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BOUNDARY VALUE METHODS VIA A MULTISTEP METHOD WITH VARIABLE COEFFICIENTS FOR SECOND ORDER INITIAL AND BOUNDARY VALUE PROBLEMS

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Abstract: Interpolation and collocation procedures are used to construct a linear multistep method (LMM) with variable coefficients from which LMMs with constant coefficients are reproduced. The LMMs are applied as boundary value methods (BVMs) to solve the general second order initial and boundary value problems without first reducing the ordinary differential equation(ODE) into an equivalent first order system. The order, error constant, zero stability and the interval of absolute stability for the LMMs are discussed. We use the specific cases $k = 4$ and $k = 5$ to illustrate the process. Numerical experiments are performed to show the efficiency of the methods.

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1. Introduction

In the past decades LMMs have been extensively used for solving first order initial value problems (IVPs) and are conventionally applied to solve higher

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order IVPs by first reducing the ODE into an equivalent first order system. This approach is extensively discussed in the literature and in this paper we cite just a few notable ones such as Lambert [17], [19], Brugnano and Trigiante [5], Onumanyi et al [22], [21], Fatunla [9], and Jennings [15]. LMMs of the Adams-Moulton type for solving BVPs are due to Onumanyi et al [21], and Brugnano and Trigiante [5]. However, these methods are applicable by first reducing the higher order ODE to an equivalent system of first-order ODEs, which involves more human effort and computer time as discussed in Awoyemi [3]. We note that LMMs based on the Numerov's type method have been considered by Yusuph and Onumanyi [23], Lambert [19], Lambert and Watson [18] and Henrici [11] for solving directly the special case $y'' = f(x, y)$ with Dirichlet boundary conditions.

Recently, Amodio and Iavernaro [1] introduced symmetric BVMs for special second order ODEs of the Hamiltonian type. However, these methods were not extended to solve the general second order ODE of the form

$$Dy \equiv y'' = f(x, y, y')\,, \tag{1}$$

subject to initial conditions

$$y(a) = y_0,\ y'(a) = \delta_0\,,$$

or boundary conditions

$$y(a) = y_0,\ y(b) = y_N\,,$$

where δ_0, y_0, and y_N are real constants and f is a continuous function and satisfies a Lipschitz condition as discussed in Henrici [11] for IVPs. We note that for BVPs Keller [16] has given the theorem and the proof of the general conditions which ensure that the solution to (1) subject to boundary conditions will exist and be unique. The boundary value version of problem (1) is conventionally solved by methods such as the Shooting Method (SHM) and the Standard Finite Difference Methods (SFDMs). The shooting method works by reducing the BVP to an initial value problem (IVP) which is solved by varying the initial slope until the function satisfies the condition at the endpoint. Although, the SHM takes advantage of the numerous existing codes available for solving IVPs, it suffers from numerical instability when used for IVPs with growing modes even when the BVP itself is well posed and stable. The SFDMs are more robust than the SHM, but higher order SFDMs are more tedious to derive and implement. Thus, the LMMs presented in this paper are of high order and more robust than the SFDMs and can be extended to more general boundary conditions such as $y'(a) = \delta_0$, $y'(b) = y_{N1}$, $y(a) = y_0$, $y'(b) = y_{N1}$, and $y'(a) = \delta_0$, $y(b) = y_N$, $y_{N1} \epsilon \mathbb{R}$ without creating any essential difficulties.

It is worthwhile to note that our method is not restricted to the scalar form (1) since it can be extended to solve a system of such equations by obvious notational modifications.

Jator [13] and Jator and Li [14] proposed LMMs for the direct solution of the general second order IVPs, which were shown to be zero stable and implemented without the need for either predictors or starting values from other methods. Therefore, an attempt has been made to use LMMs as BVMs in the sense of Amodio and Iavernaro [1] to solve (1) directly. We emphasize that the continuous k-step LMM is derived through interpolation and collocation, see Lie and Norsett [20], Atkinson [2], Onumanyi et al [21], and Gladwell and Sayers [10].

The paper is organized as follows. In Section 2, we derive a continuous approximation $Y(x)$ for the exact solution $y(x)$. Section 3 is devoted to the specification of the methods and how the LMMs are obtained. The analysis of the LMMs are discussed in Section 4. The computational aspects are given in Section 5 to demonstrate how the LMMs are applied as BVMs to (1), while the numerical examples are given in Section 6 to show the efficiency of the BVMs. Finally, the conclusion of the paper is discussed in Section 7.

2. Derivation of the Methods

In this section, our objective is to derive a k-LMM of the form

$$y_{n+k} = -\sum_{j=0}^{k-1} \alpha_j y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j f_{n+j}\,, \tag{2}$$

where $\alpha_k = 1$, $\beta_k \neq 0$, and α_0 and β_0 do not both vanish. Since we are considering the direct solution of (1), we also seek the derivative formula of the form

$$y'_{n+k} = \frac{1}{h}\left(-\sum_{j=0}^{k-1} \alpha'_j y_{n+j} + h^2 \sum_{j=0}^{k} \beta'_j f_{n+j}\right)\,. \tag{3}$$

We begin the derivation process of deriving (2) by seeking to approximate the exact solution $y(x)$ by a continuous method $Y(x)$ of the form

$$Y(x) = \sum_{j=0}^{r+s-1} \lambda_j \Upsilon_j(x)\,, \tag{4}$$

where $x \in [a,b]$, λ_j's are unknown coefficients and $\Upsilon_j(x)$'s are polynomial basis functions of degree $r+s-1$. The number of interpolation points r and the number of distinct collocation points s are chosen to satisfy $2 \leq r \leq k$, and $0 < s \leq k+1$ respectively. The positive integer $k \geq 2$ denotes the step number of the method. We then construct a k-step multistep collocation method by imposing the following conditions

$$Y\left(x_{n+j}\right) = y_{n+j}, \quad j = 0, 1, 2, \ldots, r-1\,, \tag{5}$$

$$DY\left(x_{n+j}\right) = f_{n+j}, \quad j = 0, 1, 2, \ldots, s-1\,, \tag{6}$$

where y_{n+j} is the numerical approximation to the analytical solution $y(x_{n+j})$, $f_{n+j} = f(x_{n+j}, y_{n+j})$, and n is a grid index. Equations (5) and (6) lead to a system of $(r+s)$ equations, which is solved to obtain the λ_j's. We proceed by considering the following notations.

We define the interpolation/collocation matrix $\bigvee$ of dimension $(r+s)\times(r+s)$ as

$$\bigvee = \begin{pmatrix} P_0(x_n) & \cdots & P_{r+s-1}(x_n) \\ P_0(x_{n+1}) & \cdots & P_{r+s-1}(x_{n+1}) \\ \vdots & & \vdots \\ P_0(x_{n+r-1}) & \cdots & P_{r+s-1}(x_{n+r-1}) \\ DP_0(x_n) & \cdots & DP_{r+s-1}(x_n) \\ DP_0(x_{n+1}) & \cdots & DP_{r+s-1}(x_{n+1}) \\ \vdots & & \vdots \\ DP_0(x_{n+s-1}) & \cdots & DP_{r+s-1}(x_{n+s-1}) \end{pmatrix}$$

and consider further notations by defining the following vectors:

$$\bigwedge = \left(y_n, y_{n+1}, \ldots, y_{n+r-1}, f_n, f_{n+1}, \ldots, f_{n+s-1}\right)^T,$$

$$\Upsilon(x) = \left(P_0(x), P_1(x), \ldots, P_{r+s-1}(x)\right)^T,$$

$$\bigsqcup = \left(\lambda_0, \lambda_1, \ldots, \lambda_{r+s-1}\right)^T,$$

where T denotes the transpose of the vectors. It is worth noting that $\Upsilon(x)$ represents a vector of arbitrary basis functions. We emphasize that the collocation points are selected from the extended set Ω, where

$$\Omega = \{x_n, \ldots, x_{n+k}\} \bigcup \{x_{n+k-1}, x_{n+k}\}.$$

From the above definitions, if we let $Y(x)$ satisfy conditions (3) and (4), then, the continuous k-step LMM is constructed as in [23] from the following equation:

$$Y(x) = \bigwedge{}^T \left(\bigvee{}^{-1}\right)^T \Upsilon(x)\,. \tag{7}$$

The continuous k-step LMM is obtained from (7) after some manipulation and expressed in the form

$$Y(x) = \sum_{j=0}^{r-1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{s-1} \beta_j(x) f_{n+j}\,, \tag{8}$$

where the $\alpha_j(x)$'s and $\beta_j(x)$'s are the continuous coefficients. The continuous k-step LMM (8) is used to generate discrete LMMs of the form (2) and derivative formulas of the form (3), which are applied as simultaneous numerical integrators to provide the discrete solution to (1). In this light, we seek a solution on the mesh

$$\pi_N : a = x_0 < x_1 < x_2 < \ldots < x_N = b\,,$$

where π_N is a partition of $[a, b]$ such that $x_n = a + nh$, $n = 0, 1, 2, \ldots, N = (b-a)/h$, h is the constant step-size of the partition of π_N. We note that y_n is the numerical approximation to the analytical solution $y(x_n)$ and $f_n = f(x_n, y_n)$.

3. Specification of the Methods

In this section, we give some specific examples of the form (2) produced by the continuous LMM (8). In addition to producing LMMs of the form (2), (8) is also used to obtain the formula for the derivative given by

$$Y'(x) = \frac{1}{h}\left(\sum_{j=0}^{r-1} \alpha'_j(x) y_{n+j} + h^2 \sum_{j=0}^{s-1} \beta'_j(x) f_{n+j}\right) \tag{9}$$

which is effectively applied by imposing that

$$Y'(x) = \delta(x), \quad Y'(a) = \delta_0\,, \tag{10}$$

to produce derivative formulas of the form (3).

In particular, we use (8) to obtain a continuous k-step LMM by specifying r, s, k, and $\Upsilon_j(x)$. We emphasize that the main methods of the form (2) are obtained by evaluating (8) at $x = x_{n+k}$. We also express $\alpha_j(x)$ and $\beta_j(x)$ as functions of t for convenience, where $t = (x - x_{n+k-1})/h$. The coefficients $\alpha'_j(x)$ and $\beta'_j(x)$ are easily obtained from the first derivatives of $\alpha_j(x)$ and $\beta_j(x)$. We discuss details of specific methods next.

Case $k = 4$. We use (8) to obtain a continuous 4-step method with the following specifications: $r = 3$, $s = 5$, $k = 4$, $\Upsilon_i(x) = x^i$, $i = 0, 1, \ldots, 7$. We also express $\alpha_j(x)$ and $\beta_j(x)$ as functions of t for convenience, where $t = (x - x_{n+3})/h$ in what follows:

$\alpha_0(t) = \frac{1}{42}(40t - 84t^3 - 35t^4 + 21t^5 + 14t^6 + 2t^7)$
$\alpha_1(t) = \frac{1}{21}(-21 - 61t + 84t^3 + 35t^4 - 21t^5 - 14t^6 - 2t^7)$
$\alpha_2(t) = \frac{1}{42}(84 + 82t - 84t^3 - 35t^4 + 21t^5 + 14t^6 + 2t^7)$
$\beta_0(t) = \frac{1}{10080}(-42 - 725t + 1456t^3 + 630t^4 - 357t^5 - 252t^6 - 38t^7)$
$\beta_1(t) = \frac{1}{1260}(126 - 957t + 2247t^3 + 910t^4 - 567t^5 - 364t^6 - 51t^7)$
$\beta_2(t) = \frac{1}{720}(582 + 707t - 96t^3 + 50t^4 + 15t^5 - 8t^6 - 2t^7)$
$\beta_3(t) = \frac{1}{1260}(126 + 519t + 630t^2 + 217t^3 - 70t^4 - 63t^5 - 14t^6 - t^7)$
$\beta_4(t) = \frac{1}{10080}(-42 - 149t + 336t^3 + 350t^4 + 147t^5 + 28t^6 + 2t^7)$.

The LMMs are obtain for $k = 4$ by evaluating (8) at $x = \{x_{n+4}, x_{n+3}\}$, which is equivalent to $t = \{1, 0\}$ to obtain the following formulas

$$y_{n+4} - 2y_{n+2} + y_n = \frac{h^2}{15}(f_n + 16f_{n+1} + 26f_{n+2} + 16f_{n+3} + f_{n+4}), \quad (11)$$

$$y_{n+3} - 2y_{n+2} + y_{n+1} = \frac{h^2}{240}(-f_n + 24f_{n+1} + 194f_{n+2} + 24f_{n+3} - f_{n+4}). \quad (12)$$

We emphasize that for $k = 4$, the following derivatives which are of the form (3) are obtained from (9) subject to the conditions (10) by imposing that $\delta(x_{n+\tau}) = \delta_{n+\tau}, \tau = 0, \ldots, 4$.

$h\delta_n = -\frac{149}{42}y_n + \frac{128}{21}y_{n+1} - \frac{107}{42}y_{n+2} + \frac{h^2}{1260}(-67f_n + 2256f_{n+1}434f_{n+2} - 48f_{n+3} + 5f_{n+4})$

$h\delta_{n+1} = \frac{20}{21}y_n - \frac{61}{21}y_{n+1} + \frac{41}{21}y_{n+2} + \frac{h^2}{10080}(-613f_n - 11464f_{n+1} - 2870f_{n+2} + 344f_{n+3} - 37f_{n+4})$

$h\delta_{n+2} = -\frac{37}{42}y_n + \frac{16}{21}y_{n+1} + \frac{5}{42}y_{n+2} + \frac{h^2}{1260}(73f_n + 1136f_{n+1} + 574f_{n+2} - 48f_{n+3} + 5f_{n+4})$

$h\delta_{n+3} = \frac{20}{21}y_n - \frac{61}{21}y_{n+1} + \frac{41}{21}y_{n+2} + \frac{h^2}{10080}(-725f_n - 7656f_{n+1} + 9898f_{n+2} + 4152f_{n+3} - 149f_{n+4})$

$h\delta_{n+4} = -\frac{149}{42}y_n + \frac{128}{21}y_{n+1} - \frac{107}{42}y_{n+2} + \frac{h^2}{1260}(325f_n + 4048f_{n+1} + 1106f_{n+2} + 1744f_{n+3} - 149f_{n+4})$

Case $k = 5$. We use (8) to obtain a continuous 5-step method with the following specifications: $r = 3$, $s = 6$, $k = 5$, $\Upsilon_i(x) = x^i$, $i = 0, 1, \ldots, 8$. We also express $\alpha_j(x)$ and $\beta_j(x)$ as functions of t where $t = (x - x_{n+4})/h$ in what follows:

$\alpha_0(t) = \frac{1}{442}(-186 - 413t + 672t^3 + 364t^4 - 126t^5 - 140t^6 - 36t^7 - 3t^8)$
$\alpha_1(t) = \frac{1}{221}(-256 + 192t - 672t^3 - 364t^4 + 126t^5 + 140t^6 + 36t^7 + 3t^8)$
$\alpha_2(t) = \frac{1}{442}(1140 + 29t + 672t^3 + 364t^4 - 126t^5 - 140t^6 - 36t^7 - 3t^8)$

$\beta_0(t) = \frac{1}{2227680}(51744+137992t-235452t^3-129857t^4+42987t^5+49826t^6+13166t^7+1134t^8)$

$\beta_1(t) = \frac{1}{2227680}(1252608+2087360t-3073168t^3-1644006t^4+585501t^5+633024t^6+160214t^7+13167t^8)$

$\beta_2(t) = \frac{1}{1113840}(1919904+929192t+157416t^3+38857h^2t^4-45759t^5-15778t^6-698t^7+126t^8)$

$\beta_3(t) = \frac{1}{1113840}(1150464+1371904t-470064t^3-68978t^4+97419t^5+45332t^6+7502t^7+441t^8)$

$\beta_4(t) = \frac{1}{2227680}(180768+847432t+1113840t^2+486892h^2t^3-70161t^4-131901t^5-45486t^6-6746t^7-378t^8)$

$\beta_5(t) = \frac{1}{2227680}(-5376-24256t+60144t^3+69706t^4+35133t^5+9128t^6+1198t^7+63t^8)$.

The LMMs for $k=5$ are obtain by evaluating (8) at $x=\{x_{n+5},x_{n+4},x_{n+3}\}$, which is equivalent to $t=\{1,0,-1\}$ to obtain the following formulas

$$y_{n+5}-\frac{950}{221}y_{n+2}+\frac{795}{221}y_{n+1}-\frac{66}{221}y_n=\frac{h^2}{5304}(-163f_n+35f_{n+1}+14206f_{n+2} + 10162f_{n+3}+5653f_{n+4}+347f_{n+5}), \quad (13)$$

$$y_{n+4}-\frac{570}{221}y_{n+2}+\frac{256}{221}y_{n+1}+\frac{93}{221}y_n=\frac{h^2}{3315}(77f_n+1864f_{n+1}+5714f_{n+2} + 3424f_{n+3}+269f_{n+4}-8f_{n+5}), \quad (14)$$

$$y_{n+3}-\frac{411}{221}y_{n+2}+\frac{159}{221}y_{n+1}+\frac{31}{221}y_n=\frac{h^2}{53040}(337f_n+11783f_{n+1}+42998f_{n+2} + 5738f_{n+3}-407f_{n+4}+31f_{n+5}). \quad (15)$$

We emphasize that for $k=5$, the following derivatives which are of the form (3) are obtained from (9) subject to the conditions (10) by imposing that $\delta(x_{n+\tau})=\delta_{n+\tau}, \tau=0,\ldots,5$.

$h\delta_n = -\frac{1437}{442}y_n+\frac{1216}{221}y_{n+1}-\frac{995}{442}y_{n+2}+\frac{h^2}{278460}(-20999f_n+426680f_{n+1}+94538f_{n+2}-15424f_{n+3}+3169f_{n+4}-344f_{n+5})$

$h\delta_{n+1} = \frac{17}{26}y_n-\frac{30}{13}y_{n+1}+\frac{43}{26}y_{n+2}+\frac{h^2}{131040}(-5035f_n-114965f_{n+1}-36658f_{n+2}+6754f_{n+3}-1459f_{n+4}+163f_{n+5})$

$h\delta_{n+2} = -\frac{253}{442}y_n+\frac{32}{221}y_{n+1}+\frac{189}{442}y_{n+2}+\frac{h^2}{278460}(9689f_n+176234f_{n+1}+125422f_{n+2}-15620f_{n+3}+3253f_{n+4}-358f_{n+5})$

$h\delta_{n+3} = \frac{129}{442}y_n-\frac{350}{221}y_{n+1}+\frac{571}{442}y_{n+2}+\frac{h^2}{2227680}(-49867f_n-410597f_{n+1}+2211982f_{n+2}+1003426f_{n+3}-69715f_{n+4}+6131f_{n+5})$

$h\delta_{n+4} = -\frac{413}{442}y_n + \frac{192}{221}y_{n+1} + \frac{29}{442}y_{n+2} + \frac{h^2}{278460}(17249f_n + 260920f_{n+1} + 232298f_{n+2} + 342976f_{n+3} + 105929f_{n+4} - 3032f_{n+5})$

$h\delta_{n+5} = \frac{101}{34}y_n - \frac{118}{17}y_{n+1} + \frac{135}{34}y_{n+2} + \frac{h^2}{171360}(-36359f_n - 442745f_{n+1} + 189158f_{n+2} + 77066f_{n+3} + 251729f_{n+4} + 51871f_{n+5})$.

4. Analysis of the Methods

In order to analyze the specific methods obtained in Section 3, the discrete method (2) is rewritten in conventional form as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} \,, \tag{16}$$

or compactly written in the form

$$\rho(E)y_n = h^2\sigma(E)f_n \,, \tag{17}$$

where $\rho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j$ and $\sigma(\zeta) = \sum_{j=0}^{k} \beta_j \zeta^j$ are the characteristic polynomials, $\zeta \in \mathbb{C}$, and $E^j y_n = y_{n+j}$ is a shift operator.

Following Fatunla [9] and Lambert [17] we define the local truncation error associated with (16) to be the linear difference operator

$$L[y(x);h] = \sum_{j=0}^{k} \{\alpha_j y(x+jh) - h^2\beta_j y''(x+jh)\} \,. \tag{18}$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (18) as a Taylor series about the point x to obtain the expression

$$L[y(x);h] = C_0 y(x) + C_1 h y' + \ldots, + C_q h^q y^q(x) + \ldots, \tag{19}$$

where the constant coefficients C_q , $q = 0, 1, \ldots$ are given as follows:

$C_0 = \sum_{j=0}^{k} \alpha_j$,

$C_1 = \sum_{j=1}^{k} j\alpha_j$,

$\vdots$

$C_q = \frac{1}{q!}[\sum_{j=1}^{k} j^q \alpha_j - q(q-1)\sum_{j=1}^{k} j^{q-2}\beta_j]$.

According to Henrici [11], we say that the method (16) has order p if

$$C_0 = C_1 = \ldots = C_p = C_{p+1} = 0, \; C_{p+2} \neq 0,$$

therefore, C_{p+2} is the error constant and $C_{p+2}h^{p+2}y^{(p+2)}(x_n)$ is the principal local truncation error at the point x_n.

Definition 4.1. According to Lambert and Watson [18], a LMM having an even step number is characterized by the following symmetries in the coefficients of the polynomials $\rho(\zeta)$ and $\sigma(\zeta)$:

$$\alpha_j = \alpha_{k-j}, \; \beta_j = \beta_{k-j}, \quad j = 0, 1 \ldots, k.$$

Remark 4.2. It follows that the main method (11) and additional method (12) are symmetric. In particular, the main method (11) can be used to solve periodic IVPs in the conventional way as discussed in Lambert and Watson [18]. However, in this paper the main method (11) and the additional method (12) are used as BVMs to solve (1).

Definition 4.3. A polynomial $\vartheta(\zeta)$ of degree $k = k_1 + k_2$ is said to be an $N_{k_1k_2}$ polynomial if its roots satisfy

$$|\zeta_1| \leq |\zeta_2| \ldots |\zeta_{k_1}| \leq 1 < |\zeta_{k_1+1}| \leq \ldots \leq |\zeta_k|,$$

where the roots of unit modulus are of multiplicity at most 2 (see Brugnano [6]), k_1 is the number of zeros that lie in or on the unit circle and k_2 is the number of zeros the lie outside the unit circle.

Definition 4.4. A BVM used with (k_1, k_2)-boundary conditions is said to be $0_{k_1k_2}$-stable if the polynomial $\rho(\zeta)$ is an $N_{k_1k_2}$ polynomial as discussed in Brugnano [6].

It is easily shown that the main method (11)$(k = 4, k_1 = 4, k_2 = 0)$ is not $0_{k_1k_2}$-stable, but zero-stable in the sense of Lambert [19], while the main method (13) $(k = 5, k_1 = 3, k_2 = 2)$ is $0_{k_1k_2}$-stable.

In what follows, we discuss the regions of absolute stability via the boundary locus method by considering the test equation $Dy = \lambda y, \lambda, y \,\epsilon\, \mathbb{R}$. On applying (16) to the test equation yields the difference equation $(\rho(E) - \hbar\sigma(E))y_n = 0$, $\hbar = h^2\lambda$, $\lambda < 0$, whose solution can be generated by the roots of the stability polynomial $\Phi(\zeta, \hbar) = \rho(\zeta) - \hbar\sigma(\zeta)$. The roots of the stability polynomial are generally complex numbers, hence we assume that $\hbar$ is also a complex number. We define a region of stability Ω with boundary $\delta\Omega$ to be a region of the complex $\hbar$-plane such that the roots of $\Phi(\zeta, \hbar) = 0$ lie within the unit circle whenever $\hbar$ lies in the interior of the region (see [19]). Since the roots of $\Phi(\zeta, \hbar) = 0$ are continuous functions of $\hbar$, $\hbar$ will lie on $\delta\Omega$ when one of the roots of $\Phi(\zeta, \hbar) = 0$ lies on the boundary of the unit circle. Thus, $\Phi(e^{i\theta}, \hbar) = \rho(e^{i\theta}) - \hbar\sigma(e^{i\theta}) = 0$. Hence, the locus of $\delta\Omega$ is expressed as

$$\hbar(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}$$

which after some manipulation can be written in the form $\hbar(\theta) = X_1(\theta) +$

$iX_2(\theta)$. Our calculations show that methods (11) to (15) yield the vectors $X_1(0) = (0,0,0,0)^T$, $X_1(\pi) = (0,-20/3,-762/143,-128/13,-5700/793)^T$, $X_2(0) = (0,0,0,0)^T$, $X_2(\pi) = (0,0,0,0)^T$. We note that for real $\bar{h}$, the end-points of the intervals of absolute stability are given by the points at which $\delta\Omega$ cuts the real axis. The intervals of absolute stability for the methods (11)-(15) are summarized in Table 1. It is worth noting that the methods (11)-(15) have high orders and relatively small error constants as displayed in Table 1.

Method	Order p	Error Constant C_{p+3}	Interval of Absolute Stability
(11)	6	$-2/945$	$(0,0)$
(12)	6	$31/60480$	$(-20/3,0)$
(13)	7	$-13121/6683040$	$(-762/143,0)$
(14)	7	$62/208845$	$(-128/13,0)$
(15)	7	$-961/13366080$	$(-5700/793,0)$

Table 1: Orders, error constants, and intervals of absolute stability for BVMs

Remark 4.5. We observe that $\rho(\zeta)$ for the main method (11) is a Von Neumann polynomial (conservative) and therefore (11) is not recommended for initial value methods since its region of absolute stability is empty as shown in Table 1. However, the method performs efficiently when used as a BVM with $(1,1)$ boundary conditions for $\lambda < 0$.

5. Computational Aspects

In this section, we implement the LMMs as BVMs in the sense of Amodio and Iavernaro [1]. Thus, we rewrite the main method (11) as

$$y_{n+2} - 2y_n + y_{n-2} = \frac{h^2}{15}(f_{n-2} + 16f_{n-1} + 26f_n + 16f_{n+1} + f_{n+2}), \quad n = 2, \ldots, N-2,$$

which is used together with the following initial method ($n = 0$) obtained from (12)

$$y_3 - 2y_2 + y_1 = \frac{h^2}{240}(-f_0 + 24f_1 + 194f_2 + 24f_3 - f_4)$$

and final method obtained from (12) by applying it at the end point in the reverse direction.

$$-y_{N-3}+2y_{N-2}-y_{N-1}=\frac{h^2}{240}(f_N-24f_{N-1}-194f_{N-2}-24f_{N-3}+f_{N-4})\,.$$

In the same vein, we rewrite the the main method (13) as

$$y_{n+3}-\frac{950}{221}y_{n-1}+\frac{795}{221}y_{n-2}-\frac{66}{221}y_{n-3}=\frac{h^2}{5304}(-163f_{n-3}+35f_{n-2}+14206f_{n-1} + 10162f_n+5653f_{n+1}+347f_{n+2}),\quad n=3,\ldots,N-2\,,$$

which is used together with the following initial method ($n=0$) obtained from (15)

$$y_3-\frac{411}{221}y_2+\frac{159}{221}y_1+\frac{31}{221}y_0=\frac{h^2}{53040}(337f_0+11783f_1+42998f_2 + 5738f_3-407f_4+31f_5)$$

and the final methods

$$-y_{N-4}+\frac{570}{221}y_{N-2}-\frac{256}{221}y_{N-1}-\frac{93}{221}y_N=\frac{h^2}{3315}(-77f_N-1864f_{N-1} - 5714f_{N-2}-3424f_{N-3}-269f_{N-4}+8f_{N-5})$$

and

$$-y_{N-3}+\frac{411}{221}y_{N-2}-\frac{159}{221}y_{N-1}-\frac{31}{221}y_N=\frac{h^2}{53040}(-337f_N-11783f_{N-1} - 42998f_{N-2}-5738f_{N-3}+407f_{N-4}-31f_{N-5})$$

obtained from (14) and (15) by applying them at the end point in the reverse direction. We emphasized that the first k derivatives for $n=0$ are provided by $\delta(x_\tau)=\delta_\tau, \tau=0,\ldots,k-1$ and the rest are provided by the derivatives of the main methods given by $\delta(x_{n+k})=\delta_{n+k}, n=0,\ldots,N-k$.

The methods are implemented as BVMs efficiently by combining the main methods and the additional methods as simultaneous numerical integrators for second order IVPs and BVPs. In particular, for linear problems, we can solve (1) directly from the start with Gaussian elimination using partial pivoting, and for nonlinear problems, we use a modified Newton-Raphson method. In each case, the the main methods and the additional methods are combined as BVMs to give a single matrix of finite difference equations which simultaneously provides the values of the solution and the first derivatives generated by the sequences $\{y_n\}, \{y'_n\}, n=0,\ldots,N$, where the single block matrix equation is solved while adjusting for boundary conditions.

6. Numerical Examples

In this section, we have tested the performance of the methods on five problems by considering three linear and two nonlinear problems. For each example we find absolute errors of the approximate solution in π_N. All computations were carried out using our written *Mathematica* code in *Mathematica 6.0* (see Tables 2 to 7).

Example 6.1. We consider the IVP given by

$$y'' - 4y' + 8y = x^3, \; y(0) = 2, \; y'(0) = 4, \quad 0 \le x \le 1$$

$$\text{Exact}: y(x) = e^{2x}(2\cos(2x) - \frac{3}{64}\sin(2x)) + \frac{3}{32}x + \frac{3}{16}x^2 + \frac{1}{8}x^3 .$$

The BVMs ($k = 4, 5$) were applied to this problem and the results were compared with the theoretical solution. As expected the BVM ($k = 5$) performs better than the BVM ($k = 4$). In addition, a direct comparison is made between the self-starting method (SSM) for $k = 4$ discussed in [14] and BVM for $k = 4$, since both methods have the same step number, $k = 4$. Hence, the methods are comparable in the level of accuracy at the mesh points. It is observed that the BVM ($k = 4$) performs better than the method in [14]. Therefore, for this example, our method is clearly superior. The details of the numerical results are displayed in Table 2.

x	SSM [14]($k = 4$)	BVM($k = 4$)	BVM ($k = 5$)
0.0	0.00000000	0.00000000	0.00000000
0.1	5.11×10^{-6}	6.13×10^{-7}	8.14×10^{-8}
0.2	1.50×10^{-5}	1.85×10^{-6}	2.44×10^{-7}
0.3	2.79×10^{-5}	3.42×10^{-6}	4.55×10^{-7}
0.4	4.29×10^{-5}	5.55×10^{-6}	7.29×10^{-7}
0.5	6.70×10^{-5}	8.39×10^{-6}	1.06×10^{-6}
0.6	1.03×10^{-4}	1.23×10^{-5}	1.45×10^{-6}
0.7	1.45×10^{-4}	1.74×10^{-5}	1.93×10^{-6}
0.8	1.91×10^{-4}	2.35×10^{-5}	2.47×10^{-6}
0.9	2.40×10^{-4}	3.09×10^{-5}	3.08×10^{-6}
1.0	2.95×10^{-4}	3.86×10^{-5}	4.06×10^{-6}

Table 2: Absolute errors, $|y(x) - y|$, $h = 0.1$ for Example 6.1

Example 6.2. Consider the linear BVP which has also been solved by Burden and Faires [8].

$$y'' = -\frac{2}{x}y' + \frac{2}{x^2}y + \frac{\sin(\ln x)}{x^2}, \quad 1 \le x \le 2, \quad y(1) = 1, \quad y(2) = 2$$

$$\text{Exact}: y(x) = c_1 x + \frac{c_2}{x^2} - \frac{3}{10}\sin(\ln x) - \frac{1}{10}\cos(\ln x)$$

$$c_2 = \frac{1}{70}[8 - 12\sin(\ln 2) - 4\cos(\ln 2)], \quad c_1 = \frac{11}{10} - c_2 .$$

The BVMs ($k = 4, 5$) were applied to this problem and the results were compared with the theoretical solution. As expected the BVM ($k = 5$) performs better than the BVM($k = 4$). Furthermore, the BVM for $k = 4$, was compared with SFDM and SHM given in Burden and Faires [8]. It is observed that our method performs better than the SFDM and SHM given in Burden and Faires [8]. Specifically, the maximum error for BVM ($k = 4$) is 3.08×10^{-8}, while the maximum errors for the SFDM and SHM are 4.55×10^{-5} and 1.43×10^{-7} respectively. Hence, for this example, our method is clearly superior. The details of the numerical results are given in Table 3.

x	SFDM [8]	SHM [8]	BVM ($k = 4$)	BVM ($k = 5$)
1.0	0.00000000	0.00000000	0.00000000	0.00000000
1.1	2.88×10^{-5}	1.43×10^{-7}	3.08×10^{-8}	2.04×10^{-8}
1.2	4.17×10^{-5}	1.34×10^{-7}	1.19×10^{-8}	1.54×10^{-8}
1.3	4.55×10^{-5}	9.78×10^{-8}	8.49×10^{-9}	1.52×10^{-8}
1.4	4.39×10^{-5}	6.02×10^{-8}	2.67×10^{-8}	1.85×10^{-8}
1.5	3.92×10^{-5}	3.06×10^{-8}	2.35×10^{-8}	2.39×10^{-8}
1.6	3.26×10^{-5}	1.08×10^{-8}	3.55×10^{-8}	2.54×10^{-8}
1.7	2.49×10^{-5}	5.43×10^{-10}	2.15×10^{-8}	2.13×10^{-8}
1.8	1.68×10^{-5}	5.05×10^{-9}	2.63×10^{-8}	1.43×10^{-8}
1.9	8.41×10^{-6}	4.41×10^{-9}	1.73×10^{-8}	6.26×10^{-9}
2.0	0.00000000	0.00000000	0.00000000	0.00000000

Table 3: Absolute errors, $|y(x) - y|$, for Example 6.2, $h = 0.1$

Example 6.3. Our third test problem is the non-linear IVP

$$y'' - x(y')^2 = 0, \quad y(0) = 1, \quad y'(0) = 1/2$$

$$\text{Exact}: y(x) = 1 + \frac{1}{2}\ln((2 + x)/(2 - x)) ,$$

which was solved by the BVMs ($k = 4, 5$) and the results were compared with the theoretical solution. As expected the BVM ($k = 5$) performs better than the BVM ($k = 4$). In addition, the level of accuracy at the mesh points for BVM ($k = 4$) was compared the SSM ($k = 4$) discussed in [14]. It is noticed that the BVM performs better than the method in [14]. Hence, for this example, our method is clearly superior. The details of the numerical results are displayed in Table 4.

x	SSM [14]($k=4$)	BVM ($k=4$)	BVM ($k=5$)
0.1	7.51×10^{-9}	6.67×10^{-9}	1.03×10^{-9}
0.2	1.80×10^{-8}	1.60×10^{-8}	2.51×10^{-9}
0.3	2.88×10^{-8}	2.55×10^{-8}	3.97×10^{-9}
0.4	3.65×10^{-8}	3.23×10^{-8}	5.71×10^{-9}
0.5	7.05×10^{-8}	4.45×10^{-8}	7.62×10^{-9}
0.6	1.20×10^{-7}	4.82×10^{-8}	9.31×10^{-9}
0.7	1.73×10^{-7}	6.27×10^{-8}	1.27×10^{-8}
0.8	2.14×10^{-7}	6.09×10^{-8}	1.26×10^{-8}
0.9	5.82×10^{-7}	7.42×10^{-8}	2.00×10^{-8}
1.0	1.15×10^{-6}	5.55×10^{-8}	3.46×10^{-8}

Table 4: Absolute errors, $\|y - \overline{y}\|$, for Example 6.3, $h = 0.1$, where $y(x) = (\sin x)^2$

Example 6.4. We consider the nonlinear BVP which has also been solved by Burden and Faires [8].

$$y'' = \frac{1}{8}(32 + 2x^3 - yy'), \quad 1 \leq x \leq 3, \quad y(1) = 17, \quad y(3) = \frac{43}{3}$$

$$\text{Exact}: y(x) = x^2 + \frac{16}{x}.$$

In this example, a comparison is made between the SHM [8] and the BVM ($k = 4$) with respect to computational demands and the level of accuracy at the mesh points. In the area of computational work, the SHM involves four (eight for a 2 by 2 system) function evaluations per step using the Runge-Kutta method, while the BVM ($k = 4$) involves only one function evaluation per step. Moreover, the SHM works by reducing the BVP to an initial value problem (IVP) which is solved by varying the initial slope until the function satisfies the condition at the endpoint. On the other hand, our method is implemented by combining the main methods and the additional methods as BVMs to give a single matrix of finite difference equations which simultaneously provides the values of the solution at the mesh points. Hence, the BVM ($k = 4$) is considerably faster and requires less computational work. Thus, the computational cost is lower in the BVM ($k = 4$) than the SHM method in [8]. In terms of accuracy, our method performs better than the SHM [8]. Our method is also highly efficient since it involves less computational work and yields highly accurate results. Therefore, for this example, our method is clearly superior. The details of the numerical results are given in Table 5.

Example 6.5. We consider the linear BVP which has also been solved in

x	SHM [8]	BVM ($k = 4$)
1.0	0.00000000	0.00000000
1.1	4.06×10^{-5}	1.50×10^{-6}
1.2	5.60×10^{-5}	3.97×10^{-7}
1.3	5.94×10^{-5}	8.12×10^{-8}
1.4	5.71×10^{-5}	1.57×10^{-6}
1.5	5.23×10^{-5}	1.83×10^{-6}
1.6	4.64×10^{-5}	1.79×10^{-6}
1.7	4.06×10^{-5}	1.58×10^{-6}
1.8	3.41×10^{-5}	7.93×10^{-7}
1.9	2.84×10^{-5}	1.07×10^{-6}
2.0	2.32×10^{-5}	3.26×10^{-7}
2.1	1.84×10^{-5}	6.42×10^{-7}
2.2	1.40×10^{-5}	1.27×10^{-6}
2.3	1.01×10^{-5}	2.18×10^{-7}
2.4	6.68×10^{-6}	1.84×10^{-6}
2.5	3.61×10^{-6}	3.82×10^{-7}
2.6	9.17×10^{-7}	1.86×10^{-6}
2.7	1.43×10^{-6}	1.35×10^{-6}
2.8	3.47×10^{-6}	1.22×10^{-6}
2.9	5.21×10^{-6}	2.75×10^{-6}
3.0	6.69×10^{-6}	0.00000000

Table 5: Absolute errors, $|y(x) - y|$, for Example 6.4, $h = 0.1$

Bramble and Hubbard [4], Jator and Sinkala [12], and Collatz [7].

$$y'' - 4y = 4\cosh(1), \quad y(0) = y(1) = 0$$

$$\text{Exact}: y(x) = \cosh(2x-1) - \cosh(1)\,.$$

The errors in solution were obtained using our BVMs ($k = 4, 5, h = 1/10$). As expected the BVM ($k = 5$) performs better than the BVM ($k = 4$), see Table 6. We note that the maximum absolute errors for our BVMs ($k = 4, h = 1/5, 1/10$) were compared with the results given in [4], [12], and [7] as reproduced in Table 7. For this example, our method is more accurate than those given in [4], [12]. The method in [12] performs better than the BVM ($k = 4$) for $h = 1/5$ and for $h = 1/10$, the BVM ($k = 4$) is more accurate. The details of the numerical results are reported in Table 7.

x	BVM($k=4$)	BVM ($k=5$)
0.0	0.00000000	0.00000000
0.1	6.63×10^{-9}	2.97×10^{-10}
0.2	8.56×10^{-9}	1.76×10^{-10}
0.3	1.24×10^{-8}	1.39×10^{-10}
0.4	1.21×10^{-8}	1.66×10^{-10}
0.5	1.41×10^{-8}	2.42×10^{-10}
0.6	1.21×10^{-8}	3.91×10^{-10}
0.7	1.24×10^{-8}	4.12×10^{-10}
0.8	8.56×10^{-9}	4.47×10^{-10}
0.9	6.63×10^{-9}	6.38×10^{-10}
1.0	0.00000000	0.00000000

Table 6: Absolute errors, $|y(x) - y|$, $h = 0.1$ for Example 6.5

x	Collatz	Bramble-Hubbard	BVM ($k=4$)	Jator-Sinkala
1/5	7.61×10^{-4}	4.23×10^{-5}	1.71×10^{-6}	2.87×10^{-8}
1/10	1.65×10^{-6}	1.64×10^{-4}	1.41×10^{-8}	3.90×10^{-8}

Table 7: Absolute maximum errors, $|y(x) - y|$, for different methods for Example 6.5

7. Conclusions

We have proposed a family of LMMS which are applied as BVMs for solving the general second-order IVPs and BVPs directly without first adapting the second order ODE to an equivalent first order system. The LMMs methods are derived through the interpolation and collocation procedures by the matrix inversion approach. An essential ingredient in the methods involves the way in which they are generated and applied. For instance, the same continuous scheme is used to generate not only the main methods, but also additional methods, which are applied at the endpoints thereby avoiding some of the stability problems encountered in the application of traditional LMMs. The fact that BVMs can be used to solve both initial and boundary value problems make them more competitive with existing methods in the literature, such as those given in Burden and Faires [8] and Jator and Li [14]. Our future research will be focused on developing an automatic code for BVMs with an error estimation strategy for solving second order IVPs and BVPs.

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