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X_d-FRAMES IN BANACH SPACES AND THEIR DUALS

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Abstract: We consider consequences of the lower and the upper X_d-frame conditions. The lower X_d-frame condition is proved to be necessary for existence of some series expansions. Our main interest is on duals and dual*s. We consider connection between dual and dual* of an X_d-Bessel sequence, and necessary and sufficient conditions for their existence. If X_d has the canonical vectors as a Schauder basis, then an X_d-Bessel sequence, having a dual or dual*, is moreover a Banach frame.

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1. Introduction

Very useful property of a frame $\{g_i\}_{i=1}^{\infty}$ for a Hilbert space $\mathcal{H}$ is that $\{g_i\}_{i=1}^{\infty}$ has a dual frame $\{f_i\}_{i=1}^{\infty}$, i.e. there exists a frame $\{f_i\}_{i=1}^{\infty}$ for $\mathcal{H}$ such that every element $f \in \mathcal{H}$ can be written as $f = \sum_{i=1}^{\infty}\langle f, f_i\rangle g_i = \sum_{i=1}^{\infty}\langle f, g_i\rangle f_i$. Natural generalizations of frames to Banach spaces are the so called X_d-frames and Banach frames (see Definition 3.1). In general, a Banach frame $\{g_i\}_{i=1}^{\infty} \in (X^*)^{\mathbb{N}}$ for a Banach space X does not guarantee validity of reconstruction formulas in the form

$$f = \sum_{i=1}^{\infty} g_i(f) f_i, \ \forall f \in X, \tag{1}$$

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$$g = \sum_{i=1}^{\infty} g(f_i) g_i, \ \forall g \in X^*. \tag{2}$$

There exist spaces X and Banach frames $\{g_i\}_{i=1}^{\infty}$ for X such that no family $\{f_i\}_{i=1}^{\infty} \in X^{\mathbb{N}}$ satisfies (1) and (2) (e.g. [3, Example 2.8]). Sufficient conditions for validity of (1) and (2) in general Banach spaces can be found in [3, 4]; for certain Banach spaces as coorbit spaces and shift-invariant subspaces of L^p, one may look at [7] and [1], respectively.

In the present paper we define two concepts for "duals", namely, *dual* and *dual**, based on (1) and (2), respectively, and based on the upper X_d^*-frame condition (see Definition 4.2). In Section 4, connection between these two concepts for X_d-Bessel sequences is considered. If X_d has the canonical vectors as a basis and the X_d-Bessel sequence $\{g_i\}_{i=1}^{\infty}$ has a dual or dual*, then $\{g_i\}_{i=1}^{\infty}$ is an X_d-frame (Lemma 4.4). One assertion from [3] is improved by the use of weaker assumption (see Theorem 4.5). Section 5 contains an example of series expansions in the form (1) and (2) via an X_d-frame $\{g_i\}_{i=1}^{\infty}$ and sequence $\{f_i\}_{i=1}^{\infty}$, which is not a dual of $\{g_i\}_{i=1}^{\infty}$. As preliminary results, Section 3 contains investigation of the upper and the lower X_d-frame conditions separately. Some results in Section 3 are from the Ph.D. Thesis [15] and their proofs are included here for the sake of completeness.

2. Basic Notations

For the entire paper, X and Y denote Banach spaces, X^* denotes the dual of X. The notion *operator* is used for a linear mapping. It is said that an operator G is defined from X *onto* Y if its range $\mathcal{R}(G)$ coincides with Y. The restriction of an operator $G : X \to Y$ on a subset M of X is denoted by $G|_M$. The notation $\{g_i\}_{i=1}^{\infty} \subset Y$ is used with the meaning $g_i \in Y$, $\forall\, i \in \mathbb{N}$. The n-th canonical vector $\{\delta_{ni}\}_{i=1}^{\infty}$ is denoted by e_n, $n \in \mathbb{N}$.

Throughout the paper, X_d denotes a Banach space of scalar sequences and X_d^* denotes the dual of X_d. Recall that X_d is called: *BK-space*, if the coordinate functionals are continuous, or equivalently, if convergence in X_d implies convergence by coordinates; *CB-space*, if the canonical vectors form a Schauder basis for X_d; *RCB-space*, if it is reflexive CB-space. An operator G, given by $G\{c_i\}_{i=1}^{\infty} := \sum_{i=1}^{\infty} c_i g_i$ ($g_i \in Y, i \in \mathbb{N}$), is called *well defined from X_d into Y* if the series $\sum_{i=1}^{\infty} c_i g_i$ converges in Y for every $\{c_i\}_{i=1}^{\infty} \in X_d$.

Lemma 2.1. (see [10, p. 201]) *If X_d is a CB-space, then the space $X_d^{\circledast} := \{\{Ge_i\}_{i=1}^{\infty} : G \in X_d^*\}$ with the norm $\|\{Ge_i\}_{i=1}^{\infty}\|_{X_d^{\circledast}} := \|G\|_{X_d^*}$ is a*

BK-space isometrically isomorphic to X_d^*.

Throughout the paper, when X_d is a CB-space, X_d^* is identified with $X_d^{\circledast}$.

Let X_d be a CB-space and let $\{E_i\}_{i=1}^{\infty}$ denote the sequence of the coefficient functionals, associated to the canonical basis $\{e_i\}_{i=1}^{\infty}$. For every $n\in\mathbb{N}$, the n-th canonical vector $\{E_n(e_i)\}_{i=1}^{\infty}$ belongs to $X_d^{\circledast}$, but $X_d^{\circledast}$ is not necessarily a CB-space. Take for example the CB-space ℓ^1, whose dual is the nonseparable space ℓ^{∞}. The space X_d will be called a *$CBCB$-space* if both X_d and $X_d^{\circledast}$ are CB-spaces. Note that if X_d is an RCB-space, then the coefficient functionals $\{E_i\}_{i=1}^{\infty}$ form a Schauder basis of X_d^* (see e.g. [8]) and thus the canonical vectors $\{E_n(e_i)\}_{i=1}^{\infty}$, $n\in\mathbb{N}$, form a Schauder basis of $X_d^{\circledast}$, i.e. X_d is a $CBCB$-space. A $CBCB$-space is not necessarily an RCB-space, take for example the non-reflexive CB-space c_0, whose dual is the CB-space ℓ^1. Finally, note that every CB-space is a BK-space; the converse does not hold in general, take for example the BK-spaces ℓ^{∞} and C (the subspace of ℓ^{∞}, consisting of all convergent sequences), which are not CB-spaces. Therefore, the classes of sequence spaces under consideration are connected via the implications:

$$RCB \underset{\not\Leftarrow}{\Rightarrow} CBCB \underset{\not\Leftarrow}{\Rightarrow} CB \underset{\not\Leftarrow}{\Rightarrow} BK.$$

Remark. It is easy to see that Lemma 2.1 holds in the following more general case: *If Y is a Banach space and $\{y_i\}_{i=1}^{\infty}$ is a complete system in Y, then $Y^{\circledast} := \{\{Gy_i\}_{i=1}^{\infty} : G \in Y^*\}$ normed by $\|\{Gy_i\}_{i=1}^{\infty}\|_{Y^{\circledast}} := \|G\|_{Y^*}$ is a BK-space, isometrically isomorphic to Y^*.* Thus, the dual of every separable Banach space can be considered as a BK-space, because every separable Banach space has a complete system, see [13, p. 219].

3. X_d-Frames

Recall that $\{g_i\}_{i=1}^{\infty} \subset \mathcal{H}$ is called a *(Hilbert) frame* for the Hilbert space $\mathcal{H}$ if there exist constants $0 < A \le B < \infty$ such that $A\|f\|_{\mathcal{H}}^2 \le \sum_{i=1}^{\infty} |\langle f, g_i\rangle|^2 \le B\|f\|_{\mathcal{H}}^2$, $\forall f \in \mathcal{H}$. We consider the following generalizations of frames to Banach spaces, introduced in [1, 6, 7].

Definition 3.1. Let X_d be a BK-space and $\{g_i\}_{i=1}^{\infty} \subset X^*$. The sequence $\{g_i\}_{i=1}^{\infty}$ is called a *Banach frame for X with respect to X_d*, if:

(*a*) $\{g_i(f)\}_{i=1}^{\infty} \in X_d, \quad \forall f \in X$,

(*b*) $\exists \quad 0 < A \le B < \infty \quad : \quad A\|f\|_X \le \|\{g_i(f)\}_{i=1}^{\infty}\|_{X_d} \le B\|f\|_X, \forall f \in X$,

(c) $\exists$ bounded operator $S : X_d \to X$ such that $S\{g_i(f)\}_{i=1}^{\infty} = f, \ \forall f \in X$.

When (a) and (b) are satisfied, $\{g_i\}_{i=1}^{\infty}$ is called an X_d-frame for X. The constant B (resp. A) is called an upper (resp. lower) bound for $\{g_i\}$.

When (a) and the upper inequality in (b) are satisfied, $\{g_i\}_{i=1}^{\infty}$ is called an X_d-Bessel sequence for X with bound B.

In particular, an ℓ^p-frame (resp. ℓ^p-Bessel sequence) for X is called a *p-frame (resp. p-Bessel sequence) for X* .

We continue with consideration of the upper and the lower X_d-frame conditions separately. To given space X_d and sequence $\{g_i\}_{i=1}^{\infty} \subset X^*$, associate the following operators:

$$Uf := \{g_i(f)\}_{i=1}^{\infty}\ , \ \ \mathcal{D}(U) = \{f \in X \ : \ \{g_i(f)\}_{i=1}^{\infty} \in X_d\}; \tag{3}$$

$$T\{d_i\}_{i=1}^{\infty} := \sum_{i=1}^{\infty} d_i g_i\ , \ \ \mathcal{D}(T) = \{\{d_i\}_{i=1}^{\infty} \in X_d^* : \sum_{i=1}^{\infty} d_i g_i \in X^*\}. \tag{4}$$

3.1. The Upper X_d-Frame Condition

When $\{g_i\}_{i=1}^{\infty}$ is an X_d-Bessel sequence for X, it is clear that U is a bounded operator from X into X_d. Below we review some needed assertions, which connect the X_d-Bessel condition to the operator T.

Proposition 3.2. (see [3]) *Let X_d be a $CBCB$-space. If $\{g_i\}_{i=1}^{\infty} \subset X^*$ is an X_d-Bessel sequence for X with bound B, then $\mathcal{D}(T) = X_d^*$ and $\|T\| \leq B$. When X_d is moreover reflexive, the converse is true.*

Lemma 3.3. *Let X_d be a $CBCB$-space and $\{g_i\} \subset X^*$ be an X_d-Bessel sequence for X. Then $U^* = T$ and $U = T^*|_X$.*

Proof. By Proposition 3.2, T is bounded from $X_d^{\circledast}$ into X^* and clearly, U is bounded from X into X_d. Let $G \in X_d^*$ and $\{d_i\}_{i=1}^{\infty}$ be its corresponding element in $X_d^{\circledast}$ (see Lemma 2.1). For every $f \in X$,

$$(U^*(G))(f) = G\left(\sum_{i=1}^{\infty} g_i(f) e_i\right) = \sum_{i=1}^{\infty} d_i g_i(f) = (T\{d_i\}_{i=1}^{\infty})(f),$$

which implies that $U^*(G) = T\{d_i\}_{i=1}^{\infty}$. Since X_d^* is identified with $X_d^{\circledast}$, one can identify U^* with T. Thus one can also write $U^{**} = T^*$, which implies that $U = T^*|_X$. $\square$

Concerning X_d^*-Bessel sequences, similar assertions hold:

Proposition 3.4. *Let X_d be a CB-space and let U' and T' are given by*

$$U'f := \{g_i(f)\}_{i=1}^\infty\,, \quad \mathcal{D}(U') = \{f \in X \ : \ \{g_i(f)\}_{i=1}^\infty \in X_d^*\};$$

$$T'\{d_i\}_{i=1}^\infty := \sum_{i=1}^\infty d_i g_i\,, \quad \mathcal{D}(T') = \{\{d_i\}_{i=1}^\infty \in X_d \ : \ \sum_{i=1}^\infty d_i g_i \in X^*\}.$$

Then the following hold.

(i) (see [3]) *The sequence $\{g_i\} \subset X^*$ is an X_d^*-Bessel sequence for X with bound B if and only if $\mathcal{D}(T') = X_d$ and $\|T'\| \le B$.*

(ii) If $\{g_i\}_{i=1}^\infty \subset X^*$ is an X_d^*-Bessel sequence for X, then $T' = U'^* \,|_{X_d}$ and $U' = T'^* \,|_X$.

3.2. The Lower X_d-Frame Condition

In this subsection we consider a sequence $\{g_i\}_{i=1}^\infty \subset X^*$ such that

$$\exists A > 0 \ : \ A\|f\|_X \le \|\{g_i(f)\}_{i=1}^\infty\|_{X_d}\,, \ \forall f \in \mathcal{D}(U). \tag{5}$$

The following lemma generalizes results from [5, 14] which concern the case $X_d = \ell^p$, $p \in (1, \infty)$.

Lemma 3.5. *Let X_d be a BK-space and $\{g_i\} \subset X^*$ satisfies (5). Then U is an injective closed operator whose range $\mathcal{R}(U)$ is closed in X_d and the inverse $U^{-1}\colon R(U) \to \mathcal{D}(U)$ is bounded with $\|U^{-1}\| \le \frac{1}{A}$.*

The proof goes in the same way as in the case $X_d = \ell^p$, see [14, Lemma 3.1].

It is known that if F is a bounded operator from a Banach space Y into a linear normed space L and there exists bounded inverse $F^{-1} : R(F) \to Y$, then $R(F)$ is closed in L, [10, p. 466]. Note that in Lemma 3.5 the domain $\mathcal{D}(U)$ is not necessarily closed in X, i.e. $\mathcal{D}(U)$ is not necessarily a Banach space, and $\mathcal{R}(U)$ is obtained to be closed. Sequences $\{g_i\}_{i=1}^\infty \subset X^*$, which satisfy the conditions of Lemma 3.5 and such that $\mathcal{D}(U)$ is not closed in X, can be seen in [14, Examples 3.1 and 5.1].

In Hilbert spaces the lower frame condition alone implies series expansions of the elements of $\mathcal{D}(U)$ — it is known [2] that $\{g_i\}_{i=1}^\infty \subset \mathcal{H}$ satisfies the lower frame-condition for $\mathcal{H}$ if and only if there exists a Bessel sequence $\{f_i\} \subset \mathcal{H}$ for $\mathcal{H}$ such that $f = \sum_{i=1}^\infty \langle f, g_i \rangle f_i$, $\forall f \in \mathcal{D}(U)$. In Banach spaces the lower X_d-frame condition (5) is not enough for the existence of series expansions of the elements of $\mathcal{D}(U)$ via an X_d^*-Bessel sequence. Necessary and sufficient condition in the case $X_d = \ell^p$ is given in [15]. Below we consider the general case of a

CB-space X_d.

Proposition 3.6. *Let X_d be a CB-space and let $\{g_i\}_{i=1}^\infty \subset X^*$ satisfy (5). Then the following assertions are equivalent.*

($\mathcal{A}_1$) There exists an X_d^*-Bessel sequence $\{f_i\}_{i=1}^\infty \subset X(\subseteq X^{**})$ for X^* such that

$$f = \sum_{i=1}^{\infty} g_i(f) f_i, \ \forall f \in \mathcal{D}(U). \tag{6}$$

($\mathcal{A}_2$) The operator $U^{-1} : \mathcal{R}(U) \to X$ has a bounded extension defined on X_d.

Proof. ($\mathcal{A}_1$)⇒($\mathcal{A}_2$). By Proposition 3.4, there exists a bounded operator $V : X_d \to X$, $V\{c_i\}_{i=1}^\infty = \sum_{i=1}^\infty c_i f_i$. For every $f \in \mathcal{D}(U)$,

$$V\{g_i(f)\}_{i=1}^\infty = \sum_{i=1}^{\infty} g_i(f) f_i = f = U^{-1}Uf = U^{-1}\{g_i(f)\}_{i=1}^\infty.$$

Therefore, V is an extension of U^{-1}.

($\mathcal{A}_2$)⇒($\mathcal{A}_1$). Let $V : X_d \to X$ be a bounded extension of U^{-1}. Define $f_i := Ve_i$, $i \in \mathbb{N}$. For every $f \in \mathcal{D}(U)$,

$$f = VUf = V\left(\sum_{i=1}^{\infty} g_i(f) e_i\right) = \sum_{i=1}^{\infty} g_i(f) f_i.$$

For every $g \in X^*$, one has $gV \in (X_d)^*$, i.e. $\{gV(e_i)\}_{i=1}^\infty \in X_d^{\circledast}$, and

$$\|\{g(f_i)\}_{i=1}^\infty\|_{X_d^{\circledast}} = \|gV\|_{X_d^*} \le \|V\| \, \|g\|_{X^*}.$$

Therefore, $\{f_i\}_{i=1}^\infty$ (considered as a sequence with elements in X^{**}) is an X_d^*-Bessel sequence for X^*. □

In some cases (6) holds for all $f \in X$, take for example the sequence $\{g_i\}_{i=1}^\infty = \{iz_i\}_{i=1}^\infty$ and $\{f_i\}_{i=1}^\infty = \{\frac{1}{i} z_i\}_{i=1}^\infty$, where $\{z_i\}_{i=1}^\infty$ denotes an orthonormal basis of a Hilbert space X and $X_d = \ell^2$. In general this is not so — examples in the Hilbert space setting can be found in [2, 14].

In the Hilbert case, if the sequence $\{g_i\}_{i=1}^\infty \subset \mathcal{H}$ satisfies the lower frame-condition, then ($\mathcal{A}_2$) is always satisfied and thus ($\mathcal{A}_1$) is always satisfied; the X_d^*-Bessel sequence $\{f_i\}_{i=1}^\infty$ for $\mathcal{H}^*$ in this case is actually a Bessel sequence for $\mathcal{H}$. Thus ($\mathcal{A}_2$) is the needed additional condition going from the Hilbert case to the Banach case.

The following lemma shows that there is sense to speak about ($\mathcal{A}_1$) only if $\{g_i\}_{i=1}^\infty \subset X^*$ satisfies the lower X_d-frame condition for X.

Lemma 3.7. *Let X_d be a CB-space and $\{g_i\}_{i=1}^\infty \subset X^*$. Assume that $(\mathcal{A}_1)$ holds. Then $\{g_i\}_{i=1}^\infty$ satisfies (5).*

Proof. Let B denote a bound for the X_d^*-Bessel sequence $\{f_i\}_{i=1}^\infty$. For every $g \in X^*$, the sequence $\{g(f_i)\}_{i=1}^\infty$ belongs to $X_d^{\circledast}$ and it can be written as $\{G_g(e_i)\}_{i=1}^\infty$ for some $G_g \in X_d^*$ (see Lemma 2.1). For every $f \in \mathcal{D}(U)$, the sequence $\{g_i(f)\}_{i=1}^\infty$ belongs to X_d and

$$\begin{aligned}
\|f\| = \left\| \sum_{i=1}^\infty g_i(f) f_i \right\| &= \sup_{g\in X^*, \|g\|\le 1} \left| g\left(\sum_{i=1}^\infty g_i(f) f_i \right) \right| \\
&= \sup_{g\in X^*, \|g\|\le 1} \left| G_g\left(\{g_i(f)\}_{i=1}^\infty \right) \right| \\
&\le \sup_{g\in X^*, \|g\|\le 1} \|\{g(f_i)\}_{i=1}^\infty\|_{X_d^*} \, \|\{g_i(f)\}_{i=1}^\infty\|_{X_d} \\
&\le B \, \|\{g_i(f)\}_{i=1}^\infty\|_{X_d}. \qquad \square
\end{aligned}$$

As a consequence of Proposition 3.6 and Lemma 3.7, the following assertion holds.

Theorem 3.8. *Let X_d be a CB-space and $\{g_i\}_{i=1}^\infty \subset X^*$. Then: $(\mathcal{A}_1)$ holds $\Leftrightarrow$ $\{g_i\}_{i=1}^\infty$ satisfies (5) and $(\mathcal{A}_2)$ holds.*

3.3. The X_d-Frame Condition

Based on Proposition 3.2 and Lemma 3.3, an equivalent characterization of the X_d-frame condition can be given as follows:

Theorem 3.9. *Let X_d be a $CBCB$-space. If $\{g_i\}_{i=1}^\infty \subset X^*$ is an X_d-frame for X, then the operator T is well defined (and hence bounded) from X_d^* onto X^*. When X_d is moreover reflexive, the converse also holds.*

The proof goes in the same way as in the case $X_d = \ell^p$, given in [6, Theorem 2.4].

Compare the definitions of a (Hilbert) frame and a 2-frame. It is natural to observe that a 2-frame for a Banach space gives a (Hilbert) frame:

Proposition 3.10. *Let $\{g_i\}_{i=1}^\infty \subset X^*$ be a 2-frame for X. Then X can be equipped with an inner product, in which X becomes a Hilbert space and $\{g_i\}_{i=1}^\infty$ - a frame for X.*

Proof. Since $\{g_i\}_{i=1}^\infty$ is a 2-frame for X, the operator U, defined by (3), is an isomorphism of X and $\mathcal{R}(U)$. By Lemma 3.5, $\mathcal{R}(U)$ is a closed subspace of

ℓ^2 and thus $\mathcal{R}(U)$ is a Hilbert space. The relation $\langle .,. \rangle_1$, defined by

$$\langle x_1, x_2 \rangle_1 := \langle Ux_1, Ux_2 \rangle_{\ell^2}, \text{ for } x_1, x_2 \in X, \tag{7}$$

determines an inner product in X and the space $X_H := (X, \|.\|_1)$, where $\|.\|_1 := \sqrt{\langle x, x \rangle_1}$, is a Hilbert space.

Let A be a lower bound for the 2-frame $\{g_i\}_{i=1}^{\infty}$. For every $i \in \mathbb{N}$ and every $f \in X$,

$$|g_i(f)| \le \|g_i\|_{X^*} \|f\|_X \le \|g_i\|_{X^*} \frac{1}{A} \|Uf\|_2 = \frac{1}{A} \|g_i\|_{X^*} \|f\|_1.$$

Therefore, the functional g_i, considered as an operator defined on X_H, is bounded on X_H. By definition,

$$\|f\|_1 = \|Uf\|_{\ell^2} = \|\{g_i(f)\}_{i=1}^{\infty}\|_{\ell^2}, \ \forall f \in X_H,$$

which implies that $\{g_i\}_{i=1}^{\infty} \subset (X_H)^* = X_H$ is a frame for X_H. □

We finish the section with few simple examples.

Example 3.11. Let $1 < r < \infty$ and let $\{E_i\}_{i=1}^{\infty}$ denote the coefficient functionals, associated to the canonical basis $\{e_i\}_{i=1}^{\infty}$ of ℓ^r.

1. $\{E_i\}_{i=1}^{\infty}$ is an r-frame for ℓ^r.

2. If $1 < r < p < \infty$, then $\{E_i\}_{i=1}^{\infty}$ is a p-Bessel sequence for ℓ^r, but not a p-frame for ℓ^r.

Indeed, for every $f \in \ell^r$, we have that $\{E_i(f)\}_{i=1}^{\infty} = f \in \ell^r \subset \ell^p$ and $\|\{E_i(f)\}_{i=1}^{\infty}\|_p = \|f\|_p \le \|f\|_r$ (see [9], [12, p. 31]). Therefore, $\{E_i\}_{i=1}^{\infty}$ is a p-Bessel sequence for ℓ^r. Since the set $\mathcal{R}(U) = \{\{E_i(d)\}_{i=1}^{\infty} \mid d \in \ell^r\}$ consists of the elements of ℓ^r and ℓ^r is not closed in ℓ^p, Lemma 3.5 implies that $\{G_i\}_{i=1}^{\infty}$ does not satisfy the lower p-frame condition for ℓ^r.

3. If $1 < p < r < \infty$, then $\{E_i\}_{i=1}^{\infty}$ satisfies the lower p-frame condition for $\mathcal{D}(U)$, but fails the upper one.

Indeed, in this case $\mathcal{D}(U) = \{f \in \ell^r : \{E_i(f)\}_{i=1}^{\infty} \in \ell^p\}$, i.e. $\mathcal{D}(U)$ consists of the elements of ℓ^p. For every $d \in \mathcal{D}(U)$, one has that $\|d\|_r \le \|d\|_p = \|\{E_i(d)\}_{i=1}^{\infty}\|_p$. For $f_n = e_1 + e_2 + \cdots + e_n \in \mathcal{D}(U)$, $n \in \mathbb{N}$,

$$\|Uf_n\|_{\ell^p} = \|f_n\|_{\ell^p} = n^{\frac{1}{p}} = n^{\frac{1}{p}-\frac{1}{r}} \|f_n\|_r \ \text{ and } \ \lim_{n\to\infty} n^{\frac{1}{p}-\frac{1}{r}} = \infty,$$

which implies that $\{E_i\}_{i=1}^{\infty}$ fails the upper p-frame condition for $\mathcal{D}(U)$.

4. Duals of X_d-Bessel Sequences

As it was recalled in the introduction, if $\{g_i\}_{i=1}^{\infty} \subset \mathcal{H}$ is a frame for the Hilbert space $\mathcal{H}$, then there exists a frame $\{f_i\}_{i=1}^{\infty}$ for $\mathcal{H}$ such that $h = \sum_{i=1}^{\infty} \langle f, g_i \rangle f_i = \sum_{i=1}^{\infty} \langle f, f_i \rangle g_i, \forall h \in \mathcal{H}$; $\{f_i\}_{i=1}^{\infty}$ is called a *dual frame of* $\{g_i\}_{i=1}^{\infty}$. By Theorem 3.9, when X_d is an RCB-space, the X_d-frame condition for $\{g_i\}$ alone implies existence of reconstruction series in X^*, namely, every $g \in X^*$ can be written as $g = \sum_{i=1}^{\infty} c_i^g g_i$ with some coefficients $\{c_i^g\}_{i=1}^{\infty} \in X_d^*$. However, if one wants series expansions in the form (1) or (2), the X_d-frame property is not enough — Casazza has proved that there exist p-frames $\{g_i\}_{i=1}^{\infty}$, for which no family $\{f_i\}_{i=1}^{\infty}$ satisfying (1) exists. In order to have (1) or (2), the Banach frame condition is important in some cases. Let $\{g_i\}_{i=1}^{\infty} \subset X^*$ and let X_d be a BK-space. Consider the following conditions:

($\mathcal{P}_1$) $\{g_i\}_{i=1}^{\infty}$ is a Banach frame for X with respect to X_d.

($\mathcal{P}_2$) There exists an X_d^*-Bessel sequence $\{f_i\}_{i=1}^{\infty} \subset X$ ($\subseteq X^{**}$) for X^* such that

$$f = \sum_{i=1}^{\infty} g_i(f) f_i, \quad f \in X. \tag{8}$$

($\mathcal{P}_3$) There exists an X_d^*-Bessel sequence $\{f_i\}_{i=1}^{\infty} \subset X$ ($\subseteq X^{**}$) for X^* such that

$$g = \sum_{i=1}^{\infty} g(f_i) g_i, \quad g \in X^*. \tag{9}$$

Proposition 4.1. (see [3]) *Let $\{g_i\}_{i=1}^{\infty} \subset X^*$ be an X_d-frame for X. The following assertions hold.*

(i) If X_d is a CB-space, then $(\mathcal{P}_1) \Leftrightarrow (\mathcal{P}_2)$.

(ii) If X_d is a $CBCB$-space, then $(\mathcal{P}_1) \Leftrightarrow (\mathcal{P}_2) \Leftrightarrow (\mathcal{P}_3)$ and in each of the cases $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$, $\{f_i\}_{i=1}^{\infty}$ is an X_d^*-frame for X^*.

One of our aims is to show that Proposition 4.1 holds under the weaker assumption that $\{g_i\}_{i=1}^{\infty}$ is an X_d-Bessel sequence for X (see Theorem 4.5). Based on Proposition 4.1, we consider two kind of "duals":

Definition 4.2. Let $\{g_i\}_{i=1}^{\infty} \subset X^*$. A sequence $\{f_i\}_{i=1}^{\infty} \subset X$ is called:

— *dual of* $\{g_i\}_{i=1}^{\infty}$, if $\{f_i\}_{i=1}^{\infty}$ is an X_d^*-Bessel sequence for X^* and (8) holds;

— *dual* of* $\{g_i\}_{i=1}^{\infty}$, if $\{f_i\}_{i=1}^{\infty}$ is an X_d^*-Bessel sequence for X^* and (9) holds.

Note that if X_d is a CB-space, $\{g_i\}_{i=1}^\infty \subset X^*$ is an X_d-frame for X and U^{-1} has a bounded linear extension $V : X_d \to X$, then the sequence $\{f_i\}_{i=1}^\infty := \{Ve_i\}_{i=1}^\infty$ is a dual of $\{g_i\}_{i=1}^\infty$ (see the proof of [3, Proposition 3.4]). Let us now consider the connection between *dual* and *dual**.

Lemma 4.3. *Let $\{g_i\}_{i=1}^\infty$ be an X_d-Bessel sequence for X_d.*

(i) If X_d is a CB-space, then a dual of $\{g_i\}_{i=1}^\infty$ is a dual of $\{g_i\}_{i=1}^\infty$, however a dual is not necessarily a dual*.*

(ii) If X_d is a $CBCB$-spaces, then a sequence $\{f_i\}_{i=1}^\infty$ is a dual of $\{g_i\}_{i=1}^\infty$ if and only if $\{f_i\}_{i=1}^\infty$ is a dual of $\{g_i\}_{i=1}^\infty$.*

Proof. Validity of this statement follows from calculations in the proof of [3, Proposition 3.4], but for the sake of completeness we give a proof here.

(i) Let X_d be a CB-space. Assume that $\{f_i\}_{i=1}^\infty$ is a dual* of $\{g_i\}_{i=1}^\infty$ and let $\widetilde{B}$ denote an X_d^*-Bessel bound for $\{f_i\}_{i=1}^\infty$. For every $g \in X^*$, $\{g(f_i)\}_{i=1}^\infty$ can be written as $\{G_g(e_i)\}_{i=1}^\infty$ for some $G_g \in X_d^*$ (see Lemma 2.1). Thus, for every $f \in X$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\left\| f - \sum_{i=1}^n g_i(f) f_i \right\|_X &= \sup_{g\in X^*, ||g||=1} \left| g(f) - \sum_{i=1}^n g(f_i) g_i(f) \right| \\
&= \sup_{g\in X^*, ||g||=1} \left| G_g\left(\sum_{i=n+1}^\infty g_i(f) e_i \right) \right| \\
&\leq \sup_{g\in X^*, ||g||=1} ||G_g|| \, \left\| \sum_{i=n+1}^\infty g_i(f) e_i \right\| \\
&= \sup_{g\in X^*, ||g||=1} \|\{g(f_i)\}_1^\infty\| \, \left\| \sum_{i=n+1}^\infty g_i(f) e_i \right\| \\
&\leq \widetilde{B} \left\| \sum_{i=n+1}^\infty g_i(f) e_i \right\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Therefore, (8) holds.

Consider the CB-space $X_d = \ell^1$. Let $X = \ell^1$ and $\{E_i\}_{i=1}^\infty \in X^*$ be the sequence of the coordinate functionals, associated to the canonical basis $\{e_i\}_{i=1}^\infty$ of ℓ^1. Then $\{E_i\}_{i=1}^\infty \in X^*$ is an ℓ^1-frame for ℓ^1, $\{e_i\}_{i=1}^\infty$ is an X_d^*-Bessel sequence for X^* and $f = \sum_{i=1}^\infty E_i(f)e_i$, $\forall f \in X$. However, not all $g \in X^* = \ell^\infty$ can be written as $g = \sum_{i=1}^\infty g(e_i)E_i$. Thus $\{e_i\}_{i=1}^\infty$ is a dual of $\{E_i\}_{i=1}^\infty$, but not a dual* of $\{E_i\}_{i=1}^\infty$.

(ii) Let X_d be a $CBCB$-space, B denote a Bessel bound for $\{g_i\}_{i=1}^\infty$ and $\{z_i\}_{i=1}^\infty$ denote the canonical basis for X_d^*. Assume that $\{f_i\}_{i=1}^\infty$ is a dual of $\{g_i\}_{i=1}^\infty$. For every $g \in X^*$ and $n \in \mathbb{N}$,

$$\begin{aligned}\left\|g - \sum_{i=1}^n g(f_i)g_i\right\|_{X^*} &= \sup_{f\in X, ||f||=1} \left|g(f) - \sum_{i=1}^n g(f_i)g_i(f)\right| \\ &= \sup_{f\in X, ||f||=1} \left|\sum_{i=n+1}^\infty g(f_i)g_i(f)\right| \\ &\leq B \left\|\sum_{i=n+1}^\infty g(f_i)z_i\right\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

and thus (9) holds. □

Lemma 4.4. *Let X_d be a CB-space and $\{g_i\}_{i=1}^\infty \subset X^*$ be an X_d-Bessel sequence for X. If $\{g_i\}_{i=1}^\infty$ has a dual or dual*, then $\{g_i\}_{i=1}^\infty$ is an X_d-frame for X.*

Proof. If $\{g_i\}_{i=1}^\infty$ has a dual, then Lemma 3.7 implies that $\{g_i\}_{i=1}^\infty$ satisfies the lower X_d-frame inequality for all $f\in X$.

Let now $\{g_i\}_{i=1}^\infty$ has a dual*. By Lemma 4.3 (i), $\{g_i\}_{i=1}^\infty$ has a dual and the rest follows again from Lemma 3.7. □

Theorem 4.5. *Let $\{g_i\}_{i=1}^\infty \subset X^*$ be an X_d-Bessel sequence for X. Then :*

(i) the conclusion of Proposition 4.1 hold;

(ii) when X_d is an RCB-space, any one of $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$ implies that $\{f_i\}_{i=1}^\infty$ is a Banach frame for X^* with respect to X_d^*.

Proof. Validity of (i) follows from Lemma 4.4 and Proposition 4.1.

(ii) Assume now that X_d is a reflexive CB-space, which implies that X_d^* is a CB-space. By Lemma 4.3(ii), $(\mathcal{P}_2)$ is equivalent to $(\mathcal{P}_3)$. Let $(\mathcal{P}_3)$ hold. By Lemma 4.4, $\{g_i\}_{i=1}^\infty$ is an X_d-frame for X. Therefore, X is also reflexive, because X is isomorphic to the closed subspace $\mathcal{R}(U)$ of X_d. By (i), $\{f_i\}_{i=1}^\infty$ is an X_d^*-frame for X^*. Now, applying the equivalence of $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ with the changed role of $\{f_i\}_{i=1}^\infty$ and $\{g_i\}_{i=1}^\infty$, we obtain that $\{f_i\}_{i=1}^\infty$ is a Banach frame for X^* with respect to X_d^*. □

5. Series Expansions via X_d-Frames and Sequences, which are not Duals

Li and Ogawa [11] gave an example of a frame $\{g_i\}_{i=1}^{\infty}$ for a Hilbert space $\mathcal{H}$ and a sequence $\{f_i\}_{i=1}^{\infty}$, which satisfies $h=\sum_{i=1}^{\infty}\langle h,g_i\rangle f_i=\sum_{i=1}^{\infty}\langle h,f_i\rangle g_i$, $\forall h\in\mathcal{H}$, but which is not a frame for $\mathcal{H}$. We give an example of an X_d-frame $\{g_i\}_{i=1}^{\infty}$ and a sequence $\{f_i\}_{i=1}^{\infty}$, which satisfies (1) and (2), but which is not a dual of $\{g_i\}_{i=1}^{\infty}$.

Example 5.1. Let $X=X_d=\ell^p$, $1<p<\infty$, and let $\{E_i\}_{i=1}^{\infty}$ be the sequence of the coefficient functionals associated to the canonical basis $\{e_i\}_{i=1}^{\infty}$ of X. Denote

$$\{g_i\}_{i=1}^{\infty}:=\left\{\frac{1}{2}E_1,E_2,\frac{1}{2^2}E_1,E_3,\frac{1}{2^3}E_1,E_4,\dots\right\},$$

$$\{f_i\}_{i=1}^{\infty}:=\{e_1,e_2,e_1,e_3,e_1,e_4,\dots\}.$$

Then $\{g_i\}_{i=1}^{\infty}$ is a Banach frame for X with respect to ℓ^p, $\{f_i\}_{i=1}^{\infty}$ satisfies the lower q-frame condition for X^*, but does not satisfy the upper one, and

$$f=\sum_{i=1}^{\infty}g_i(f)f_i,\ \forall f\in X,\quad\text{and}\quad g=\sum g(f_i)g_i,\ \forall g\in X^*. \tag{10}$$

Proof. For every $g\in X^*$, $(\sum_{i=1}^{\infty}|g(f_i)|^q)^{1/q}\geq(\sum_{i=1}^{\infty}|g(e_i)|^q)^{1/q}=\|g\|_{(\ell^p)^*}$. Therefore, $\{f_i\}_{i=1}^{\infty}$ satisfies the lower q-frame condition for X^*. If $g\in X^*$ is such that $g(e_1)\neq 0$, then $\sum_{i=1}^{\infty}|g(f_i)|^q=\infty$, which implies that $\{f_i\}_{i=1}^{\infty}$ fails the upper q-frame condition for X^*.

Fix $f\in X$ and consider $S_n(f)=\sum_{i=1}^{n}|\,g_i(f)\,|^p$, $n\in\mathbb{N}$. For every $n\in\mathbb{N}$,

$$\begin{aligned}S_{2n}(f)&=|\,E_1(f)\,|^p\sum_{i=1}^{n}\frac{1}{2^{ip}}+\sum_{i=2}^{n+1}|\,E_i(f)\,|^p\\&=|\,E_1(f)\,|^p\left(\sum_{i=1}^{n}\frac{1}{2^{ip}}-1\right)+\sum_{i=1}^{n+1}|\,E_i(f)\,|^p\\&=|\,E_1(f)\,|^p\ \frac{1}{2^p-1}\left(2-2^p-\frac{1}{2^{pn}}\right)+\sum_{i=1}^{n+1}|\,E_i(f)\,|^p,\end{aligned}$$

$$S_{2n+1}(f)=|\,E_1(f)\,|^p\ \frac{1}{2^p-1}\left(2-2^p-\frac{1}{2^{p(n+1)}}\right)+\sum_{i=1}^{n+1}|\,E_i(f)\,|^p\ .$$

The increasing sequence $\{S_n(f)\}_{n=1}^{\infty}$ is bounded, because

$$0 \leq S_{2n}(f) \leq S_{2n+1}(f) \leq \sum_{i=1}^{n+1} |E_i(f)|^p \leq \|f\|^p, \quad \forall n \in \mathbb{N}.$$

Therefore, $\{S_n(f)\}_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} S_n(f) \leq \|f\|^p$, which implies that $\{g_i\}_{i=1}^{\infty}$ is a p-Bessel sequence for X.

For every $f \in X$,

$$\begin{aligned} 0 &\leq \left\| f - \sum_{i=1}^{2n} g_i(f) f_i \right\| = \left\| f + \frac{1}{2^n} E_1(f) e_1 - \sum_{i=1}^{n+1} E_i(f) e_i \right\| \\ &\leq \left\| f - \sum_{i=1}^{n+1} E_i(f) e_i \right\| + \frac{1}{2^n} \|E_1(f) e_1\| \to 0 \ \text{ as } n \to \infty, \\ 0 &\leq \left\| f - \sum_{i=1}^{2n+1} g_i(f) f_i \right\| \\ &\leq \left\| f - \sum_{i=1}^{n+1} E_i(f) e_i \right\| + \frac{1}{2^{n+1}} \|E_1(f) e_1\| \to 0 \ \text{ as } n \to \infty. \end{aligned}$$

Similar inequalities hold for $g \in X^*$. Therefore, (10) holds.

The sequence $\{x_i\}_{i=1}^{\infty} = \{2e_1, e_2, 0, e_3, 0, e_4, \ldots\}$ is a q-Bessel sequence for X^* and $f = \sum_{i=1}^{\infty} g_i(f) x_i$ for all $f \in X$. By Theorem 4.5, $\{g_i\}_{i=1}^{\infty}$ is a Banach frame for X with respect to ℓ^p. □

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