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ON TORIC QUANTUM CODES

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Abstract: In this paper we present new toric quantum codes derived from quadratic forms. The new class of codes is optimum in the sense that its codeword length is the smallest among the already known codes. The parameters of the new codes are $[[d^2, 2, d]]$.

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1. Introduction

Quantum bits or qubits, unlike the bits, may assume an infinite number of distinct states called *superpositions*. These superpositions are very fragile and may be destroyed by interactions of the quantum system with the surrounding environment. A solution to this problem is to isolate the qubits in order to minimize the aforementioned interactions.

An alternative solution is to use a subclass of stabilizer codes called topological quantum error-correcting codes, introduced by Kitaev [5]. Each code

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in this class is associated with a tessellation of a surface so that each edge of the tessellation corresponds injectively to a qubit. The advantage of topological quantum codes or surface codes is related to their topology and with the locality of their stabilizer operators.

Kitaev's toric code [5] was the first proposed topological quantum code. This code is defined on an $l \times l$ square lattice on the torus. There are a total of $2l^2$ edges, that is, $2l^2$ qubits. The stabilizer operators are associated to each vertex and to each face of the lattice, and the parameters of the codes are $[[2l^2, 2, l]]$.

In a similar way, Freedman and Meyer in [3] proposed a construction of topological quantum codes in the projective plane $\mathbb{R}P^2$. They showed that Shor's repetition code $[[9, 1, 3]]$ is a particular example of these codes.

Bombin and Martin-Delgado in [1] present a new class of toric codes with parameters $[[d^2 + 1, 2, d]]$, called homological quantum codes. In this construction a new regular lattice is employed on the torus to define the code.

A new class of toric codes with parameters $[[d^2, 2, d]]$ is provided in this paper. These codes are constructed on a lattice different from that in [1]. The new lattice is determined by using quadratic forms.

2. Toric Codes

Kitaev's toric code is defined on an $l \times l$ square lattice on the torus (Figure 1). The lattice is the self-dual tessellation $\{4, 4\}$ of the torus. The parameters of this class of codes are $[[2l^2, 2, l]]$, where the length of the codeword is the number of edges of the lattice $n = |E| = 2l^2$. The number of encoded qubits is given by the following rule $k = 2g$ (for the torus, the genus is $g = 1$), the distance is the minimum between the number of edges contained in the shortest homological nontrivial cycle of the lattice and the number of edges contained in the shortest homological nontrivial cycle of the dual lattice. Since a homological nontrivial cycle of edges on the lattice is an edge-path on the lattice, which cannot be contracted to a face, it follows that the shortest path corresponds to the orthogonal axis of the lattice, therefore $d = l$ [2].

The class of toric codes is a sub-class of the stabilizer codes, and its stabilizer operators are associated to each vertex and to each face of the lattice (Figure 1). Given a vertex $v \in V$, the vertex operator A_v is defined as the tensor product of σ_x, corresponding to each one of four edges which have v as a common vertex and the identity operator acting on the remaining qubits. Analogously, given

Figure 1: The $\{4,4\}$ tessellation drawn on the torus

a face $f \in F$, the face operator B_f is defined as the tensor product σ_z, which corresponds to each one of the four edges of the border of the face f and the identity operator acting on the remaining qubits, that is, $A_v = \bigotimes_{j \in E_v} \sigma_x^j$ and $B_f = \bigotimes_{j \in E_f} \sigma_z^j$. The toric code consists of the space fixed by the operators A_v and B_f, then $\mathcal{C} = \{|\psi\rangle : A_v|\psi\rangle = |\psi\rangle,\ B_f|\psi\rangle = |\psi\rangle \ \ \forall\ v, f\}$. The dimension of $\mathcal{C}$ is four, thus $\mathcal{C}$ encodes $k = 2$ qubits.

Kitaev's code may be characterized as the set of cosets of the quotient group $\mathbb{Z}^2/l\mathbb{Z}^2 \cong \mathbb{Z}_l \times \mathbb{Z}_l$. Identification of the opposite edges of the region limited by $\mathbb{Z}_l \times \mathbb{Z}_l$ yields the identification with the flat torus. Thus, the area associated with the sublattice $\mathbb{Z}_l \times \mathbb{Z}_l$ is l^2. Therefore, there are $4l^2$ possible edges in $\mathbb{Z}_l \times \mathbb{Z}_l$. However, as each edge belongs to two faces of the sub-lattice, just one half of the edges of the sub-lattice $\mathbb{Z}_l \times \mathbb{Z}_l$ effectively constitutes the qubits associate to the codeword length. Then, we have $n = 2l^2$ qubits. In the case of two qubits to be encoded, one is related to the sub-lattice $\mathbb{Z}_l \times \mathbb{Z}_l$ and the other one is related to the dual sub-lattice of $\mathbb{Z}_l \times \mathbb{Z}_l$. Thus, $k = 2$. The minimum distance of the code is associated with the smallest number of edges to be covered between the edges representing each coset $(d = l)$.

In [1], Bombin and Martin-Delgado used another regular sub-lattice on the torus to define the homological quantum codes. The sub-lattice is based on the Lee sphere. It can be shown that l Lee spheres with radius r, in the two dimensional case, may be used to tessellate the $l \times l$ torus, where $l = 2r^2 + 2r + 1$ [4]. The lattices used in [1] demand a half of the number of qubits and it keeps the same properties of the original Kitaev's code, for instance, the vertex and face operators acting on four qubits. Figure 2 shows both systems of sub-lattices. The torus is a result of the quotient $\mathbb{Z}^2/l\mathbb{Z}^2$, where $l = 2r^2 + 2r + 1$. These sub-lattices provide codes with parameters $[[d^2 + 1, 2, d]]$.

The tiling produces a sub-lattice of the torus, where the code is defined. The polyomino is a fundamental region with area l of a sub-lattice of the $l \times l$ torus. However, the error patterns corrected by such a code depend on the shape of the polyomino, [4].

Figure 2: The toric code and the homological quantum code for $d = 3$

The polyomino used in the construction of codes in [1], when $r = 1$, is generated by taking a representative of the codeword and displacing the component by one unity either to the left or to the right, or upward or downward. The resulting figure is a Lee sphere with radius one, with its center in the representative of the codeword in the Lee metric (Figure 2).

A Lee sphere of radius r, in two dimensions, consists of $l = 2r^2 + 2r + 1$ cells [4]. The close-packed codes of Lee radius r, where $l = 2r^2 + 2r + 1$, are in a correspondence with a tiling of the $l \times l$ torus with polyominoes, or equivalently, Lee spheres of radius r.

3. A New Class of Toric Codes

Quadratic forms are a useful tool to study lattices under the arithmetic point of view. In the construction being proposed, polyominoes are employed to tessellate the $l \times l$ torus. The polyominoes, as well as the representatives of the code, will be determined by the solutions of a quadratic form.

In particular, the square lattice $\mathbb{Z}^2$ has generator matrix $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and the corresponding quadratic form is $x^2 + y^2$.

Let us consider the quadratic form $x^2 + y^2 = N$, where $x, y, N \in \mathbb{N}$. The polyomino with area N, where the code will be defined, must tessellate the $N \times N$ torus, that is, N squares of the lattice will constitute the fundamental region of the sub-lattice on the torus. The solutions $\{\pm x, \pm y\}$ of $x^2 + y^2 = N$, for a given N, can be arranged in the following matrix $A = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$, where the rows define a path within the cells of the lattice. The representative of the sub-lattice is thus determined: x unities horizontally and y unities vertically; y unities horizontally in the opposite way (indicated by the minus sign) and x unities vertically. Thus, N representatives of the sub-lattice are obtained and each polyomino which contains only one representative has area N. The code

Figure 3: Two representations of code $[[16, 2, 4]]$

may be characterized as cosets from the quotient group $\mathbb{Z}^2/N\mathbb{Z}^2$.

We may derive the values of the parameters n and k of a topological toric code for each one of the polyominoes. The codelength is given by the number of edges of the polyomino minus half the number of the edges in the border of the polyomino, since these edges belong to the two faces of the lattice simultaneously. The number of encoded qubits is 2, since we are on the torus. The minimum distance is the smallest path between the two marks X, shown in Figure 3. The distance may be defined as the sum of the values in the first row of matrix A, $d = x + y$.

Considering the particular case of the quadratic form given by $N = x^2 + x^2$, we have $A = \begin{bmatrix} x & x \\ -x & x \end{bmatrix}$. Hence, the parameters of the code are $d = 2x$, $n = 2N = 4x^2 = d^2$ and $k = 2$. This construction generates codes with parameters $[[d^2, 2, d]]$. Several polyomino shapes may be considered as the fundamental region of the sub-lattice where the code is defined. In Figure 3 two shapes are given which originate the code $[[16, 2, 4]]$ on the 8×8 torus. These two polyomino shapes may be generalized for all codes $[[d^2, 2, d]]$. Therefore, the codes $[[d^2, 2, d]]$ correspond to the tessellations of the $N \times N$ torus with polyomino shapes whose area is $N = 8, 18, 32, 50, 72, \ldots$. It must be observed that different polyomino shapes determine codes with the same parameters, since the value of N is the same.

As mentioned previously, the error correction pattern depends on the polyomino shape. In particular, the shapes used for the codes $[[d^2, 2, d]]$ have a smaller number of symmetries than those associated with the Lee spheres. This suggests that these codes may be used in noise channels with unequal variances.

4. Conclusions

A new class of toric codes with parameters $[[d^2, 2, d]]$ was presented. These codes have better parameters than the already known toric codes. This class of codes suggests a more general construction of codes defined on the torus.

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