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A NUMERICAL SOLVING OF A NON LINEAR INTEGRAL EQUATION OF HAMMERSTEIN TYPE

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1. Introduction

Among the founders of the linear integral equation's theory we will mention, beside Volterra and Fredholm, also David Hilbert (1862-1943) and Erhart Schmidt (1876-1958).

It is important to remember the Romanian mathematician Traian Lalescu, who, in his doctorate thesis entitled "Sur l'equation de Volterra" sustained in Paris in 1908 used for the first time the succesive approximation method for the integration of Volterra equation. He also wrote the first book from the entire world about integral equations published in Bucharest in 1911 in romanian language and again in Paris one year later in french.

Nonlinear integral equations are a kind of equations in which the unknown function y can be found under the sign of the integral in some complicated way.

For example: $y(s) - \int_0^1 g(s,t)\,[y(t)]^2\,dt = h(s)$.

1.1. Hammerstein Type Equations

A. Hammerstein studied nonlinear integral equations looking like

$$\Psi(x) + \int_0^1 K(x,y)\, f[y, \Psi(y)]\, dy = 0.$$

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They can also be extended to on n-dimesional spaces but this not involve fundamental differences.

We will use the following important hypothesis:

— Fredholm Theorem is true for the linear integral equation having the kernel K.

— The kernel K is symmetrical: $K(x,y) = K(y,x)$.

— The kernel is positive which means that all its eigenvalues are of the positive kind.

If these conditions are fulfilled we can say that the integral equation really is of the Hammerstein type. Hammerstein used the fact that

$$\Psi(x) = \int_0^1 K(x,y)\, g(y)\, dy \quad \text{with} \quad g(y) = -f[y, \Psi(y)] \;.$$

If it exists $g(y) \in L^2$ then $\Psi(x)$ can be represented like an uniform convergent series having the form

$$\Psi(x) = \sum_{m=1}^{\infty} c_m \Phi_m(x) \;.$$

Using $\Phi_1(x), \Phi_2(x), ...$ like the ortonormalised eigenvalues for the kernel $K(x,y)$ corresponding to the eigenvalues $\lambda_1, \lambda_2, ...$ and $c_1, c_2, ...$ being unknown constants.

Then because:

$$c_m = \int_0^1 \Psi(x)\, \Phi_m(x)\, dx = -\int_0^1 \Phi_m(x)\, dx \int_0^1 K(x,y)\, f[y, \Psi(y)]\, dy$$

$$= -\int_0^1 f[y, \Psi(y)]\, dy \int_0^1 K(x,y)\, \Phi_m(x)\, dx = -\frac{1}{\lambda} \int_0^1 f[y, \Psi(y)]\, \Phi_m(y)\, dy \,,$$

the problem of solving the given equation is equivalent with the one of solving an infinite system of equations having an infinite number of unknowns:

$$c_m = -\frac{1}{\lambda_m} \int_0^1 f\left[y, \sum_{h=1}^{\infty} c_h \Phi_h(y)\right] \Phi_m(y)\, dy \,, \qquad m = 1, 2, ... \,.$$

It is normal now to consider the approximate solution

$$\Psi_n(x) = \sum_{m=1}^{\infty} c_{n,m} \Phi_m(x) \;,$$

with the constants $c_{n,1}, c_{n,2}, ...$ having to verify the system with n equations and

n unknowns:

$$c_{n,m} = -\frac{1}{\lambda}\int_0^1 f\left[y, \sum_{h=1}^{n} c_{n,h}\Phi_h(y)\right]\Phi_m(y)\,dy\,, \quad m = 1, 2, \ldots .$$

We will ask about the existence of the solutions for this system. Hammerstein showed very nicely that the system considered have at least one solution by demonstrating that the function $f(x,u)$ is a continuous one and verifies a condition of the following type:

$$|f(x,u)| \leq C_1 |u| + C_2\,,$$

with C_1 and C_2 are two positive constants and C_1 is less then the first eigenvalue λ_1 of the positive kernel $K(x,y)$. Evan if the relation can be relaxed Hammerstein demonstrated that the condition $C_1 \prec \lambda_1$ cannot be generally enlarged.

For showing this Hammerstein used the continuous function

$$H(x_1, x_2, ..., x_n) = \sum_{m=1}^{n} \lambda_m x_m^2 + 2\int_0^1 F\left[y, \sum_{h=1}^{n} x_h\Phi_h(y)\right]dy\,,$$

with

$$F(y,u) = \int_0^u f(y,v)\,dv\,,$$

and $H(x_1, x_2, ..., x_n)$ is a function having partial derivations closely related with the solutions for the system because:

$$\frac{1}{2\lambda_m}\frac{\vartheta H}{\vartheta x_m} = x_m + \frac{1}{\lambda_m}\int_0^1 f\left[y, \sum_{h=1}^{n} x_h\Phi_h(y)\right]\Phi_m(y)\,dy.$$

2. Dead Water Theory

The end of the 19-th century and the beginning of the 20-th century represented for the flow mechanics a period of extreme intense investigations. These generated important works in this field of activity.

Among them is the cavity theory whom origin is in the works of H. Helmholtz (1868) and G. Kirchhoff (1869) a theory elaborated with the purpose to explain the D‘Alembert paradox.

D‘Alembert paradox represents the contradiction between the theoretical result saying that during a straight and uniform moving of a body through an ideal fluid will be no resistance from the fluid and the experimental observation that the resistance exists.

Helmholtz created a mathematic model and so he created a theory which became an important one usually referred as dead water theory. In this theory there are some nonlinear integral equations. One of those being of Hammerstein type is solved here using some numerical methods.

So the purpose is to solve the following equation:

$$T\left(t\right)=\frac{\lambda}{\pi}\int_{0}^{\pi}e^{-T(\sigma)}\ln\left|\frac{\sin\frac{t+\sigma}{2}}{\sin\frac{t-\sigma}{2}}\right|\left(1+\sin\sigma\right)r\left(\sigma\right)\sin\sigma d\sigma.$$

The function which we must integrate has a logarithmic singularity and this one is a week singularity.

We will rewrite the integral equation as:

$$T\left(t\right)=\frac{\lambda}{\pi}\int_{0}^{\pi}e^{-T(\sigma)}\ln\left|\frac{\left(t-\sigma\right)\sin\frac{t+\sigma}{2}}{\sin\frac{t-\sigma}{2}}\right|\left(1+\sin\sigma\right)r\left(\sigma\right)\sin\sigma d\sigma-\frac{\lambda}{\pi}$$
$$\times\int_{0}^{\pi}\left[e^{-T(\sigma)}\left(1+\sin\sigma\right)r\left(\sigma\right)\sin\sigma-e^{-T(\sigma)}\left(1+\sin t\right)r\left(t\right)\sin t\right]\ln\left|t-\sigma\right|d\sigma$$
$$-\frac{\lambda}{\pi}e^{-T(\sigma)}\left(1+\sin t\right)r\left(t\right)\sin t\int_{0}^{\pi}\ln\left|t-\sigma\right|d\sigma.$$

The first and the second integrals can be calculated using trapezium method and the third one will be analytically calculated.

We will consider in $[0,\pi]$ the nods $\{t_0,t_1,...,t_n\}$ with $t_i=\frac{i}{n}\pi$, $i=0,\ldots,n$. Using trapezium method

$$\int_{0}^{\pi}f\left(\sigma\right)d\sigma=\frac{\pi}{2n}\left[f\left(t_0\right)+2\sum_{i=1}^{n-1}f\left(t_i\right)+f\left(t_n\right)\right],$$

and because $\sin t_0=\sin t_n=0$

$$T\left(t_j\right)=\frac{\lambda}{n}\left(\sum_{i=1,i\neq j}^{n-1}e^{-T(t_i)}\ln\left|\frac{\left(t-\sigma\right)\sin\frac{t_i+t_j}{2}}{\sin\frac{t_j-t_i}{2}}\right.\right.$$
$$\left.\left.\right|\left(1+\sin t_i\right)r\left(t_i\right)\sin t_i+e^{-T(t_j)}\ln\left(2\sin t_j\right)\left(1+\sin t_j\right)r\left(t_j\right)\sin t_j\right)-\frac{\lambda}{n}$$
$$\times\sum_{i=1,i\neq j}^{n-1}\left[e^{-T(t_j)}\left(1+\sin t_i\right)r\left(t_i\right)\sin t_i-e^{-T(t_j)}\left(1+\sin t_j\right)r\left(t_j\right)\sin t_j\right]$$
$$\times\ln\left|t_i-t_j\right|-\frac{\lambda}{n}e^{-T(t_j)}\left(1+\sin t_j\right)r\left(t_j\right)\sin t_j\left[\left(\pi-t_j\right)\ln\left(\pi-t_j\right)+t_j\ln t_j-\pi\right].$$

Therefore, for $j=1,2,\ldots n-1$:

$$T(t_j) = \frac{\lambda}{n} \sum_{i=1, i \neq j}^{n-1} e^{-T(t_i)}$$
$$\times \ln \left| \frac{\sin \frac{t_i + t_j}{2}}{\sin \frac{t_j - t_i}{2}} \right| (1 + \sin t_i) \, r(t_i) \sin t_i + \frac{\lambda}{n} e^{-T(t_j)} (1 + \sin t_j) \, r(t_j) \sin t_j$$
$$\times \left[\frac{1}{n} \ln(2 \sin t_j) + \sum_{i=1, i \neq j}^{n-1} \ln |t_i - t_j| - \frac{\pi - t_j}{\pi} \ln(\pi - t_j) - \frac{t_j}{\pi} \ln t_j + 1 \right].$$

So, finally, we have to solve the algebraic system:

$$T_j = \lambda \sum_{i=1}^{n-1} w_{ji} e^{-T_i}, \qquad (*)$$

with $T_i = T(t_i)$, $i = 1, ..., n$ and

$$w_{ji} = \frac{1}{n} \ln \left| \frac{\sin \frac{t_j + t_i}{2}}{\sin \frac{t_j - t_i}{2}} \right| (1 + \sin t_i) \, r(t_i) \sin t_i \succeq 0, \; i \neq j,$$

respectively

$$w_{jj} = (1 + \sin t_j) \, r(t_j) \sin t_j$$
$$\times \left[\frac{1}{n} \ln(2 \sin t_j) + \frac{1}{n} \sum_{i=1, i \neq j}^{n-1} \ln |t_i - t_j| - \frac{\pi - t_j}{\pi} \ln(\pi - t_j) - \frac{t_j}{\pi} \ln t_j + 1 \right] \succeq 0.$$

The intention is to estimate the differences:

$$\left| \lambda \sum_{i=1}^{n-1} w_{ji} e^{-T_i} - \lambda \sum_{i=1}^{n-1} w_{ji} e^{-S_i} \right| \leq \lambda \sum_{i=1}^{n-1} w_{ji} \left| e^{-T_i} - e^{-S_i} \right| \leq \lambda \sum_{i=1}^{n-1} w_{ji} |T_i - S_i|.$$

Because of the definition of w_{ji} we say that:

$$\lambda \sum_{i=1}^{n-1} w_{ji} \cong \frac{1}{\pi} \int_0^\pi \ln \left| \frac{\sin \frac{t_j + \sigma}{2}}{\sin \frac{t_j - \sigma}{2}} \right| (1 + \sin \sigma) \, r(\sigma) \sin \sigma d\sigma$$
$$\leq \frac{\lambda}{\pi} \int_0^\pi \ln \left| \frac{\sin \frac{t_j + \sigma}{2}}{\sin \frac{t_j - \sigma}{2}} \right| (1 + \sin \sigma) \sin \sigma d\sigma$$
$$= \frac{\lambda}{\pi} \int_0^\pi \sum_{m \succeq 1} \frac{1}{m} \sin m t_j \sin m\sigma \, (2 \sin \sigma + 1 - \cos 2\sigma) \, d\sigma$$
$$= \frac{\lambda}{\pi} \sin t_j \int_0^\pi 2 \sin \sigma d\sigma + \frac{\lambda}{\pi} \int_0^\pi \sum_{m \succeq 1} \frac{1}{m} \sin m t_j \sin m\sigma d\sigma$$

$$= \lambda \sin t_j + \frac{\lambda}{\pi} \sum_{m \succeq 1} \sin m t_j \cos m \sigma_\pi^0 = \lambda \sin t_j + \frac{\lambda}{\pi} \sum_{m \succeq 1} \sin m t_j \left(1 - (-1)^m\right)$$

$$\leq \lambda + \frac{\lambda}{\pi} \sum_{n \succeq 0} \frac{2}{(2n+L)^2} \leq \lambda + \frac{\lambda}{\pi} \left(2 + \sum_{n \succeq 1} \frac{2}{(2n+1)(2n-1)} \right)$$

$$= \lambda + \frac{\lambda}{\pi} \left(2 + \sum_{n \succeq 1} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) \right) = \frac{\lambda (3+\pi)}{\pi}.$$

So we can say that the system (*) has only one solution for $0 \prec \lambda \prec \frac{\pi}{3+\pi}$ and this solution can be found using the successive approximation method.

3. Conclusions

Some fields of activities, for example the one regarding the flows studies, can create, after being modeled in a mathematical way, some nonlinear integrals and their solving is of equal interest for mathematicians and also for the ones working in more practical fields of activity.

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