{in} \quad \Omega, \\ \frac{\partial}{\partial \boldsymbol{\nu}} u + \gamma u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Δ is the Laplacian operator, Ω is a bounded, smooth domain in $\mathbb{R}^2$, $\boldsymbol{\nu}$ the unit out-normal vector at the boundary $\partial\Omega$, $\varepsilon > 0$ a parameter, and γ a positive constant which is called de Gennes parameter (cf. Lu and Pan [9]).

This problem is closely related to the Ginzburg-Landau equations with de

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Gennes boundary condition (cf. [8] and Bethuel et al [4]). The energy functional associated with (1.1) is

$$\mathcal{J}[\phi] = \int_\Omega \{|\nabla\phi|^2 + \frac{1}{2\varepsilon^2}(1-|\phi|^2)^2\}dx + \gamma\int_{\partial\Omega}|\phi|^2 ds\,, \tag{1.2}$$

where ds is the line element of $\partial\Omega$. We denote

$$C_0(\varepsilon) = \inf_{\phi\in W^{1,2}(\Omega)} \mathcal{J}[\phi]\,, \tag{1.3}$$

where $W^{1,2}(\Omega)$ denotes the Sobolev space. It is well known that (1.3) is achieved by some $\phi \in W^{1,2}(\Omega)$ and then ϕ satisfies the Euler equations (1.1) weakly.

In order to state the results of [8] more precisely, let

$$\mu_0(\gamma) = \inf_{0\neq\varphi\in W^{1,2}(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2 dx + \gamma\int_{\partial\Omega}|\varphi|^2 dS}{\int_\Omega |\varphi|^2 dx}. \tag{1.4}$$

Then $\mu_0(\gamma)$ is achieved by some $\phi_\gamma \in W^{1,2}(\Omega)$ which satisfies the Euler equations in the weak sense

$$\begin{cases} -\Delta\phi_\gamma = \mu_0(\gamma)\phi_\gamma & \text{in} \quad \Omega, \\ \frac{\partial}{\partial\boldsymbol{\nu}}\phi_\gamma + \gamma\phi_\gamma = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.5}$$

By the elliptic regularity and the fact the first eigenvalue $\mu_0(\gamma)$ is simple, we may assume that $\phi_\gamma \in C^\infty(\overline{\Omega}), \phi_\gamma > 0$ on $\overline{\Omega}$. We note that for any $\gamma > 0$, $0 < \mu_0(\gamma) < \lambda_0(\Omega)$ where $\lambda_0(\Omega)$ is the first eigenvalue of the Dirichlet problem for the equations:

$$\begin{cases} -\Delta\phi = \lambda\phi & \text{in} \quad \Omega, \\ \phi = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

According to [8, Proposition 2.1], the following proposition holds.

Proposition 1.1. *Let $\varepsilon > 0$ and $\gamma > 0$. Then the following holds.*

(1) *If $\varepsilon \geq 1/\sqrt{\mu_0(\gamma)}$, then* (1.1) *has no non-zero solution.*

(2) *If $0 < \varepsilon < 1/\sqrt{\mu_0(\gamma)}$, then* (1.1) *has a unique positive solution u_ε. Moreover, u_ε satisfies*

(2.i) $0 < u_\varepsilon < 1$ *on* $\overline{\Omega}$.

(2.ii) *u_ε attains its maximum in Ω and its minimum on $\partial\Omega$.*

(2.iii) *The only minimizers of* (1.3) *are u_ε and $-u_\varepsilon$.*

In this note, we shall extend the results for the problem:

$$\begin{cases} -\Delta u = \frac{1}{\varepsilon^2}f(u^2)u & \text{in} \quad \Omega, \\ \frac{\partial}{\partial\boldsymbol{\nu}}u + \gamma u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where f is some smooth function and Ω is a smooth bounded domain in $\mathbb{R}^n$. Thus we shall indicate to reach the similar results in the case where f is a rather general function and Ω is a domain in general dimensional space.

2. Main Theorem

In this section, we shall give the main theorem. In order to do so, assume that $f \in C^\infty([0,\infty))$ satisfies the following hypotheses (H):

(H. i) $f_0 := f(0) > 0$,

(H. ii) $f'(u) < 0$ in $[0,\infty)$,

(H. iii) f is concave in $(0,\infty)$, i.e., $f''(u) \le 0$ in $(0,\infty)$,

(H. iv) There exists $N = N(n) > 0$ depending on the dimension n such that $|f(u)| = O(|u|^N)$ as $u \to \infty$ where N is arbitrary for $n = 2$ and $N \le 1/(n-2)$ for $n \ge 3$.

Example 2.1. (1) $f(u) = 1 - u$ for $n \le 3$.

(2) $f(u) = 2 - u - u^2$ for $n = 2$.

(3) $f(u) = 1 - u^2$ for $n = 2$.

By the hypotheses (H), we see that $f(u)$ is strictly monotone decreasing in $[0,\infty)$ and $f'(u)$ is monotone decreasing. Thus there exists uniquely $a > 0$ such that $f(a) = 0$ and $f(u) > 0$ for $0 \le u < a$ and $f(u) < 0$ for $u > a$. Let F is a primitive of f such that

$$F(u) = \int_u^a f(v)dv.$$

Then $F(u) > 0$ for $u \ge 0$ and $u \ne a$ and $F(a) = 0$. By (H. iv), there exists a constant $C > 0$ such that

$$F(u) \le C\{1 + |u| + |u|^{N+1}\} \tag{2.1}$$

for all $u \ge 0$.

Let Ω is smooth, bounded domain in $\mathbb{R}^n$. We consider the problem:

$$\begin{cases} -\Delta u = \frac{1}{\varepsilon^2} f(u^2)u & \text{in} \quad \Omega, \\ \frac{\partial}{\partial \boldsymbol{\nu}} u + \gamma u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.2}$$

In this and next sections, we only treat real valued functions. Such type of problem is considered in Aramaki [2], [3] and Pan and Kwek [10].

The energy functional associated with (2.2) is

$$\mathcal{J}_\varepsilon[\varphi] = \int_\Omega \{|\nabla\varphi|^2 + \frac{1}{\varepsilon^2} F(\varphi^2)\}dx + \gamma \int_{\partial\Omega} |\varphi|^2 dS \tag{2.3}$$

on $W^{1,2}(\Omega)$ where dS denotes the sphere element of $\partial\Omega$. Here we note that the

Sobolev Imbedding Theorem implies

$$W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)\,,$$

where $q \in [2,\infty)$ for $n=2$ and $q \in [2, 2n/(n-2)]$ for $n \geq 3$. Since

$$F(\varphi^2) \leq C(1+\varphi^2(x)+\varphi^{2(N+1)}(x)),$$

it follows from the hypothesis (H. iv) that $2 < 2(N+1) < 2(1+2/(n-2)) = 2n/(n-2)$ in the case where $n \geq 3$. Hence we see that $F(\varphi^2) \in L^1(\Omega)$, so the functional $\mathcal{J}_\varepsilon$ is well defined. Let

$$C_0(\varepsilon) = \inf_{\varphi \in W^{1,2}(\Omega)} \mathcal{J}_\varepsilon[\varphi]. \tag{2.4}$$

We show that $C_0(\varepsilon)$ is achieved in $W^{1,2}(\Omega)$.

Proposition 2.2. *$C_0(\varepsilon)$ is achieved by a function in $W^{1,2}(\Omega)$.*

Proof. Thuogh the arguments are standard, we give a proof which consists of three steps.

Step i. $\mathcal{J}_\varepsilon[\varphi]$ is weakly lower semi-continuous on $W^{1,2}(\Omega)$.

Let $\varphi_k \to \varphi$ as $k \to \infty$ weakly in $W^{1,2}(\Omega)$. Then by the Rellich Theorem, the inclusions $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ and $W^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega)$ are compact. Therefore, $\nabla\varphi_k \to \nabla\varphi$ weakly in $L^2(\Omega)$ and so

$$\|\nabla\varphi\|_{L^2(\Omega)} \leq \liminf_{k\to\infty} \|\nabla\varphi_k\|_{L^2(\Omega)}$$

and $\|\varphi\|_{L^2(\partial\Omega)} = \lim_{k\to\infty} \|\varphi_k\|_{L^2(\partial\Omega)}$. From now on, we denote the constants which may be vary from line to line by $C, C_1, etc.$. By the hypotheses (H),

$$\begin{aligned}\Big|\int_\Omega F(\varphi_k^2)dx - \int_\Omega F(\varphi^2)dx\Big| &\leq \Big|\int_\Omega \int_{\varphi_k^2}^{\varphi^2} f(v)dvdx\Big| \leq \int_\Omega \Big|\int_{\varphi_k^2}^{\varphi^2} |f(v)|dv\Big|dx \\ &\leq C\int_\Omega \Big|\int_{\varphi_k^2}^{\varphi^2} (1+v^N)dv\Big|dx\end{aligned}$$

$$= C\int_\Omega |\varphi^2(x) - \varphi_k^2(x) + \frac{1}{N+1}(\varphi^{2(N+1)}(x) - \varphi_k^{2(N+1)}(x)|dx \leq C_N(J_k^1 + J_k^2)\,,$$

where

$$J_k^1 = \int_\Omega |\varphi(x) - \varphi_k(x)|(|\varphi(x)| + |\varphi_k(x)|)dx$$

and

$$J_k^2 = \int_\Omega |\varphi(x) - \varphi_k(x)|(|\varphi(x)|^{2N+1} + |\varphi_k(x)|^{2N+1})dx.$$

By the Schwarz inequality,

$$J_k^1 \leq C\|\varphi_k - \varphi\|_{L^2(\Omega)}(\|\varphi_k\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)})$$

and

$$J_k^2 \le C\|\varphi_k - \varphi\|_{L^2(\Omega)} \left(\int_\Omega (|\varphi_k|^{2(2N+1)} + |\varphi|^{2(2N+1)}) dx \right)^{1/2}.$$

Since $\varphi_k \to \varphi$ weakly in $W^{1,2}(\Omega)$ and $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact imbedding, we see that $\varphi_k \to \varphi$ strongly in $L^2(\Omega)$. Therefore, $J_k^1 \to 0$ as $k \to \infty$. For J_k^2, the inclusion mapping $W^{1,2}(\Omega) \hookrightarrow L^{2(2N+1)}(\Omega)$ is continuous for $n = 2$ and for $n \ge 3$, since $2(2N+1) \le 2n/(n-2)$ by the hypothesis (H.iv), we see that $J_k^2 \to 0$ as $k \to \infty$. Thus

$$\lim_{k\to\infty} \int_\Omega F(\varphi_k^2) dx = \int_\Omega F(\varphi^2) dx.$$

Hence, we see that $\mathcal{J}_\varepsilon[\varphi] \le \liminf_{k\to\infty} \mathcal{J}_\varepsilon[\varphi_k]$.

Step ii. We show that $\mathcal{J}[\varphi]$ is increasing functional. That is to say, for any $c \ge 0$, $W_c := \{\varphi \in W^{1,2}(\Omega); \mathcal{J}_\varepsilon[\varphi] \le c\}$ is bounded set in $W^{1,2}(\Omega)$.

Let

$$\mathcal{J}_\varepsilon[\varphi] = \int_\Omega |\nabla\varphi|^2 dx + \gamma \int_{\partial\Omega} |\varphi|^2 dS + \frac{1}{\varepsilon^2} \int_\Omega F(\varphi^2) dx \le c.$$

Then $\|\nabla\varphi\|_{L^2(\Omega)} \le \sqrt{c}$ and $\int_\Omega F(\varphi^2)dx \le c\varepsilon^2$. By the Taylor expansion of F at the origin, we see that

$$c\varepsilon^2 \ge \int_\Omega F(0)dx + \int_\Omega F'(0)\varphi^2(x)dx + \frac{1}{2}\int_\Omega F''(\theta(x)\varphi^2(x))\varphi^4(x)dx$$

for some $0 < \theta(x) < 1$. Here we note that $F(0) = \int_0^a f(v)dv, F'(0) = -f(0) = -f_0$. Moreover, since $F''(u) = -f'(u)$ and f' is monotone decreasing, $f'(0) \ge f'(\theta(x)\varphi^2(x))$. Therefore, we have

$$\begin{aligned} c\varepsilon^2 &\ge |\Omega| \int_0^a f(v)dv - f_0 \int_\Omega \varphi^2 dx - \frac{1}{2} f'(0) \int_\Omega \varphi^4(x)dx \\ &\ge -f_0 \int_\Omega \varphi^2 dx - \frac{1}{2} f'(0) \int_\Omega \varphi^4(x)dx. \end{aligned}$$

Since $f'(0) < 0$, it follows from the Schwarz inequality that we have

$$\int_\Omega \varphi^4(x)dx \le C + C_1 \int_\Omega \varphi^2(x)dx \le C_2 + \frac{1}{2}\int_\Omega \varphi^4(x)dx.$$

Hence $\int_\Omega \varphi^4(x)dx \le C_3$. Using again the Schwarz inequality,

$$\int_\Omega \varphi^2(x)dx \le \left\{ \int_\Omega \varphi^4(x)dx \right\}^{1/2} |\Omega|^{1/2} \le C_4.$$

Thus we see that $\|\varphi\|_{W^{1,2}(\Omega)} \le C_5$.

Step iii. Now it is standard to show that $C_0(\varepsilon)$ is achieved by a function in $W^{1,2}(\Omega)$.

In fact, let $\{\varphi_k\} \subset W^{1,2}(\Omega)$ be a minimizing sequence, i.e.,

$$C_0(\varepsilon) = \lim_{k\to\infty} \mathcal{J}[\varphi_k].$$

Since $C_0(\varepsilon) \le \mathcal{J}_\varepsilon[0] = \frac{|\Omega|}{\varepsilon^2}\int_0^a f(v)dv < \infty$, we may assume that $\mathcal{J}_\varepsilon[\varphi_k] \le C < \infty$ for all $k = 1, 2, \ldots$. That is to say, $\mathcal{J}_\varepsilon$ is increasing functional. Therefore, $\{\varphi_k\}$ is bounded in $W^{1,2}(\Omega)$ and so $\{\varphi_k\}$ has a weakly convergent subsequence $\{\varphi_{k_l}\}$ such that $\varphi_{k_l} \to \varphi$ weakly in $W^{1,2}(\Omega)$ as $l \to \infty$. From the weakly lower semi-continuity of $\mathcal{J}_\varepsilon$,

$$C_0(\varepsilon) \le \mathcal{J}_\varepsilon[\varphi] \le \liminf_{l\to\infty} \mathcal{J}_\varepsilon[\varphi_{k_l}] = C_0(\varepsilon).$$

Thus φ is a minimizer of $\mathcal{J}_\varepsilon$. This completes the proof. □

We are in a positon to state the main theorem.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain. Assume that f satisfies* (H) *and $\varepsilon > 0, \gamma > 0$. Then we have:*

(1) *If $\varepsilon^2 \ge f_0/\mu_0(\gamma)$, then equation* (2.2) *has no non-zero solution.*

(2) *If $\varepsilon^2 < f_0/\mu_0(\gamma)$, then* (2.2) *has a unique positive solution u_ε. Moreover, u_ε satisfies:*

(2.i) $0 < u_\varepsilon < \sqrt{a}$ *on* $\overline{\Omega}$.

(2.ii) u_ε *attains its maximum in Ω and its minimum on $\partial\Omega$.*

(2.iii) *The only minimizers of* (2.4) *are $u_\varepsilon, -u_\varepsilon$.*

Remark 2.4. When $f(u) = 1 - u$, (H) holds for $n \le 3$. Then clearly $a = 1$. Thus we see that our results contain the result of [8] and also hold for the case where $n = 3$.

3. Proof of the Main Theorem 2.3

In this section, we prove of the main Theorem 2.3.

First, assume that (2.2) has a non-zero solution φ. If we multiply the first equation of (2.2) by φ, then integrate it over Ω, taking the boundary condition into consideration, we have

$$\frac{1}{\varepsilon^2}\int_\Omega f(\varphi^2)\varphi^2 dx = \int_\Omega |\nabla\varphi|^2 dx + \gamma\int_{\partial\Omega} |\varphi|^2 dS. \tag{3.1}$$

Since f is strictly monotone decreasing and $\varphi \not\equiv 0$, we see that

$$\frac{f_0}{\varepsilon^2}\int_\Omega \varphi^2 dx > \frac{1}{\varepsilon^2}\int_\Omega f(\varphi^2)\varphi^2 dx.$$

By the definition of $\mu_0(\gamma)$ and (3.1), we get

$$\frac{f_0}{\varepsilon^2}\int_\Omega \varphi^2 dx > \frac{1}{\varepsilon^2}\int_\Omega f(\varphi^2)\varphi^2 dx$$
$$= \int_\Omega |\nabla\varphi|^2 dx + \gamma\int_{\partial\Omega}\varphi^2 dS \geq \mu_0(\gamma)\int_\Omega \varphi^2 dx.$$

Thus we have $f_0/\varepsilon^2 > \mu_0(\gamma)$. Therefore Theorem 2.3 (1) holds. In this case, we note that

$$C_0(\varepsilon) = \mathcal{J}_\varepsilon[0] = \frac{|\Omega|}{\varepsilon^2}\int_0^a f(v)dv.$$

In particular, in the case where $f(v) = 1 - v$, $C_0(\varepsilon) = |\Omega|/(2\varepsilon^2)$.

Assume that $\varepsilon^2 < f_0/\mu_0(\gamma)$. Let $\phi_\gamma \in C^\infty(\overline{\Omega})$, $\phi_\gamma > 0$ on $\overline{\Omega}$ be the eigenfunction corresponding to the first eigenvalue $\mu_0(\gamma)$ of (1.5). For $t > 0$, if we take $t\phi_\gamma$ as a test function for $\mathcal{J}_\varepsilon$, we have

$$\begin{aligned}\mathcal{J}_\varepsilon[t\phi_\gamma] &= t^2\int_\Omega |\nabla\phi_\gamma|^2 dx + \frac{1}{\varepsilon^2}\int_\Omega F(t^2\phi_\gamma^2)dx + t^2\gamma\int_{\partial\Omega}\phi_\gamma^2 dS \\ &= \mu_0(\gamma)t^2\int_\Omega \phi_\gamma^2 dx + \frac{1}{\varepsilon^2}\int_\Omega F(t^2\phi_\gamma^2)dx.\end{aligned}$$

Using the Taylor expansion of F, since

$$\begin{aligned}F(t^2\phi_\gamma^2) &= F(0) + t^2\phi_\gamma^2 F'(0) + \frac{1}{2}t^4\phi_\gamma^4 F''(\theta t^2\phi_\gamma) \\ &= \int_0^a f(v)dv - t^2\phi_\gamma^2 f(0) - \frac{1}{2}t^4\phi_\gamma^4 f'(\theta t^2\phi_\gamma^2),\end{aligned}$$

we see that

$$\begin{aligned}\mathcal{J}_\varepsilon[t\phi_\gamma] &= \frac{|\Omega|}{\varepsilon^2}\int_0^a f(v)dv - \left(\frac{f_0}{\varepsilon^2} - \mu_0(\gamma)\right)t^2\int_\Omega \phi_\gamma^2 dx \\ &\quad - \frac{1}{2\varepsilon^2}t^4\int_\Omega \phi_\gamma^4 f'(\theta t^2\phi_\gamma^2)dx.\end{aligned}$$

Since $t^2\phi_\gamma^2 < a$ and f' is monotone decreasing, for small $t > 0$, we have

$$\mathcal{J}_\varepsilon[t\phi_\gamma] < \frac{|\Omega|}{\varepsilon^2}\int_0^a f(v)dv = \mathcal{J}_\varepsilon[0].$$

Therefore, if $\varepsilon^2 < f_0/\mu_0(\gamma)$, any minimizer u of (2.3) satisfies

$$C_0(\varepsilon) = \mathcal{J}_\varepsilon[u] < \frac{|\Omega|}{\varepsilon^2}\int_0^a f(v)dv = \mathcal{J}_\varepsilon[0].$$

Thus $u \not\equiv 0$ in Ω.

The proof of Theorem 2.3 (2) consists of three lemmas. First we show that u has no zero in $\overline{\Omega}$.

Lemma 3.1. *$u(x) \neq 0$ for all $x \in \overline{\Omega}$.*

Proof. Assume that $Z(u) := \{x \in \overline{\Omega}; u(x) = 0\} \neq \emptyset$. Let $v(x) = |u(x)|$. Then we have

$$\begin{aligned} \mathcal{J}_\varepsilon[v] &= \int_\Omega |\nabla|u||^2 dx + \gamma \int_{\partial\Omega} |u|^2 dS + \frac{1}{\varepsilon^2}\int_\Omega F(u^2)dx \\ &= \int_\Omega |\nabla u|^2 dx + \gamma \int_{\partial\Omega} |u|^2 dS + \frac{1}{\varepsilon^2}\int_\Omega F(u^2)dx \\ &= \mathcal{J}_\varepsilon[u]. \end{aligned}$$

Thus v is also a minimizer of (2.3) and so v is a non-negative solution of (2.2). That is to say,

$$\begin{cases} -\Delta v = f(v^2)v & \text{in} \quad \Omega, \\ \frac{\partial}{\partial \boldsymbol{\nu}} v + \gamma v = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.2}$$

Here $f(v^2)v \leq C(v + v^{2N+1})$. Since $v \in W^{1,2}(\Omega)$, it follows from the Sobolev Imbedding Theorem that $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ for any q if $n = 2$ and $q \in [2, p^*]$ where $p^* = 2n/(n-2)$ if $n \geq 3$. By the hypothesis (H.iv), $2(2N+1) \leq 2(1+2/(n-2)) = 2n/(n-2)$ for $n \geq 3$, we see that $f(v^2)v \in L^2(\Omega)$. Therefore, by the standard elliptic regularity theorem, $v \in W^{2,2}(\Omega)$. Applying again the Sobolev Imbedding Theorem, we see that $W^{2,2}(\Omega) \hookrightarrow W^{1,q}(\Omega)$ for any $q \geq 2$ if $n = 2$ and $q = 2n/(n-2) > 2$ if $n \geq 3$. Thus by the bootstrap method, we have $f(v^2)v \in W^{1,q}(\Omega)$ for any $q \geq 2$. If we choose q sufficiently large so that $1 - n/q > 0$, we see that $f(v^2)v \in C^\alpha(\overline{\Omega})$ for any $\alpha \in (0,1)$. By the Schauder theory, we have $v \in C^{2+\alpha}(\overline{\Omega})$. By the bootstrap method, we get $v \in C^\infty(\overline{\Omega})$ (cf. Agmon et al [1], Du [5] and Gilbarg and Trudinger [6]).

We claim that v cannot attain its maximum on $\partial\Omega$. In fact, assume that v has a maximum at $x_0 \in \partial\Omega$. Since $v \not\equiv 0$ and $v \geq 0$, we see that $v(x_0) > 0$ and $(\frac{\partial}{\partial \boldsymbol{\nu}} v)(x_0) \geq 0$. However, these facts contradict to the boundary condition $(\frac{\partial}{\partial \boldsymbol{\nu}} v)(x_0) + \gamma v(x_0) = 0$. Therefore, there exists $x_0 \in \Omega$ such that $\max_{x \in \overline{\Omega}} v(x) = v(x_0) > 0$. Thus since

$$f(v(x_0)^2)v(x_0) = -\varepsilon^2(\Delta v)(x_0) \geq 0,$$

we have $f(v(x_0)^2)v(x_0) \geq 0$. Therefore, $0 < v(x_0)^2 \leq a$. That is to say, $0 < v(x_0) \leq \sqrt{a}$. Thus we see that $0 \leq v(x) \leq \sqrt{a}$ for all $x \in \overline{\Omega}$. Moreover, by the hypothesis (H), we get $\Delta v = -\frac{1}{\varepsilon^2} f(v^2)v \leq 0$ in Ω. By the assumption $Z(v) \neq \emptyset$ and the fact $v \not\equiv 0$, v is not constant. It follows from the Strong maximum principle that v cannot achieve its minimum in the interior of Ω. This means that there exists $x_1 \in \partial\Omega$ such that $v(x) \geq v(x_1) = \min_{x \in \overline{\Omega}} v(x)$ for any $x \in \overline{\Omega}$ and $v(x) > v(x_1)$ for any $x \in \Omega$. By the Hopf Boundary Lemma, we have $\left(\frac{\partial v}{\partial \boldsymbol{\nu}}\right)(x_1) < 0$. Then by the boundary condition, we have

$$v(x_1) = -\frac{1}{\gamma}\left(\frac{\partial v}{\partial \boldsymbol{\nu}}\right)(x_1) > 0.$$

Thus $v(x) > 0$ on $\overline{\Omega}$. Hence we see that $Z(v) = \emptyset$. This leads to a contradiction. That is to say, u is strictly positive or strictly negative on $\overline{\Omega}$. □

Lemma 3.2. *Let $u \not\equiv 0$ be a non-negative solution of* (2.2). *Then $0 < u(x) < \sqrt{a}$ for all $x \in \overline{\Omega}$.*

Proof. We have already showed that u is a smooth function on $\overline{\Omega}$ and $0 < u(x) \leq \sqrt{a}$ for all $x \in \overline{\Omega}$. Thus it is sufficient to show that $u(x) < \sqrt{a}$ for all $x \in \overline{\Omega}$. Suppose that

$$\max_{x \in \overline{\Omega}} u(x) = \sqrt{a}.$$

Define $w(x) = \sqrt{a} - u(x)$. Then $0 \leq w(x) < \sqrt{a}$ on $\overline{\Omega}$ and w satisfies

$$\begin{cases} \Delta w = \frac{1}{\varepsilon^2} f((\sqrt{a} - w)^2)(\sqrt{a} - w) & \text{in} \quad \Omega, \\ \frac{\partial}{\partial \boldsymbol{\nu}} w + \gamma w = \gamma \sqrt{a} & \text{on} \quad \partial\Omega. \end{cases} \tag{3.3}$$

By the Taylor expansion of f at a,

$$f(u^2) = (u^2 - a) f'(a) + \frac{1}{2}(u^2 - a)^2 f''(a + \theta(u^2 - a)).$$

Therefore,

$$\begin{aligned} f(u^2)u &= -f'(a)(a - u^2)u + \frac{1}{2}(u^2 - a)^2 u f''(a + \theta(u^2 - a)) \\ &= -f'(a) w(\sqrt{a} + u)u + \frac{1}{2}(u^2 - a)^2 u f''(a + \theta(u^2 - a)). \end{aligned}$$

We define $c(x) = \frac{1}{\varepsilon^2} f'(a)(\sqrt{a} + u)u < 0$. Since $f''(u) \leq 0$ in $(0, \infty)$, we can rewrite (3.3) into the form

$$\begin{cases} \Delta w + c(x) w = \frac{1}{2\varepsilon^2}(u^2 - a)^2 u f''(a + \theta(u^2 - a)) \leq 0 & \text{in} \quad \Omega, \\ \frac{\partial}{\partial \boldsymbol{\nu}} w + \gamma w = \gamma \sqrt{a} & \text{on} \quad \partial\Omega. \end{cases} \tag{3.4}$$

If w cannot achieve its minimum in Ω, there exists $x_0 \in \partial\Omega$ such that $w(x) > w(x_0) = 0$ for all $x \in \Omega$. By the Hopf Boundary Lemma, we see that $\left(\frac{\partial w}{\partial \boldsymbol{\nu}}\right)(x_0) < 0$. This contradict to the boundary condition

$$\left(\frac{\partial w}{\partial \boldsymbol{\nu}}\right)(x_0) + \gamma w(x_0) = \gamma \sqrt{a}.$$

Thus w has a minimum 0 in Ω. Applying the strong maximum principle to (3.4), w cannot achieve a non-positive minimum in Ω unless it is not a constant. Since w is not constnat, w cannot have its minimum 0 in Ω. This leads to a contradiction. Thus $0 < u(x) < \sqrt{a}$ for all $x \in \overline{\Omega}$. □

Lemma 3.3. *The positive solution of* (2.2) *is unique.*

Proof. Let u_1, u_2 be positive solutions of (2.2). In order to prove the lemma, it is sufficient to show that $u_1 \geq u_2$. From the preceding lemmas, we see that $u_i \in C^\infty(\overline{\Omega})$ and $u_i > 0$ on $\overline{\Omega}$ for $i = 1, 2$. Define $u_\lambda = \lambda u_1$. Then if $\lambda > 0$ is sufficiently large, we clearly see that $u_\lambda > u_2$. If $\lambda \geq 1$, u_λ is a supersolution of (2.2), i.e., u_λ satisfies

$$\begin{cases} -\Delta u_\lambda \geq \frac{1}{\varepsilon^2} f(u_\lambda^2) u_\lambda & \text{in} \quad \Omega, \\ \frac{\partial}{\partial \boldsymbol{\nu}} u_\lambda + \gamma u_\lambda = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Put

$$\overline{\lambda} = \inf\{\lambda \geq 1; u_\lambda \geq u_2 \quad \text{on} \quad \overline{\Omega}\}.$$

We want to show that $\overline{\lambda} = 1$. Suppose that $\overline{\lambda} > 1$. Then $\overline{u} = u_{\overline{\lambda}} = \overline{\lambda} u_1$ is a strict supersolution of (2.2), i.e.,

$$\begin{cases} -\Delta \overline{u} > \frac{1}{\varepsilon^2} f(\overline{u}^2) \overline{u} & \text{in} \quad \Omega, \\ \frac{\partial}{\partial \boldsymbol{\nu}} \overline{u} + \gamma \overline{u} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We note that $\overline{u} \geq u_2$ on $\overline{\Omega}$ and $\inf_{\overline{\Omega}}(\overline{u} - u_2) = 0$. Define a function

$$c(x) = -\frac{1}{\varepsilon^2}(\overline{u} + u_2)\overline{u} \int_0^1 f'(u_2^2 + \theta(\overline{u}^2 - u_2^2)) d\theta > 0$$

and an operator $L\phi = \Delta\phi - c(x)\phi$. Since $-\Delta u_2 = \frac{1}{\varepsilon^2} f(u_2^2) u_2$, it follows from the mean value theorem that

$$\begin{aligned} \Delta(\overline{u} - u_2) &< \frac{1}{\varepsilon^2}\big(f(u_2^2)u_2 - f(\overline{u}^2)\overline{u}\big) \\ &= \frac{1}{\varepsilon^2}\Big\{ f(u_2^2)u_2 - \big(f(u_2^2) + (\overline{u}^2 - u_2^2)\int_0^1 f'(u_2^2 + \theta(\overline{u}^2 - u_2^2))d\theta\big)\overline{u} \Big\} \\ &= \frac{1}{\varepsilon^2} f(u_2^2)(u_2 - \overline{u}) + c(x)(\overline{u} - u_2). \end{aligned}$$

Hence, we see that $L(\overline{u} - u_2) = \frac{1}{\varepsilon^2} f(u_2^2)(u_2 - \overline{u}) \leq 0$. Since $\overline{u} - u_2$ is not a constant, it follows from the strong maximum principle that $\overline{u} - u_2$ has no non-positive minimum in the interior of Ω. Since $\overline{u} - u_2$ attains its minimum 0 in $\overline{\Omega}$, there exists $x_0 \in \partial\Omega$ such that $\overline{u}(x_0) - u_2(x_0) = 0$. By the boundary condition, we have $\frac{\partial \overline{u}}{\partial \boldsymbol{\nu}}(x_0) - \frac{\partial u_2}{\partial \boldsymbol{\nu}}(x_0) = 0$. On the other hand, since $L(\overline{u} - u_2) \leq 0$ in Ω and $(\overline{u} - u_2)(x) > \overline{u}(x_0) - u_2(x_0) = 0$ for all $x \in \Omega$, from the Hopf Boundary Lemma, we get $\frac{\partial \overline{u}}{\partial \boldsymbol{\nu}}(x_0) - \frac{\partial u_2}{\partial \boldsymbol{\nu}}(x_0) < 0$. This is a contradiction. Therefore, $\overline{\lambda} = 1$. □

The proof of Theorem 2.3 follows from the above lemmas.

4. The Case where the Magnetic Field Vanishes Identically in the Two Dimensional Space

In this section, we consider the Ginzburg-Landau system with zero magnetic field in a two dimensional domain. In order to do so, let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded domain and f satisfies (H). We consider the Ginzburg-Landau system:

$$\begin{cases} (\nabla - i\boldsymbol{A})^2\psi + \frac{1}{\varepsilon^2}f(|\psi|^2)\psi = 0 & \text{in} \quad \Omega, \\ \operatorname{curl}^2 \boldsymbol{A} = -\frac{i}{2}(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}) - |\psi|^2\boldsymbol{A} + \operatorname{curl} H & \text{in} \quad \Omega\,, \end{cases} \tag{4.1}$$

with the boundary condition

$$\begin{cases} \frac{\partial}{\partial \boldsymbol{\nu}}\psi - i\boldsymbol{A}\psi\cdot\boldsymbol{\nu} + \gamma\psi = 0 & \text{on} \quad \partial\Omega, \\ (\operatorname{curl}\boldsymbol{A} - H)\times\boldsymbol{\nu} = 0 & \text{in} \quad \partial\Omega. \end{cases} \tag{4.2}$$

Then the corresponding Gibbs free energy is

$$\mathcal{J}L(\psi, \boldsymbol{A}, \varepsilon) = \int_\Omega \big\{|\nabla - i\boldsymbol{A})\psi|^2 + |\operatorname{curl}\boldsymbol{A} - H|^2 + \frac{1}{\varepsilon^2}F(|\psi|^2)\big\}dx + \gamma\int_{\partial\Omega}|\psi|^2 ds \tag{4.3}$$

on $\mathcal{W} := W^{1,2}(\Omega;\mathbb{C}) \times W^{1,2}(\Omega;\mathbb{R}^2)$. For $H \in W^{1,2}(\Omega)$, define

$$C(\varepsilon, H) = \inf_{(\psi,\boldsymbol{A})\in\mathcal{W}} \mathcal{J}L(\psi, \boldsymbol{A}, \varepsilon).$$

We call $(\psi, \boldsymbol{A}) \in \mathcal{W}$ a least-energy solution of (4.1) and (4.2), if $C(\varepsilon, H)$ achieved by $(\psi, \boldsymbol{A})$.

Then we have

Lemma 4.1. *Let $\varepsilon > 0, \gamma > 0$ and $H \in W^{1,2}(\Omega)$. Then the equations* (4.1) *and* (4.2) *have a least-energy solution.*

Proof. That $\mathcal{J}L$ is weakly lower semi-continuous in $\mathcal{W}$ and increasing functional can be proved by similar arguments as in Proposition 2.2. Let $\{(\psi_m, \boldsymbol{A}_m)\}$ be a minimizing sequence in $\mathcal{W}$. By the Hölder gauge transformation, we may assume that $\operatorname{div}\boldsymbol{A}_m = 0$ in Ω and $\boldsymbol{A}_m\cdot\boldsymbol{\nu} = 0$ on $\partial\Omega$ (cf. Jaffe and Taubes [7]). Since $\{(\psi_m, \boldsymbol{A}_m)\}$ is bounded in $\mathcal{W}$, passing to a subsequence, we may assume that $(\psi_m, \boldsymbol{A}_m) \to (\psi, \boldsymbol{A})$ weakly in $\mathcal{W}$. Since $\mathcal{J}L$ is weakly lower semi-continuous, we have

$$\mathcal{J}L(\psi, \boldsymbol{A}, \varepsilon) \le \liminf_{m\to\infty} \mathcal{J}L(\psi_m, \boldsymbol{A}_m, \varepsilon) = C(\varepsilon, H).$$

This completes the proof. □

Then we shall prove the following proposition.

Proposition 4.2. (1) *When $H(x) \equiv 0$, any least-energy solution $(\psi, \boldsymbol{A})$ of* (4.1) *and* (4.2) *satisfies that $|\psi| \equiv u_\varepsilon$ where u_ε is the positive solution in Theorem* 2.3.

(2) *Assume that* $H(x) \equiv 0$ *and* Ω *is smooth, simply connected bounded domain in* $\mathbb{R}^2$. *Then any least-energy solution* $(\psi, \boldsymbol{A})$ *of* (4.1) *and* (4.2) *is of the form* $\psi = u_\varepsilon e^{i\chi}$ *and* $\boldsymbol{A} = \nabla\chi$ *where* χ *is a real and Hölder continuous function.*

Remark 4.3. When Ω is simply connected and $H(x) \equiv 0$, the least-energy solutions of (4.1) and (4.2) are completely determined by the unique solution u_ε of (2.2).

Proof. First we claim $C(\varepsilon, 0) = C_0(\varepsilon)$ where $C_0(\varepsilon)$ is defined by (2.4).

In fact, when $H(x) \equiv 0$, we see that $\mathcal{JL}(\varphi, 0, \varepsilon) = \mathcal{J}_\varepsilon[\varphi]$ for all real function $\varphi \in W^{1,2}(\Omega)$. Therefore, we have

$$C(\varepsilon, 0) = \inf_{(\psi,\boldsymbol{A})\in\mathcal{W}} \mathcal{JL}(\psi, \boldsymbol{A}, \varepsilon) \le \inf_{\varphi\in W^{1,2}(\Omega;\mathbb{R})} \mathcal{JL}(\varphi, 0, \varepsilon) = C_0(\varepsilon).$$

Conversely, let $(\psi, \boldsymbol{A})$ be a minimizer of $C(\varepsilon, 0)$. After a Hölder gauge transformation, we may assume that $(\psi, \boldsymbol{A})$ is smooth. From the celebrated Kato inequality ([7, Proposition 6.1]), we have

$$\int_\Omega |\nabla|\psi||^2 dx \le \int_\Omega |(\nabla - i\boldsymbol{A})\psi|^2 dx.$$

Hence

$$\begin{aligned} C(\varepsilon, 0) &= \mathcal{JL}(\psi, \boldsymbol{A}, \varepsilon) \\ &= \int_\Omega \Big\{|(\nabla - i\boldsymbol{A})\psi|^2 + |\mathrm{curl}\,\boldsymbol{A}|^2 + \frac{1}{\varepsilon^2} F(|\psi|^2)\Big\} dx + \gamma \int_{\partial\Omega} |\psi|^2 ds \\ &\ge \int_\Omega \Big\{|\nabla|\psi||^2 + |\mathrm{curl}\,\boldsymbol{A}|^2 + \frac{1}{\varepsilon^2} F(|\psi|^2)\Big\} dx + \gamma \int_{\partial\Omega} |\psi|^2 ds \ge \mathcal{J}_\varepsilon[|\psi|] \\ &\ge C_0(\varepsilon). \end{aligned} \tag{4.4}$$

Therefore, we get $C(\varepsilon, 0) = C_0(\varepsilon)$. Moreover, we get $C_0(\varepsilon) = \mathcal{J}_\varepsilon[|\psi|]$. Since $|\psi(x)|$ is a minimizer of $C_0(\varepsilon)$. we see that $|\psi(x)| = u_\varepsilon(x)$. Thus (1) holds. Furthermore, from (4.4), we see that

$$\int_\Omega |(\nabla - i\boldsymbol{A})\psi|^2 dx = \int_\Omega |\nabla|\psi||^2 dx,$$

and $\mathrm{curl}\,\boldsymbol{A} = 0$ almost everywhere in Ω. Taking the Hölder gauge transformation, we may assume that $(\psi, \boldsymbol{A})$ is smooth. Hence $\mathrm{curl}\,\boldsymbol{A} \equiv 0$ in Ω. Since Ω is simply connected, there exists a real function χ such that $\boldsymbol{A} = \nabla\chi$.

When $\varepsilon \ge f_0/\mu_0(\gamma)$, since $u_\varepsilon \equiv 0$, $\psi \equiv 0$. Then clealy (2) holds.

When $0 < \varepsilon < f_0/\mu_0(\gamma)$, since $|\psi(x)| = u_\varepsilon(x) > 0$ on $\overline{\Omega}$, we see that $\deg(\psi) = 0$. Since Ω is simply connected, there exists a real fucntion η such that $\psi(x) = u_\varepsilon(x)e^{i\eta(x)}$. Therefore, we have

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^2 dx &= \int_\Omega |(\nabla - i\boldsymbol{A})\psi|^2 dx \\ &= \int_\Omega \left\{|\nabla u_\varepsilon|^2 + u_\varepsilon^2|\nabla\eta - \boldsymbol{A}|^2\right\} dx.\end{aligned}$$

Thus we get $u_\varepsilon^2|\nabla\eta - \boldsymbol{A}|^2 = 0$ almost everywhere in Ω. So we obtain $\nabla\eta = \boldsymbol{A} = \nabla\chi$. Thus (2) holds. This completes the proof. □

References

[1] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I., *Comm. Pure Appl. Math.*, **12** (1959), 623-727.

[2] J. Aramaki, On an elliptic problem with general nonlinearity associated with superheating field in the theory of superconductivity, *Int. J. Pure Appl. Math.*, **28**, No. 1 (2006), 125-148.

[3] J. Aramaki, On an elliptic model with general nonlinearity associated with superconductivity, *Int. J. Differ. Equ. Appl.*, **10**, No. 4 (2006), 449-466.

[4] F. Bethuel, H. Brézis, E. Hélen, *Ginzburg-Landau Vortices*, Birkhäuser, Boston-Basel-Berlin (1994).

[5] Y. Du, *Order Structure and Topological Method in Nonlinear Partial Differential Equations*, World Scientific, New Jersey-London-Singapore-Beijing-Shanghai-Hong Kong-Taipei-Chennai (2006).

[6] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, New York (1983).

[7] A. Jaffe, N.S. Taubes, *Vortives and Monopoles*, Birkhäuser, Boston and Basel (1980).

[8] K. Lu, X.-B. Pan, Ginzburg-Landau equation with de Gennes boundary condition, *J. Diff. Eq.*, **129** (1996), 136-165.

[9] K. Lu, X.-B. Pan, Estimates of upper critical field for the Ginzburg-Landau equations of superconductivity, *Physica D*, **127** (1999), 73-104.

[10] X.-B. Pan, K.H. Kwek, On a problem related to vortex nucleation of superconductivity, *J. Diff. Eq.*, **182** (2002), 141-168.