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GROWTH OF ENTIRE FUNCTION SATISFYING DIFFERENTIAL EQUATION

Sanjib Kumar Datta[1 §], Arup Ratan Das[2]

[1]Department of Mathematics
University of North Bengal
Raja Rammohunpur, District Darjeeling
PIN Code 734013, West Bengal, INDIA
e-mail: sanjib_kr_datta@yahoo.co.in

[2]Pathardanga Osmania High Madrasah
P.O.-Panchgram, P.S. Nabagram, District Murshidabad
PIN Code 742184, West Bengal, INDIA
e-mail: arupratan2007@rediffmail.com

Abstract: In this paper we study the comparative growth of composite entire functions which satisfy second order linear differential equations.

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1. Introduction, Definitions and Notations

Let f and g be two transcendental entire functions defined in the open complex plane $\mathbb{C}$. It is well known that $\lim_{r\to\infty} \frac{M(r,f\circ g)}{M(r,f)} = \lim_{r\to\infty} \frac{M(r,f\circ g)}{M(r,g)} = \infty$. Clunie [3] discussed on the behaviour of $\frac{\log M(r,f\circ g)}{\log M(r,f)}$ and $\frac{\log M(r,f\circ g)}{\log M(r,g)}$ as $r \to \infty$. Song and Yang [11] worked on $\frac{\log^{[2]} M(r,f\circ g)}{\log^{[2]} M(r,f)}$ and $\frac{\log^{[2]} M(r,f\circ g)}{\log^{[2]} M(r,g)}$ as $r \to \infty$, where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

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§Correspondence address: 25, School Road, Kalianibash, Barrackpore, P.O.: Nonachandanpukur, Dist.: 24 Parganas (North), P.S.: Titagarh, PIN CODE: 743102, West Bengal, INDIA

Replacing maximum modulus functions by Nevanlinna's characteristic functions, Clunie [3] proved for any two transcendental entire functions defined in the open complex plane $\mathbb{C}$,

$$\lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,f)}=\infty \text{ and } \lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,g)}=\infty.$$

Singh [12] proved some comparative growth properties of $\log T(r,f\circ g)$ and $T(r,f)$. He [12] also raised the problem of investigating the comparative growth of $\log T(r,f\circ g)$ and $T(r,g)$ and some results on the comparative growth of $\log T(r,f\circ g)$ and $T(r,g)$ are proved in [8].

Since $M(r,f)$ and $M(r,g)$ are increasing functions of r, Singh and Baloria [13] asked whether for any two entire functions f,g and for sufficiently large $R=R(r)$, $\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f\circ g)}{\log^{[2]}M(R,f)}<\infty$ and $\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f\circ g)}{\log^{[2]}M(R,g)}<\infty$. Singh and Baloria [13], Lahiri and Sharma [9], Liao and Yang [10] worked on this question. We do not explain the standard notations and definitions of the theory of entire and meromorphic functions because those are available in [14] and [6].

Let f be an entire function defined in the open complex plane $\mathbb{C}$. Kwon [7], Chen [4], Chen and Yang [5] studied the growth of entire functions satisfying the second order linear differential equation. The purpose of this paper is to study on the growth of the solutions $f\not\equiv 0$ of the second order linear differential equation

$$f^{''}+A(z)f^{'}+B(z)f=0,$$

where $A(z)$ and $B(z)\not\equiv 0$ are entire functions.

The following definitions are well known.

Definition 1. The order ρ_f and lower order λ_f of an entire function f is defined as

$$\rho_f=\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r} \text{ and } \lambda_f=\liminf_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r}.$$

If f is meromorphic, one can easily verify that,

$$\rho_f=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r} \text{ and } \lambda_f=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

Definition 2. The type σ_f of an entire function f is defined as follows

$$\sigma_f=\limsup_{r\to\infty}\frac{\log M(r,f)}{r^{\rho_f}}, 0<\rho_f<\infty.$$

When f is meromorphic, then

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}, 0 < \rho_f < \infty.$$

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. (see [1]) *If f is meromorphic and g is entire then for all sufficiently large values of r,*

$$T(r, f\circ g) \le \{1+o(1)\}\frac{T(r,g)}{\log M(r,g)} T(M(r,g), f).$$

Lemma 2. (see [2]) *Let f be meromorphic and g be entire and also suppose that $0 < \mu \le \rho_g \le \infty$. Then for a sequence of values of r tending to infinity,*

$$T(r, f\circ g) \ge T(\exp(r^\mu), f).$$

3. Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f be an entire function satisfying the second order linear differential equation $f^{''} + A(z)f^{'} + B(z)f = 0$, where $A(z)$ and $B(z) \not\equiv 0$ are entire functions. If (i) ρ_A and ρ_B are both finite and $(ii)\lambda_f$ is positive, then for each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty} \frac{\{\log T(r, A\circ B)\}^{1+\alpha}}{\log T(\exp(r^{p^{'}}), f)} = 0 \quad if \quad p^{'} > (1+\alpha)\rho_g.$$

Proof. If $1+\alpha \le 0$, the theorem is trivial. So we take $1+\alpha > 0$.

Since $T(r,B) \le \log^+ M(r,B)$, by Lemma 1, we get for all sufficiently large values of r,

$$T(r, A\circ B) \le \{1+o(1)\}T(M(r,B), A)\,,$$

$$\log T(r, A\circ B) \le \log\{1+o(1)\} + \log T(M(r,B), A)\,,$$

$$\log T(r, A\circ B) \le \{1+o(1)\} + (\rho_A+\epsilon) r^{\rho_B+\epsilon}\,,$$

$$\log T(r, A\circ B) \le r^{\rho_B+\epsilon}\{(\rho_A+\epsilon) + o(1)\}. \tag{1}$$

Again for all sufficiently large values of r we obtain that

$$\log T(\exp(r^{p'}), f) \geq (\lambda_f - \epsilon)\log(\exp(r^{p'}))$$

$$\log T(\exp(r^{p'}), f) \geq (\lambda_f - \epsilon) r^{p'}. \tag{2}$$

Now from (1) and (2) we get for all large values of r,

$$\frac{\{\log T(r, A \circ B)\}^{1+\alpha}}{\log T(\exp(r^{p'}), f)} \leq \frac{r^{(\rho_B+\epsilon)(1+\alpha)}\{(\rho_A + \epsilon) + o(1)\}^{1+\alpha}}{(\lambda_f - \epsilon) r^{p'}} \tag{3}$$

from which the theorem follows because we can choose ϵ such that $0 < \epsilon < \min\{\lambda_A, \frac{p}{1+\alpha} - \rho_B\}$.

This proves the theorem. □

Remark 1. If we take $\rho_f > 0$ instead of $\lambda_f > 0$ and the other conditions remain the same, the conclusion of Theorem 1 remains valid with 'limit' replaced by 'limit inferior'.

Lahiri [8] proved the following theorem on the comparative growth of $\log T(r, f \circ g)$ and $T(r, f)$.

Theorem A. *Let f and g be two non-constant entire functions such that $\lambda_g < \lambda_f \leq \rho_f < \infty$.*

Then

$$\liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, f)} = 0.$$

In the line of Theorem A we may prove the following theorem.

Theorem 2. *If f is an entire function satisfying the second order linear differential equation $f'' + A(z)f' + B(z)f = 0$ with $A(z)$ and $B(z) \not\equiv 0$ as entire functions and (i)ρ_A and λ_B are both finite, i.e.$\rho_A < \infty$, $\lambda_B < \infty$ and (ii)$\lambda_B < \lambda_f$, then*

$$\liminf_{r \to \infty} \frac{\log\{T(r, A \circ B)\log M(r, B)\}}{T(r, f)} = 0.$$

Proof. In view of Lemma 1, for all sufficiently large values of r, we have,

$$T(r, A \circ B)\log M(r, B) \leq \{1 + o(1)\}T(r, B).T(M(r, B), A),$$

$$\log\{T(r, A \circ B)\log M(r, B)\} \leq \log\{1 + o(1)\} + \log T(r, B) + \log T(M(r, B), A).$$

From above we get for a sequence of values of r tending to infinity,

$$\log\{T(r, A \circ B)\log M(r, B)\}$$

$$\begin{aligned} &\leq o(1) + (\lambda_B + \epsilon)\log r + (\rho_A + \epsilon)\log M(r, B) \\ &\leq o(1) + (\lambda_B + \epsilon)\log r + (\rho_A + \epsilon) r^{\lambda_B + \epsilon}. \qquad (4) \end{aligned}$$

Again for all sufficiently large values of r, we obtain that

$$T(r, f) \geq r^{(\lambda_f - \epsilon)}. \qquad (5)$$

In view of (4) and (5) we get for a sequence of values of r tending to infinity,

$$\frac{\log\{T(r, A \circ B)\log M(r, B)\}}{T(r, f)} < \frac{o(1) + (\lambda_B + \epsilon)\log r + (\rho_A + \epsilon) r^{\lambda_B + \epsilon}}{r^{(\lambda_f - \epsilon)}}. \qquad (6)$$

Now as $\lambda_B < \lambda_f$,we can choose $\epsilon (> 0)$ in such a way that $\lambda_B + \epsilon < \lambda_f - \epsilon$ and the theorem follows from (6). □

Remark 2. If we take $\rho_B < \infty$ instead of $\lambda_B < \infty$ and $\rho_B < \lambda_f$ instead of $\lambda_B < \lambda_f$ and the other conditions remain the same, the conclusion of Theorem 2 remains valid with 'limit inferior' replaced by 'limit' as we see in the following theorem.

Theorem 3. *Let f be an entire function satisfying the second order linear differential equation $f'' + A(z)f' + B(z)f = 0$ where $A(z)$ and $B(z) \not\equiv 0$ are entire functions. If (i) ρ_A and ρ_B are both finite i.e. $\rho_A < \infty, \rho_B < \infty$ and $(ii)\rho_B < \lambda_f$, then*

$$\lim_{r \to \infty} \frac{\log\{T(r, A \circ B)\log M(r, B)\}}{T(r, f)} = 0.$$

Proof. In view of Lemma 1, we get for all sufficiently large values of r,

$$\begin{aligned} \log\{T(r, A \circ B)\log M(r, B)\} &\leq \log\{1 + o(1)\} + \log T(r, B) \\ &\quad + \log T(M(r, B), A) \\ &\leq o(1) + (\rho_B + \epsilon)\log r \\ &\quad + (\rho_A + \epsilon)\log M(r, B) \\ &\leq o(1) + (\rho_B + \epsilon)\log r + (\rho_A + \epsilon) r^{\rho_B + \epsilon}. \qquad (7) \end{aligned}$$

Now combining (5) and (7) it follows for all sufficiently large values of r,

$$\begin{aligned} &\frac{\log\{T(r, A \circ B)\log M(r, B)\}}{T(r, f)} \\ \leq\ &\frac{o(1) + (\rho_B + \epsilon)\log r + (\rho_A + \epsilon) r^{\rho_B + \epsilon}}{r^{(\lambda_f - \epsilon)}}. \qquad (8) \end{aligned}$$

As $\rho_B < \lambda_f$ we can choose $\epsilon (> 0)$ in such a manner that $\rho_B + \epsilon < \lambda_f - \epsilon$ and thus the theorem follows from (8). □

Theorem 4. *If f be an entire function satisfying the second order linear differential equation $f'' + A(z)f' + B(z)f = 0$ with $A(z)$ and $B(z) \not\equiv 0$ as entire* *functions and* $(i) 0<\lambda_A \leq \rho_A<\infty,(ii) 0<\lambda_B \leq \rho_B<\infty,(iii) \sigma_B<\infty$ *and* (iv) $0<\lambda_f<\infty$ *then*

$$\limsup_{r \rightarrow \infty} \frac{\log T(r, A \circ B)}{\log T\left(\exp \left(r^{\rho_B}\right), f\right)}<\infty .$$

Proof. Since $T(r, B) \leq log^{+} M(r, B)$, by Lemma 1 we get for all sufficiently large values of r,

$$\log T(r, A \circ B) \leq \log T(M(r, B), A)+o(1) .$$

So we obtain that

$$\begin{aligned}
\limsup_{r \rightarrow \infty} \frac{\log T(r, A \circ B)}{\log T\left(\exp \left(r^{\rho_B}\right), f\right)} & \leq \limsup_{r \rightarrow \infty} \frac{\log T(M(r, B), A)+o(1)}{\log T\left(\exp \left(r^{\rho_B}\right), f\right)} \\
& \leq \limsup_{r \rightarrow \infty} \frac{\log T(M(r, B), A)}{\log M(r, B)} . \limsup_{r \rightarrow \infty} \frac{\log M(r, B)}{r^{\rho_B}} . \\
& \quad \limsup_{r \rightarrow \infty} \frac{\log \left(\exp \left(r^{\rho_B}\right)\right)}{\log T\left(\exp \left(r^{\rho_B}\right), f\right)} \\
& = \rho_A \sigma_B \frac{1}{\lambda_f}<\infty .
\end{aligned}$$

This proves the theorem. □

Theorem 5. *Let f be an entire function satisfying the second order linear differential equation $f^{\prime \prime}+A(z) f^{\prime}+B(z) f=0$ where $A(z)$ and $B(z) \not\equiv 0$ are entire functions. Also let h be meromorphic and k be entire such that $\lambda_h>0$ and $\rho_B<\rho_k$. Then for every entire solution f of finite order and $\rho_A<\infty$,*

$$\limsup_{r \rightarrow \infty} \frac{T(r, h \circ k)}{\log T(r, A \circ B) T(r, f)}=\infty .$$

Proof. In view of Lemma 2, we obtain for a sequence of values of r tending to infinity,

$$T(r, h \circ k) \geq T\left(\exp \left(r^\mu\right), h\right), \; 0<\mu<\rho_k . \tag{9}$$

Again for all sufficiently large values of r,

$$\log T\left(\exp \left(r^\mu\right), h\right) \geq\left(\lambda_h-\epsilon\right) \log \left\{\exp \left(r^\mu\right)\right\}$$

$$T\left(\exp \left(r^\mu\right), h\right) \geq \exp \left\{\left(\lambda_h-\epsilon\right) r^\mu\right\} . \tag{10}$$

Now combining (9) and (10) we get for a sequence of values of r tending to infinity,

$$T(r, h \circ k) \geq \exp \left\{\left(\lambda_h-\epsilon\right) r^\mu\right\} . \tag{11}$$

Since $T(r, B) \leq \log ^{+} M(r, B)$, for all sufficiently large values of r we obtain from Lemma 1

$$\log T(r, A \circ B) \leq \log\{1 + o(1)\} + \log T(M(r, B), A)$$

$$\log T(r, A \circ B) \leq \log\{1 + o(1)\} + (\rho_A + \epsilon) \log M(r, B)$$

$$\log T(r, A \circ B) \leq \{1 + o(1)\}. \exp\{(\rho_A + \epsilon) r^{\rho_B + \epsilon}\}. \tag{12}$$

Again we have for all sufficiently large values of r,

$$T(r, f) \leq r^{\rho_f + \epsilon}. \tag{13}$$

From (12) and (13) it follows for all large values of r,

$$\log T(r, A \circ B).T(r, f) \leq \{1 + o(1)\} r^{\rho_f + \epsilon}. \exp\{(\rho_A + \epsilon) r^{\rho_B + \epsilon}\}. \tag{14}$$

Combining (11) and (14) we get for a sequence of values of r tending to infinity,

$$\frac{T(r, h \circ k)}{T(r, A \circ B) T(r, f)} \geq \frac{\exp\{(\lambda_h - \epsilon) r^{\mu}\}}{\{1 + o(1)\} r^{\rho_f + \epsilon}. \exp\{(\rho_A + \epsilon) r^{\rho_B + \epsilon}\}}. \tag{15}$$

Since $\rho_B < \rho_k$, we can choose $\epsilon (> 0)$ in such a manner that

$$\rho_B + \epsilon < \mu < \rho_k. \tag{16}$$

Thus the theorem follows from (15) and (16). □

Theorem 6. *If f be an entire function satisfying the second order linear differential equation $f^{''} + A(z) f^{'} + B(z) f = 0$ with $A(z)$ and $B(z) \not\equiv 0$ as entire functions and $\rho_f < \infty$, $\lambda_{A \circ B} = \infty$, then for every $\alpha (> 0)$,*

$$\lim_{r \to \infty} \frac{\log T(r, A \circ B)}{\log T(r^{\alpha}, f)} = \infty.$$

Proof. If possible, let there exists a constant β such that for a sequence of values of r tending to infinity,

$$\log T(r, A \circ B) \leq \beta \log T(r^{\alpha}, f). \tag{17}$$

Again we obtain for all sufficiently large values of r,

$$\log T(r^{\alpha}, f) \leq (\rho_f + \epsilon) \log r^{\alpha}$$

$$\log T(r^{\alpha}, f) \leq (\rho_f + \epsilon). \alpha \log r. \tag{18}$$

Now combining (17) and (18), we have for a sequence of values of r tending to infinity,

$$\log T(r, A \circ B) \leq \beta.(\rho_f + \epsilon) \alpha. \log r$$

$$\lambda_{A \circ B} \leq \beta. \alpha. (\rho_f + \epsilon),$$

which contradicts the condition $\lambda_{A \circ B} = \infty$. So for all sufficiently large values of r we get

$$\log T(r, A \circ B) > \beta \log T(r^{\alpha}, f),$$

from which the theorem follows. □

Corollary 1. *Under the assumptions of Theorem* 6,

$$\lim_{r\to\infty} \frac{T(r, A \circ B)}{T(r^{\alpha}, f)} = \infty.$$

Proof. By Theorem 6 we obtain for all sufficiently large values of r and for $K > 1$,

$$\log T(r, A \circ B) > K \log T(r^{\alpha}, f)$$

$$i.e., T(r, A \circ B) > \{T(r^{\alpha}, f)\}^{K},$$

from which the corollary follows. □

Theorem 7. *Let f be an entire function satisfying the second order linear differential equation $f^{''} + A(z)f^{'} + B(z)f = 0$ where $A(z)$ and $B(z) \not\equiv 0$ are entire functions. If $\rho_A < \infty, 0 < \lambda_B \leq \rho_B < \infty$ and $\lambda_f < \infty$ then*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(r, A \circ B)}{\log T(r, f)} \leq \frac{\rho_B}{\lambda_f}.$$

Proof. In view of Lemma 1 and since $T(r, B) \leq log^{+} M(r, B)$, we get for all sufficiently large values of r,

$$\begin{aligned} \log T(r, A \circ B) &\leq \log T(M(r, B), A) + \log\{1 + o(1)\} \\ \log T(r, A \circ B) &\leq (\rho_A + \epsilon) \log M(r, B) + o(1) \end{aligned}$$

$$\log T(r, A \circ B) \leq \{(\rho_A + \epsilon) + o(1)\} \log M(r, B)$$

$$\log^{[2]} T(r, A \circ B) \leq \log^{[2]} M(r, B) + o(1)$$

$$\frac{\log^{[2]} T(r, A \circ B)}{\log T(r, f)} \leq \frac{\{\log^{[2]} M(r, B) + o(1)\}}{\log r} . \frac{\log r}{\log T(r, f)}$$

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log^{[2]} T(r, A \circ B)}{\log T(r, f)} &\leq \limsup_{r\to\infty} \frac{\log^{[2]} M(r, B) + o(1)}{\log r} . \\ & \quad \limsup_{r\to\infty} \frac{\log r}{\log T(r, f)} \\ &= \rho_B . \frac{1}{\lambda_f} = \frac{\rho_B}{\lambda_f}. \end{aligned}$$

This proves the theorem. □

Theorem 8. *If f be an entire function satisfying the second order linear* *differential equation* $f^{''}+A(z)f^{'}+B(z)f=0$ *with* $A(z)$ *and* $B(z)\not\equiv 0$ *as entire functions and* $\lambda_A>0$, $0<\rho_f<\infty$ *then*

$$\limsup_{r\to\infty}\frac{\log T(r,A\circ B)}{\log T(\exp(r^{\mu}),f)}=\infty,$$

where $0<\mu<\rho_B$.

Proof. Let $0<\mu<\mu^{'}<\rho_B$. Then in view of Lemma 2, for a sequence of values of r tending to infinity, we get

$$T(r,A\circ B)\geq T(\exp(r^{\mu^{'}}),A)$$

$$\log T(r,A\circ B)\geq \log T(\exp(r^{\mu^{'}}),A)$$

$$\log T(r,A\circ B)\geq (\lambda_A-\epsilon)\log\{\exp(r^{\mu^{'}})\}$$

$$\log T(r,A\circ B)\geq (\lambda_A-\epsilon)r^{\mu^{'}}. \tag{19}$$

Again since $\rho_f<\infty$ then for $\epsilon(>0)$ and for all sufficiently large values of r,

$$\log T(\exp(r^{\mu}),f)\leq(\rho_f+\epsilon)\log\{\exp(r^{\mu})\}$$

$$\log T(\exp(r^{\mu}),f)\leq(\rho_f+\epsilon)r^{\mu}. \tag{20}$$

So combining (19) and (20) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log T(r,A\circ B)}{\log T(\exp(r^{\mu}),f)}\geq\frac{(\lambda_A-\epsilon)r^{\mu^{'}}}{(\rho_f+\epsilon)r^{\mu}}. \tag{21}$$

Since $\mu^{'}>\mu$, the theorem follows from (21). □

Corollary 2. *Under the assumptions of Theorem* 8,

$$\limsup_{r\to\infty}\frac{T(r,A\circ B)}{T(\exp(r^{\mu}),f)}=\infty,\ \ 0<\mu<\rho_B.$$

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Proof. In view of Theorem 8, we get for a sequence of values of r tending to infinity,

$$\log T(r,A\circ B)\geq K\log T(\exp(r^{\mu}),f),\quad \text{for}\quad K>1\,,$$

$$T(r,A\circ B)\geq\{T(\exp(r^{\mu}),f)\}^{K},$$

from which the corollary follows. □

Theorem 9. *Let* f *be an entire function satisfying the second order linear differential equation* $f^{''}+A(z)f^{'}+B(z)f=0$ *where* $A(z)$ *and* $B(z)\not\equiv 0$ *are entire functions. If* (i) $\rho_A<\infty,(ii)$ $0<\rho_B<\infty,(iii)$ $\sigma_B<\infty,(iv)$ $0<\rho_f<$

∞, (v) $\sigma_f > 0$ *and* (vi) $\rho_B = \rho_f$, *then*

$$\liminf_{r\to\infty} \frac{\log T(r, A\circ B)}{T(r,f)} \le \rho_A . \frac{\sigma_B}{\sigma_f}$$

Proof. In view of the inequality $T(r,B) \le \log^+ M(r,B)$ and Lemma 1 we get for all sufficiently large values of r,

$$T(r, A\circ B) \le \{1+o(1)\}T(M(r,B), A)$$

$$\log T(r, A\circ B) \le \log\{1+o(1)\} + \log T(M(r,B), A)$$

$$\log T(r, A\circ B) \le o(1) + (\rho_A + \epsilon)\log M(r,B)$$

$$\liminf_{r\to\infty} \frac{\log T(r, A\circ B)}{T(r,f)} \le (\rho_A + \epsilon)\liminf_{r\to\infty} \frac{\log M(r,B)}{T(r,f)}. \tag{22}$$

From the definition of type of an entire function it follows for all sufficiently large values of r,

$$\log M(r,B) \le (\sigma_B + \epsilon) r^{\rho_B}. \tag{23}$$

Again we obtain for a sequence of values of r tending to infinity,

$$T(r,f) \ge (\sigma_f - \epsilon) r^{\rho_f}. \tag{24}$$

Now it follows from (23) and (24) that for a sequence of values of r tending to infinity,

$$\frac{\log M(r,B)}{T(r,f)} \le \frac{(\sigma_B + \epsilon) r^{\rho_B}}{(\sigma_f - \epsilon) r^{\rho_f}}. \tag{25}$$

As $\rho_B = \rho_f$ we get from (25) that

$$\liminf_{r\to\infty} \frac{\log M(r,B)}{T(r,f)} \le \frac{(\sigma_B + \epsilon)}{(\sigma_f - \epsilon)}.$$

Since $\epsilon(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log M(r,B)}{T(r,f)} \le \frac{\sigma_B}{\sigma_f}. \tag{26}$$

Now from (22) and (26) we obtain that

$$\liminf_{r\to\infty} \frac{\log T(r, A\circ B)}{T(r,f)} \le (\rho_A + \epsilon)\frac{\sigma_B}{\sigma_f}. \tag{27}$$

As $\epsilon(>0)$ is arbitrary, the theorem follows from (27). □

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