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CLOSED SETS AND CLOSURES IN PRETOPOLOGY

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Abstract: In this paper, we propose different concepts of closure in pretopological spaces. We also focus on modeling possibilities they offer in the field of social sciences, in comparison with tools from graph theory or topology.

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Key Words: graph, pretopology, closure

1. Introduction

Concept of closure is a fundamental one as well in graph theory as in topology; In particular when one want to detect sets such as strong connected components. In a previous paper (4), we have already shown how concept of closure can lead to generalize that of component in pretopology and can enable handling of phenomena in social sciences with a greater facility and accuracy. In view to enlighten how this closure can be an adequate tool and to handle various situations, in this paper, we develop the ways for defining a closure, properties

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they fulfil and potential applications. First, we recall basic definitions about pretopology, closure and closed sets (Section 2 and 3). In a second part, we propose new closures (Section 4) and we continue by applying these closure concepts in a social network (Section 5) before concluding.

2. Different Types of Pretopological Spaces [1]

Definition 1. Let X is a non empty set. $\mathcal{P}(X)$ denotes the family of subsets of X. We call pseudoclosure on X any mapping a from $\mathcal{P}(X)$ onto $\mathcal{P}(X)$ such as:

$$a(\emptyset) = \emptyset$$
$$\forall A \subset X, \quad A \subset a(A)\,,$$

(X, a) is then called pretopological space.

We can define 4 different types of pretopological spaces.

1. (X, a) is a $\mathcal{V}$ type pretopological space if and only if
$$\forall A \subset X, \forall B \subset X, A \subset B \Rightarrow a(A) \subset a(B)\,.$$
2. (X, a) is a $\mathcal{V}_D$ pretopological space if and only if
$$\forall A \subset X, \forall B \subset X, a(A \cup B) = a(A) \cup a(B)\,.$$
3. (X, a) is a $\mathcal{V}_s$ pretopological space if and only if
$$\forall A \subset X, a(A) = \bigcup_{x \in A} a(\{x\})\,.$$
4. (X, a), a $\mathcal{V}_D$ pretopological space, is a topological space if and only if
$$\forall A \subset X, a(a(A)) = a(A)\,.$$

Property 1. *If (X, a) is a $\mathcal{V}_s$ space, then (X, a) is a $\mathcal{V}_D$ space. If (X, a) is a $\mathcal{V}_D$ space, then (X, a) is a $\mathcal{V}$ space.*

Examples. Let X a non empty set and $\mathcal{R}$ a binary relationship defined on X. The so called "descending pretopology", denoted by a_d, is defined by mean of the following pseudoclosure:

$$\forall A \subset X, a_d(A) = \{x \in X / \mathcal{R}(x) \cap A \neq \emptyset\} \cup A$$

with $\mathcal{R}(x) = \{y \in X / x\mathcal{R}y\}$.

The so called "ascending pretopology", denoted by a_a, is defined by:

$$\forall A \subset X, a_a(A) = \{x \in X / \mathcal{R}^{-1}(x) \cap A \neq \emptyset\} \cup A$$

with $\mathcal{R}^{-1}(x) = \{y \in X / y\mathcal{R}x\}$.

The so called "ascending-descending pretopology", denoted by a_{ad}, is defined by:

$$\forall A \subset X, a_{ad}(A) = \{x \in X/\mathcal{R}(x) \cap A \neq \emptyset \wedge \mathcal{R}^{-1}(x) \cap A \neq \emptyset\} \cup A\,.$$

Descending pretopology and ascending pretopology are $\mathcal{V}_s$ ones. Instead, ascending-descending pretopology is only a $\mathcal{V}$ one.

3. Closed Subsets and Closure in a Pretopological Space [1], [2], [3]

Definition 2. Let (X, a) a pretopological space. We put:

$$\forall A \subset X, a^0(A) = A\,,$$

and

$$\forall n, n \geq 1, a^n(A) = a(a^{n-1}(A))\,.$$

Definition 3. Let (X, a) a $\mathcal{V}$ pretopological space. Let $A \subset X$. A is a closed subset if and only if $a(A) = A$. We name closure of A, the subset of X, denoted by $F(A)$, which is the smallest closed subset which contains A.

Remark. $F(A)$ is the intersect of all closed subsets which contain A. In the case where (X, a) is a "general" pretopological space (i.e. is not a $\mathcal{V}$ space, nor a $\mathcal{V}_D$ space, nor a $\mathcal{V}_s$ space, nor a topological space) the closure may not exist.

Proposition 1. *Let (X, a) a $\mathcal{V}$ space. Let $A \subset X$. If one of the two following conditions is fulfilled:*

— *X is a finite set;*

— *a is a $\mathcal{V}_s$ space;*

then:

$$\forall A \subset X, F(A) = \bigcup_{n \geq 0} a^n(A)\,.$$

4. Other Pseudoclosures and Closures in a Pretopological Space – Properties [1], [2], [3]

Definition 4. Let (X, a) a $\mathcal{V}$ pretopological space. F', the inverse of the closure generated by a, is defined by:

$$\forall A \subset X, F'(A) = \{y \in X/F(\{y\} \cap A \neq \emptyset\}\,.$$

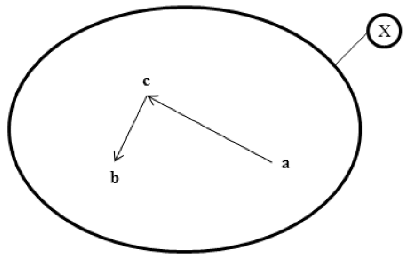

Figure 1

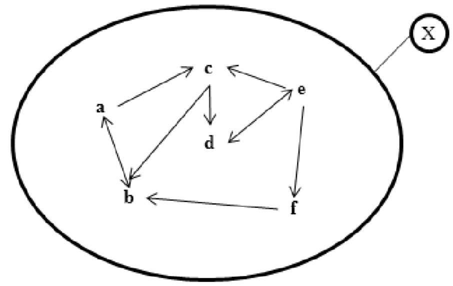

Figure 2

We note a" $= F' \circ F$ (a" is the composed of the mapping F' and F) and F" the closure according to a".

Remark. If a is of $\mathcal{V}$ type, then a^n, F, a", F" also are of $\mathcal{V}$ type and F' is of $\mathcal{V}_s$ type. If a is of $\mathcal{V}_s$ type, then a^n, F, a", F", F' also are $\mathcal{V}_s$ type.

Property 2. *1. If a is of $\mathcal{V}$ type then*

$$\forall x \in X, \forall y \in X, x \in F(\{y\}) \Leftrightarrow y \in F'(\{x\}).$$

2. If a is of $\mathcal{V}$ type then

$$\forall x \in X, \forall A \subset X, (A \cap F'(\{x\}) \neq \emptyset \Rightarrow x \in F(A)).$$

3. If a is of $\mathcal{V}_s$ type then

$$\forall x \in X, \forall A \subset X, (A \cap F'(\{x\}) \neq \emptyset \Leftrightarrow x \in F(A)).$$

Proof. If a is of $\mathcal{V}$ type, $A \cap F'(\{x\}) \neq \emptyset$ implies there exists $y \in A$ such as $y \in F'(\{x\})$ Then there exists $y \in A$, $F(\{y\}) \cap \{x\} \neq \emptyset$. Hence there exists $y \in A$, $x \in F(\{y\}) \subset F(A)$.

If a is of $\mathcal{V}_s$ type, the converse is true. Indeed, if $x \in F(A)$ then

$$x \in \bigcup_{y \in A} F(\{y\}),$$

then there exists $y \in A, x \in F(\{y\})$. So there exists $y \in A$ such as $y \in F'(\{x\})$ (Property 1). Hence $A \cap F'(\{x\}) \neq \emptyset$. In general, the converse is not true in case of a $\mathcal{V}$ type space. □

Example. Let $X = \{a, b, c\}$ endowed with the ascending descending pretopology a defined on the graph given in Figure 1. Let $A = \{a, b\}$. We get $F(A) = X$ then $c \in F(A)$ with $F'(\{c\}) = \{c\}$ hence $A \cap F'(\{c\}) = \emptyset$.

Property 3. *If a is of $\mathcal{V}$ type then $\forall x \in X, \forall y \in X, x \in F'F(\{y\})$ is* *equivalent to* $y \in F'F(\{x\})$.

Proof. $x \in F'F(\{y\})$ is equivalent to $F(\{y\}) \cap F(\{x\}) \neq \emptyset$ and $y \in F'F(\{x\})$ is equivalent to $F(\{y\}) \cap F(\{x\}) \neq \emptyset$. □

Property 4. *If* a *is of* $\mathcal{V}$ *type then* $\forall A \subset X, \forall x \in X, A \cap F'F(\{x\}) \neq \emptyset \Rightarrow x \in F'F(A)$. *If* a *is of* $\mathcal{V}_S$ *type then* $\forall A \subset X, \forall x \in X, A \cap F'F(\{x\}) \neq \emptyset \Leftrightarrow x \in F'F(A)$.

Proof. If a is of $\mathcal{V}$ type, $A \cap F'F(\{x\}) \neq \emptyset \Rightarrow \exists y \in A, x \in F'F(\{y\})$ (Property 3) and then $x \in F'F(A)$.

If a is of $\mathcal{V}_S$ type, the converse is true. Indeed, if $x \in F'F(A)$ then $F(A) \cap F(\{x\}) \neq \emptyset$ hence $F(\{x\}) \cap \bigcup_{y \in A} F(\{y\}) \neq \emptyset$ so there exists y, $y \in A, F(\{y\}) \cap F(\{x\}) \neq \emptyset$ and there exists $y \in A$ such as $y \in F'F(\{x\})$ and at last $A \cap F'F(\{x\}) \neq \emptyset$.

In general, the converse is not true in case of a $\mathcal{V}$ type space. □

Example. Let us return to the above example. We get $F'F(A) = X$ then $c \in F'F(A)$ with $F'F(\{c\}) = \{c\}$ hence $A \cap F'F(\{c\}) = \emptyset$.

Property 5. *If* a *is of* $\mathcal{V}_s$ *type then* $\forall x \in X, \forall y \in X, x \in (F'F)^n(\{y\}) \Leftrightarrow y \in (F'F)^n(\{x\})$.

Proof. The property is true for $n = 1$ according to Property 3. Let us suppose that it is true for n: If $x \in (F'F)^{n+1}(\{y\})$ then $x \in F'F(F'F)^n(\{y\}$ hence $F'F(\{x\}) \cap (F'F)^n(\{y\}) \neq \emptyset$ (Property 4). That implies there exists $z \in (F'F)^n(\{y\})$ such as $z \in F'F(\{x\})$ so there exists z such as $y \in (F'F)^n(\{z\})$ and such as $z \in F'F(\{x\})$ (true for n) and then $y \in (F'F)^n(\{z\}) \subset (F'F)^n F'F(\{x\})$ and at last $y \in (F'F)^{n+1}(\{x\})$. Conversely, according to the same way.

The remark is not true if a is of $\mathcal{V}$ type. □

Example. Let $X = \{a, b, c, d, e, f\}$ endowed with the ascending descending pretopology a defined on the Figure 2.

We get $F(\{a\}) = \{a, b, c\}$ hence $F'F(\{a\}) = \{a, b, c, d, e\}$ and $FF'F(\{a\}) = X$ hence $f \in F'FF'F(\{a\})$ but $F(\{f\}) = \{f\}$ and $F'F(\{f\}) = \{f\}$ hence $F'FF'F(\{f\}) = \{f\}$. Finally, a does not belong to $F'F(F'F(\{f\}))$.

Property 6. *If* a *is of* $\mathcal{V}_S$ *type then* $\forall x \in X, \forall y \in X, x \in F"(\{y\})$ *is equivalent to* $y \in F"(\{x\})$.

Proof. If $x \in F"(\{y\})$ then $x \in \bigcup_{n \geq 0} (F'F)^n(\{y\})$ (Proposition 1) then there exists n such as $x \in (F'F)^n(\{y\})$ so there exists n such as $y \in (F'F)^n(\{x\})$ (according to Property 5) and then $y \in \bigcup_{n \geq 0} (F'F)^n(\{x\})$ so $y \in F"(\{x\})$. Conversely, according to the same way. □

5. Closures in a Network [1], [2], [3]

In social sciences, one defines a network as a family of valued or binary relationships on a given population. In the wider framework of pretopology, we can generalize this definition by assuming that a network is a family of pretopologies on a given set. In a formal way, we can put:

Definition 5. Let X a non empty set. Let I a countable family of indices. The family $\{(X, a_i), i \in I\}$ of pretopological spaces is a network on X.

Let us now define an analytical framework for the network as defined. Three possibilities are proposed and studied in the point of view of closures: composition, union and intersect of pretopologies.

Definition 6. For any pretopologies a_1 and a_2 defined on X, for any subset A of X, we define the three following mappings:

— $(a_1 \cup a_2)(A) = a_1(A) \cup a_2(A)$ [union of pretopologies].

— $(a_1 \cap a_2)(A) = a_1(A) \cap a_2(A)$ [intersect of pretopologies].

— $(a_1 \bigodot a_2)(A) = a_1(a_2(A))$ [composition of pretopologies].

Remark. Let $\mathcal{P}_X$ the set of pretopologies on X. Let $\mathcal{P}_X(V)$ the set of $\mathcal{V}$ type pretopologies on X. Let $\mathcal{P}_X(V_D)$ the set of $\mathcal{V}_D$ type pretopologies on X. Let $\mathcal{P}_X(V_s)$ the set of $\mathcal{V}_s$ type pretopologies on X. $(\mathcal{P}_X, \bigodot)$ is a monoid ($\bigodot$ is composition law, associative, with a neutral element). $(\mathcal{P}_X(V), \bigodot)$, $(\mathcal{P}_X(V_D), \bigodot)$ and $(\mathcal{P}_X(V_s), \bigodot)$ are sub-monoids of $(\mathcal{P}_X, \bigodot)$. $(\mathcal{P}_X, \cup)$ and $(\mathcal{P}_X, \cap)$ are commutative monoids. $(\mathcal{P}_X(V), \cup)$ (respectively $(\mathcal{P}_X(V), \cap)$, is a commutative sub-monoid of $(\mathcal{P}_X, \cup)$ (respectively of $(\mathcal{P}_X, \cap)$. $(\mathcal{P}_X(V_D), \cup)$ is a commutative sub-monoid of $(\mathcal{P}_X, \cup)$ and $(\mathcal{P}_X(V_s), \cup)$ is a commutative sub-monoid of $(\mathcal{P}_X, \cup)$.

Remark. Let a_1 and a_2 two $\mathcal{V}$ type pretopologies on X. Let F_{a_1} the closure according to a_1 and let F_{a_2} the closure according to a_2. If $A \subset X, a_1(A) \subset a_2(A)$ then $F_{a_1}(A) \subset F_{a_2}(A)$.

Proposition 2. *Let $\{(X, a_i), i \in I\}$ a network such as for any $i \in I$, a_i is of $\mathcal{V}$ type. Let F_{a_i} the closure according to a_i. Let F'_{a_i} the inverse of the closure according to a_i. Let $F"_{a_i}$ the closure according to $a_i"$. Let $F_\cup$ (respectively $F_\cap$) the closure according to $\bigcup_{i \in I} a_i$ (respectively $\bigcap_{i \in I} a_i$). Let $F'_\cup$ (respectively $F'_\cap$) the inverse of the closure according to $\bigcup_{i \in I} a_i$ (respectively $\bigcap_{i \in I} a_i$). Let $F"_\cup$ (respectively $F"_\cap$) the closure according to $(\bigcup_{i \in I} a_i)"$ (respectively $(\bigcap_{i \in I} a_i)"$). Let $A \in X$.*

i. $\bigcup_{i \in I} F_{a_i}(A) \subset F_\cup(A)$, $\bigcup_{i \in I} F'_{a_i}(A) \subset F'_\cup(A)$ *and* $\bigcup_{i \in I} F"_{a_i}(A) \subset F"_\cup(A)$.

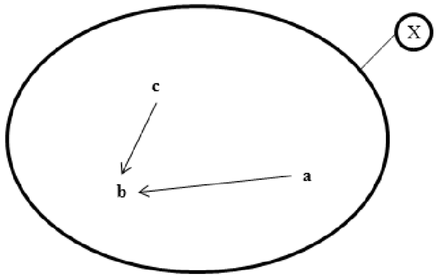

Figure 3

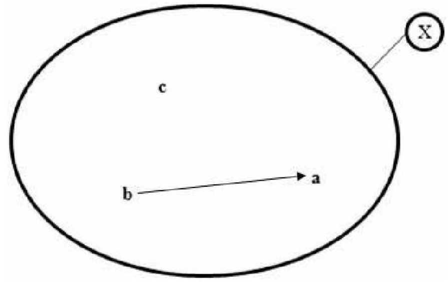

Figure 4

ii. $F_\cap(A) \subset \bigcap_{i\in I} F_{a_i}(A)$, $F'_\cap(A) \subset \bigcap_{i\in I} F'_{a_i}(A)$ *and* $F"_\cap(A) \subset \bigcap_{i\in I} F"_{a_i}(A)$.

Proof. i. $\forall i \in I, a_i(A) \subset \bigcup_{i\in I} a_i(A)$ then $\forall i \in I, F_{ai}(A) \subset F_\cup(A)$ (preliminary nota) hence $\bigcup_{i\in I} F_{a_i}(A) \subset F_\cup(A)$.

$\forall i \in I, F'_{a_i}(A) = \{x \in X/F_{ai}(x) \cap A \neq \emptyset$ then $\bigcup_{i\in I} F'_{a_i}(A) = \bigcup_{i\in I}\{x \in X/F_{a_i}(\{x\}) \cap A \neq \emptyset\}$ so $\bigcup_{i\in I} F'_{a_i}(A) = \{x \in X/\bigcup_{i\in I} F_{a_i}(\{x\}) \cap A \neq \emptyset\} \subset \{x \in X/F_\cup(\{x\}) \cap A \neq \emptyset\} \subset F'_\cup(A)$ $\forall i \in I, F_{a_i}(A) \subset \bigcup_{i\in I} F_{a_i}(A)$ then $\forall i \in I, F'_{a_i}(F_{a_i}(A)) \subset F'_{a_i}(\bigcup_{i\in I} F_{a_i}(A)) \subset (\bigcup_{i\in I} F'_{a_i})(\bigcup_{i\in I} F_{a_i})(A) \subset (F'_\cup F_\cup)(A)$ (preliminary nota) then $(\bigcup_{i\in I} F"_{a_i})(A) \subset F"_\cup(A)$.

ii. $\forall i \in I, \bigcap_{i\in I} a_i(A) \subset a_i(A)$ (by definition) then $\forall i \in I, F_\cap(A) \subset F_{a_i}(A)$ (preliminary nota) then $F_\cap(A) \subset \bigcap_{i\in I} F_{a_i}(A)$.

$F'_\cap(A) = \{x \in X/F_\cap(\{x\}) \cap A \neq \emptyset\} \subset \{x \in X/\bigcap_{i\in I} F_{a_i}(\{x\}) \cap A \neq \emptyset\} \subset \bigcap_{i\in I}\{x \in X/F_{a_i}(\{x\}) \cap A \neq \emptyset\} \subset \bigcap_{i\in I} F'_{a_i}(A)$. $\forall i \in I, \bigcap_{i\in I} F_{a_i}(A) \subset F_{a_i}(A)$, hence $(\bigcap_{i\in I} F'_{a_i})(\bigcap_{i\in I} F_{a_i}(A)) \subset (\bigcap_{i\in I} F'_{ai})(F_{a_i})(A)(\bigcap_{i\in I} F'_{a_i}$ is of $\mathcal{V}$ type). Then $\forall i \in I, (\bigcap_{i\in I} F'_{a_i})(\bigcap_{i\in I} F_{a_i})(A) \subset F'_{a_i} F_{a_i}(A)$ but $F'_\cap F_\cap(A) \subset (\bigcap_{i\in I} F'_{a_i})(\bigcap_{i\in I} F_{a_i})(A)$ then $\forall i \in I, F'_\cap F_\cap(A) \subset F'_{a_i} F_{a_i}(A)$, this implies $\forall i \in I, F"_\cap(A) \subset F"_{a_i}(A)$ and so $F"_\cap(A) \subset \bigcap_{i\in I} F"_{a_i}(A)$. Note that inclusions can be strict ones. □

Example. i. Let $\{(X, a_i), i \in I\}$ a network with $X = \{a, b, c\}$, I = $\{1, 2\}$, a_1 and a_2 respectively the ascending and the descending pretopologies defined on Figure 3:

Let $A = \{a\}$. We get $a_1(A) = \{a, b\}$ and $a_1(a_1(A)) = a1(\{a, b\}) = \{a, b\}$ hence $F_{a_1}(A) = \{a, b\}$ and $a_2(A) = \{a\}$ so $F_{a_2}(A) = \{a\}$. Finally, $(F_{a_1} \cup F_{a_2})(A) = \{a, b\}$. Moreover, $(a_1 \cup a_2)(A) = \{a, b\}$ and $(a_1 \cup a_2)(a_1 \cup a_2)(A) = (a_1 \cup a_2)(\{a, b\}) = \{a, b\} \cup X = X$ so $F_{a_1\cup a_2}(A) = X$ and then $(F_{a_1} \cup F_{a_2})(A) \neq F_{a_1\cup a_2}(A)$. We get $F'_{a_1}(A) = \{x \in X/F_{a_1}(\{x\}) \cap A \neq \emptyset\} = A$ and $F'_{a_2}(A) = \{x \in X/F_{a_2}(\{x\}) \cap A \neq \emptyset\} = \{a, b\}$. This leads to

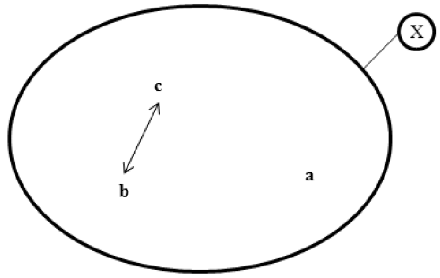

Figure 5

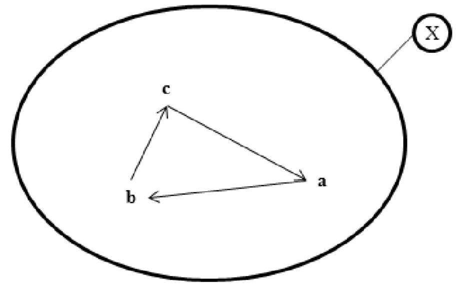

Figure 6

$(F'_{a_1} \cup F'_{a_2})(A) = \{a, b\}$. Moreover, $F'_{a_1 \cup a_2}(A) = \{x \in X / F_{a_1 \cup a_2}(\{x\}) \cap A \neq \emptyset\} = X$ then $(F'_{a_1} \cup F'_{a_2})(A) \neq F'_{a_1 \cup a_2}(A)$. Let $\{(X, a_i), i \in I\}$ a network with $X = \{a, b, c\}$, I $= \{1, 2\}$, a_1 and a_2, a_1 and a_2 the descending pretopologies respectively defined on the two following graphs: Let $A = \{a\}$. We get $F_{a_1}(A) = \{a, b\}$ and $F'_{a_1}(\{a, b\}) = \{a, b\}$, then $F"_{a_1}(A) = \{a, b\}$. We also get $F_{a_2}(A) = \{a\}$ and $F'_{a_2}(\{a\}) = \{a\}$, then $F"_{a_2}(A) = \{a\}$. This leads to $(F"_{a_1} \cup F"_{a_2})(A) = \{a, b\}$. Moreover, $F_{a_1 \cup a_2}(A) = \{a, b, c\}$ so $F"_{a_1 \cup a_2}(A) = X$ and then $(F"_{a1} \cup F"_{a2})(A) \neq F"_{a_1 \cup a_2}(A)$.

ii. Let $\{(X, a_i), i \in I\}$ a network with $X = \{a, b, c\}$, I $= \{1, 2\}$, a_1 and a_2, a_1 and a_2 respectively the ascending and the descending pretopologies respectively defined on the graph of Figure 6:

Let $A = \{a\}$. We get $a_1(A) = \{a, b\}$ and $a_1(a_1(A)) = a_1(\{a, b\}) = X$, then $F_{a_1}(A) = X$ and $a_2(A) = \{a, c\}$ so $a_2(a_2(A)) = X$, so $F_{a_2}(A) = X$.

Finally, $(F_{a_1} \cap F_{a_2})(A) = X$.

Moreover, $(a_1 \cap a_2)(A) = \{a\}$ and $(a_1 \cap a_2)(a_1 \cap a_2)(A) = (a_1 \cap a_2)(\{a\}) = \{a\}$ so $F_{a_1 \cap a_2}(A) = \{a\}$ and then $F_{a_1 \cap a_2}(A) \neq (F_{a_1} \cap F_{a_2})(A)$.

We get $F'_{a_1}(A) = \{x \in X / F_{a_1}(\{x\}) \cap A \neq \emptyset\} = X$ et $F'_{a_2}(A) = X$. This leads to $(F'_{a_1} \cap F'_{a_2})(A) = X$. And $F'_{a_1 \cap a_2}(A) = \{a\}$ so $F'_{a_1 \cap a_2}(A) \neq (F'_{a_1} \cap F'_{a_2})(A)$.

At last, we get $F"_{a_1}(A) = X$ and $F"_{a_2}(A) = X$. Finally, $(F"_{a_1} \cap F"_{a_2})(A) = X$. Moreover, $F"_{a_1 \cap a_2}(A) = \{a\}$ so $F"_{a_1 \cap a_2}(A) \neq (F"_{a_1} \cap F"_{a_2})(A)$.

Proposition 3. *Let be $\{(X, a_i), i \in I\}$ a network with for any i in I, a_i of $\mathcal{V}$ type. Let F_{ai} the closure according to a_i. Let $F_{\cup}$ (respectively $F_{\cap}$) the closure according to $\bigcup_{i \in I} a_i$ (respectively $\bigcap_{i \in I} a_i$). Let $F_{\cup F}$ (respectively $F_{\cap F}$) the closure according to $\bigcup_{i \in I} F_{a_i}$ (respectively $\bigcap_{i \in I} F_{a_i}$).*

i. $F_{\cup} = F_{\cup F}$.

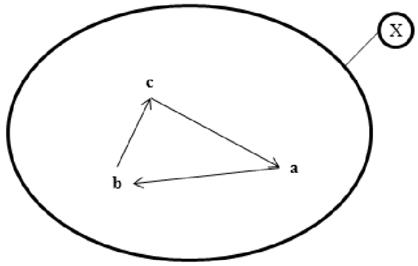

Figure 7

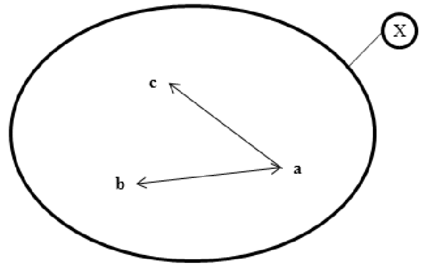

Figure 8

ii. $\forall A \subset X, F_{\cap}(A) \subset F_{\cap F}(A)$.

Proof. i. Let $A \subset X, \bigcup_{i\in I} F_{a_i}(A) \subset F_{\cup}(A)$ (Proposition 2-i) then $F_{\cup F}(A) \subset F_{\cup}(A)$ (preliminary remark).

Conversely, $\forall i \in I, a_i(A) \subset F_{a_i}(A)$ then $\bigcup_{i\in I} a_i(A) \subset \bigcup_{i\in I} F_{a_i}(A)$ then $F_{\cup}(A) \subset F_{\cup F}(A)$ (preliminary remark).

ii. Let $A \subset X$, $F_{\cap}(A) \subset \bigcap_{i\in I} F_{a_i}(A)$ (Proposition 2-ii) then $F_{\cap}(A) \subset F_{\cap F}(A)$ (preliminary remark). □

In ii. including can be strictly verified.

Example. Let be $\{(X, a_i), i \in I\}$ a network with $X = \{a, b, c\}$, $I = \{1, 2\}$, a_1 and a_2 respectively the ascending and the descending pretopologies defined on the graph of Figure 7:

Let $A = \{a\}$. We get $a_1(A) = \{a, b\}$ and $a_2(A) = \{a, c\}$ then $(a_1 \cap a_2)(A) = \{a\}$ so, $F_{a_1\cap a_2}(A) = \{a\}$. Moreover, $F_{a_1}(A) = F_{a_2}(A) = X$ then $(F_{a_1} \cap F_{a_2})(A) = X$, so $F_{F_{a_1}\cap F_{a_2}}(A) = X$ and then $F_{a_1\cap a_2}(A) \neq F_{F_{a_1}\cap F_{a_2}}(A)$.

Definition 7. Let be $\{(X, a_i), i \in I\}$ a network with for any i in I, a_i of $\mathcal{V}$ type. We define the mapping, denoted $\prod_{i\in I} a_i$, from $\mathcal{P}(X)$ into $\mathcal{P}(X)$ by: $\forall A \subset X, \prod_{i\in I} a_i(A) = \{x \in X/\exists n \in I/x \in a_n(a_{n-1}(...(a_i(A))...))\}$.

Remark. Generally speaking, composition of pretopologies is not a commutative operation.

Example. Let be $\{(X, a_i), i \in I\}$ a network with $X = \{a, b, c\}$, $I = \{1, 2\}$, a_1 and a_2 the descending pretopologies defined on the graphs of Figure 8 and Figure 9: Let $A = \{a\}$. We get $a_2(a_1(A)) = a_2(\{a, b\}) = X$. Moreover, $a_1(a_2(A)) = a_1(\{a\}) = \{a, b\}$.

Proposition 4. *Let be $\{(X, a_i), i \in I\}$ a network with for any i in I, a_i of $\mathcal{V}$ type. Let be σ a permutation of I. Let $F_{\prod}$ the closure according to $\prod_{i\in I} a_i$.*

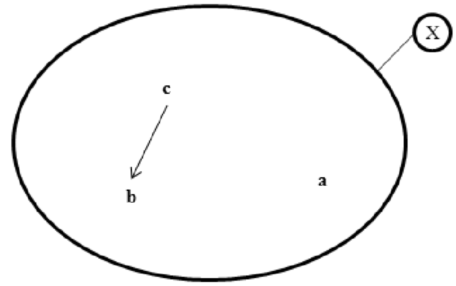

Figure 1: Figure 9

Let $F_{\prod_\sigma}$ *the closure according to* $\prod_{i\in I} a_{\sigma(i)}$. $F_{\prod} = F_{\prod_\sigma}$.

Proof. Let $A \subset X$. $\forall i \in I, a_i(A) \subset \prod_{i\in I} a_{\sigma(i)}(A)$ (according to the definition) then $\forall i \in I, a_i(A) \subset F_{\prod_\sigma}(A)$, so $\prod_{i\in I} a_i(A) \subset F_{\prod_\sigma}(A)$ (because $F_{\prod_\sigma}(A)$ is a closet set for $F_{\prod_\sigma}$) and $F_{\prod}(A) \subset F_{\prod_\sigma}(A)$ (preliminary remark).

Conversely, let $A \subset X$. $\forall i \in I, a_{\sigma(i)}(A) \subset \prod_{i\in I} a_i(A)$ (according to the definition) then $\forall i \in I, a_{\sigma(i)}(A) \subset F_{\prod}(A)$, then $\prod_{i\in I} a_{\sigma(i)}(A) \subset F_{\prod}(A)$ (because $F_{\prod}(A)$ is a closed set for $F_{\prod}$) and $F_{\prod_\sigma}(A) \subset F_{\prod}(A)$ (preliminary remark). □

Corollary 5. *Let be* $\{(X, a_i), i \in I\}$ *a network with for any* i *in* I, a_i *of* $\mathcal{V}$ *type. Let* σ *a permutation of* I. *Let* F_{a_i} *the closure according to* a_i. *Let* $F_{\prod F}$ *the closure according to* $\prod_{\in I} F_{a_i}$. *Let* $F_{\prod F_\sigma}$ *the closure according to* $\prod_{\in I} F_{a_{\sigma(i)}}$. $F_{\prod F} = F_{\prod F_\sigma}$.

Proof. $\{(X, F_{a_i}), i \in I\}$ is a network with for any i in I, F_{a_i} of $\mathcal{V}$ type. We then can apply Proposition 4 to this network. This leads to the result. □

Proposition 6. *Let be* $\{(X, a_i), i \in I\}$ *a network with for any i in I,* a_i *of* $\mathcal{V}$ *type. Let* $F_{\prod}$ *the closure according to* $\prod_{i\in I} a_i$. *Let* F_{a_i} *the closure according to* a_i. *Let* $F_{\prod F}$ *the closure according to* $\prod_{i\in I} F_{a_i}$ $F_{\prod F} = F_{\prod}$.

Proof. Let $A \subset X$. $(\prod_{\in I} a_i)(A) \subset (\prod_{\in I} F_{a_i})(A)$ (according to the definition) then $F_{\prod}(A) \subset F_{\prod F}(A)$ (preliminary remark).

Conversely, $\forall i \in I, a_i(A) \subset (\prod_{i\in I} a_i)(A)$ (according to the definition) then $\forall i \in I, F_{a_i}(A) \subset F_{\prod}(A)$ (preliminary remark) so $(\prod_{i\in I} F_{a_i})(A) \subset F_{\prod}(A)$ (because $F_{\prod}(A)$ is a closed set for $F_{\prod}$) then $F_{\prod F}(A) \subset F_{\prod}(A)$ (preliminary remark). □

Proposition 7. *Let be* $\{(X, a_i), i \in I\}$ *a network with for any i in I,* a_i *of* $\mathcal{V}$ *type. Let* F_{a_i} *the closure according to* a_i. *Let* $F_{\prod F}$ *the closure according to* $\prod_{i\in I} F_{a_i}$. *Let* $F_{\cup}$ *the closure according to* $\bigcup_{i\in I} a_i$ $F_{\prod F} = F_{\cup}$.

Proof. Let $A \subset X$. $\forall i \in I, a_i(A) \subset F_{a_i}(A)$ (according to the definition) then $\forall i \in I, a_i(A) \subset \prod_{i\in I} F_{a_i}(A)$ so $(\bigcup_{i\in I} a_i)(A) \subset F_{\prod F}(A)$ and then $F_{\cup}(A) \subset F_{\prod F}(A)$ (preliminary remark).

Conversely, let $A \subset X$. $\forall i \in I, F_{a_i}(A) \subset \bigcup_{i\in I} F_{a_i}(A)$, so $\forall i \in I, F_{a_i}(A) \subset F_{\cup}(A)$ (Proposition 2-i) then $(\prod_{i\in I} F_{a_i})(A) \subset F_{\cup}(A)$ (because $F_{\cup}(A)$ is a closed set for $F_{\cup}$) and then $F_{\prod F}(A) \subset F_{\cup}(A)$ (preliminary remark). □

6. Conclusion

In this work, we presented different concepts of closures in pretopological spaces which appear to us as quite important due to the fact that, on these concepts depend definitions of path and chain in pretopology. In fact, in a pretopological space of $\mathcal{V}$ type, we say there exists a path from a subset B of X towards a subset A of X if and only if $B \subset F(A)$. We say there exists a chain from B towards A if and only if $B \subset F"(A)$. So, starting from these previous concepts, we are able to deal with all questions related with connectivity in pretopology. Among the results we got, we include results of graph theory (for example, we can note that to say there exists a chain from x towards y is equivalent to say there exists a chain from y towards x (Property 6)). However, we also exhibit the interest of pretopology when we consider pretopologies of $\mathcal{V}$ type where, for example, to say that there exists a chain from y towards x is no more equivalent to say there exists a chain from x towards y (cf. example following the 5). In networks analysis, we saw that closure of a union of pretopologies is equivalent to the closure of the composition of the pretopologies. However, the closure of intersect is no equivalent to the closure of intersect of closures. These results lead to different ways for aggregating and analyzing spaces in terms of connectivity.

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