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SOLVING A STOCHASTIC DIFFERENTIAL ITÔ EQUATIONS

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Abstract: In the following article two differential stochastic equations are being solved. The first equation is a model for exponential growth with several white noise sources in the relative growth rate. The second one modulates the so called Brownian bridge". The equations are solved using Itô first formula.

1. Introduction

The differential and integral stochastic calculation has developed over the last years due to the necessity of modulating growth phenomena through a probabilistic way of approach where the "noisy" environment in which these phenomena occur has its own importance. There are laws for the differential and integral stochastic calculation that allow the solving of differential stochastic equations.

2. Table of Application

We remind Itô's formula, the 1 dimensional case. Let x_t be a stochastic Itô process which stochastic differential

$$dx_t = udt + vdB_t \tag{2.1}$$

and let be $g(t,x) \in C^2([0,\infty) \times R)$ then $Y_t = g(t,x_t)$ is an Itô process and its

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stochastical differential has the following form:

$$dY_t = \frac{\partial g}{\partial t}(t,x_t)dt + \frac{\partial g}{\partial x}(t,x_t)dx_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,x_t)(dx_t)^2\,, \tag{2.2}$$

where $(dx_t)^2 = (dx_t)\cdot(dx_t), dt\cdot dt = dt\cdot dB_t = dB_t\cdot dt = 0$ and B_t is the 1-dimensional Brownian mouvement.

Statement 1. *Let be $(B_1,...,B_n)$ Brownian motion in $\mathbf{R}^n$, $\alpha_1,...,\alpha_n$ real constants. The stochastic differential equation*

$$dx_t = rx_t dt + x_t(\sum_{k=1}^{n}\alpha_k dB_k(t)); x_0 > 0 \tag{2.3}$$

is a model for an exponential growth with several independents white noise sources in the relative growth rate. The solution is:

$$x_t = x_0 \exp\left[(r - \frac{1}{2}\sum_{k=1}^{n}\alpha_k^2)t + \sum_{k=1}^{n}\alpha_k(B_k(t) - B_k(0)\right]. \tag{2.4}$$

Proof. Rewrite the equation (2.3)

$$\frac{dx_t}{x_t} = rdt + \sum_{k=1}^{n}\alpha_k dB_k(t), \tag{2.5}$$

integrating, we obtain:

$$\int_0^t \frac{dx_s}{x_s} = \int_0^t rds + \sum_{k=1}^{n}\alpha_k\int_0^t dB_k(s). \tag{2.6}$$

To evaluate the integral on the left hand side we use the Itô formula for function $g(t,s) = \ln x, x > 0$ and obtain:

$$d(\ln x_t) = \frac{1}{x_t}dx_t + \frac{1}{2}(-\frac{1}{x_t^2})(dx_t)^2; \tag{2.7}$$

also

$$d(\ln x_t) = \frac{1}{x_t}[rx_t dt + x_t(\sum_{k=1}^{n}\alpha_k dB_k(t))] - \frac{1}{2x_t^2}x_t^2(\sum_{k=1}^{n}\alpha_k^2)dt\,, \tag{2.8}$$

$$d(\ln x_t) = (r - \frac{1}{2}\sum_{k=1}^{n}\alpha_k^2)dt + \sum_{k=1}^{n}\alpha_k dB_k(t). \tag{2.9}$$

To compare the result obtained whit (2.6), we deduct the following:

$$\int_0^t \frac{dx_s}{x_s} = \int_0^t d(\ln x_s) + \frac{1}{2}(\sum_{k=1}^{n}\alpha_k^2)\int_0^t ds, \tag{2.10}$$

in other mod:

$$\ln x_s \mid_0^t + \frac{1}{2}(\sum_{k=1}^{n} \alpha_k^2) = rt + \sum_{k=1}^{n} \alpha_k (B_k(t) - B_k(0)). \tag{2.11}$$

So, we get the solution:

$$x_t = x_0 \exp\left[(r - \frac{1}{2}\sum_{k=1}^{n} \alpha_k^2)t + \sum_{k=1}^{n} \alpha_k (B_k(t) - B_k(0)\right]. \tag{2.12}$$

Now, applying Itô's formula, we shall prove that the solution obtained verify the equation (2.3). Let:

$$g(t,x) = \exp\left\{(r - \frac{1}{2}\sum_{k=1}^{n} \alpha_k^2)t + \sum_{k=1}^{n} \alpha_k x_k(t)\right\},$$

where $g(t,x) \in C^2([0,\infty) \times R^n)$. So, to compute the partial differentials of $g(t,x)$, with respect to t and x_k, into Itô's formula, thus we have:

$$\frac{\partial g}{\partial t}(t,x) = c \exp\,(ct + \sum_{k=1}^{n} \alpha_k x_k(t) = cg(t,x), x = (x_1, \ldots, x_n)\,. \tag{2.13}$$

Also we get:

$$\frac{\partial g}{\partial x_k}(t,x) = \alpha_k \exp ct + \sum_{k=1}^{n} \alpha_k x_k(t) = \alpha_k g(t,x)\,. \tag{2.14}$$

Finally:

$$\frac{\partial g}{\partial x_k^2}(t,x) = \alpha_k^2 g(t,x)\,. \tag{2.15}$$

Now, we define the process $g(t,(B_1,...B_n)) = x_t$ and by introduction in equation (2.2), we get:

$$dx_t = cx_t dt + \sum_{k=1}^{n} \alpha_k x_t dB_k + \frac{1}{2}\sum_{k=1}^{n} \alpha_k^2 x_t dt, \tag{2.16}$$

where we have used $dB_i dB_j = dt$ only if only $i = j$ else it is 0. Therefore, by notation $c + \frac{1}{2}(\sum_{k=1}^{n} \alpha_k^2) = r$ we obtain:

$$dx_t = rx_t dt + x_t(\sum_{k=1}^{n} \alpha_k dB_k) \tag{2.17}$$

This proves the given solution verify the studied equation.

Statement 2. (The Brownie Bridge) *For fixed $a, b \in R$ and B_t the Brownian motion, consider the following stochastic differential 1-dimensional equa-*

tion:

$$dY_t = \frac{b - Y_t}{1-t} dt + dB_t; \quad 0 \leq t \leq 1, \quad Y_0 = a\,. \tag{2.18}$$

The solution is given by the following stochastic process

$$Y_t = a(1-t) + bt(1-t)\int_0^t \frac{1}{1-s} dB_s, \quad 0 \leq t \leq 1\,. \tag{2.19}$$

Proof. In view of Itô's formula, we define the function $g(t,x) = a(1-t) + bt + (1-t)x, g \in C^2([0,\infty) \times R)$ and by notation $x_t = \int_0^t \frac{1}{1-s} dB_s$, the process $Y_t = g(t, x_t) = a(1-t) + bt + (1-t)x_t$ verifies the given equation. We shall prove that, by computation, similarity of the partially differentials:

$$\frac{\partial g}{\partial t}(t,x) = -a - x + b,$$

$$\frac{\partial g}{\partial x}(t,x) = 1 - t,$$

$$\frac{\partial g}{\partial x^2}(t,x) = 0,$$

and including of Itô's formula. We obtain the following results, successively:

$$dY_t = (-a + b - \int_0^t \frac{1}{1-s} dB_s)dt + (1-t)dx_t$$
$$= \frac{1}{1-t}[b - bt - a(1-t) - x_t(1-t)]dt + dB_t. \tag{2.20}$$

But this $Y_t = a(1-t) + bt + (1-t)x_t$ and including in the equation, (2.21) implies (2.19), that prove Statement 2. For $t = 0, Y_0 = a$ and for $t \to 1, Y_1 = b$.

3. Conclusions

Using the stochastic differential calculus Itô we have determined the solutions of two differential stochastic equations (2.3), and (2.18).

If the stochastic differential equation is from type:

$$dx_t = rx_t dt + \alpha dB_t; \quad \alpha, r \in R \tag{3.1}$$

then the solution $X_t = X_0 \exp{(r - \frac{1}{2}\alpha^2)t + \alpha B_t)}$.

Also, if a process is type $X_t = X_0 \exp{(\mu t + \alpha B_t)}, \alpha, \mu \in R$ then it verify an stochastic differential equation by type:

$$dx_t = (\mu t + \frac{1}{2}\alpha^2)x_t dt + \alpha dB_t; \quad \alpha, \mu \in R\,. \tag{3.2}$$

In the specially literature the solutions of the such equations is called geometrical Brownian motions. Such processes are important as models for stochastically prices in economics.

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