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A POSSIBLE SOLUTION FOR AN OPEN PROBLEM IN EXTENDED THERMODYNAMICS

M.C. Carrisi[1], S. Pennisi[2 §], A.R. Pillosu[3]
[1,2,3]Dipartimento di Matematica ed Informatica
Universitá di Cagliari
Via Ospedale 72, Cagliari, 09124, ITALY
[1]e-mail: cristina.carrisi@tiscali.it
[2]e-mail: spennisi@unica.it
[3]e-mail: anri.p78@virgilio.it

Abstract: Some years ago, Professor Brini and Ruggeri exploited the ET_m^α theories, that is theories in Extended Thermodynamics of m moments and of degree α; the index α represents the approximation degree for the distribution function, through its formal expansion in the neighborhood of the Maxwellian distribution. They proved, through simple examples of stationary problems that the entropy principle fails in general, if all the non-equilibrium variables are of the same order of magnitude; this is because some "critical derivatives" are not small along all the solutions.

Here we show that the above results do not mean that in the previous theory there is something wrong. We show also other ways to overcome the above mentioned difficulties.

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1. Introduction

In the paper [1] it is emphasized that the entropy principle plays a fundamental role in Extended Thermodynamics (see also [2]). However, it is exploiting in

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§Correspondence author

the full non linear case without any assumption about the non equilibrium processes. If processes not far from an equilibrium state are considered, an approximate distribution function is usually derived through a formal expansion in the neighborhood of the Maxwellian distribution and so an ET_m^α theory is obtained (A Grad-like system when $\alpha = 1$). In [1] it is proved through some stationary examples that the entropy principle fails inevitably, if all the non-equilibrium variables are of the same order of magnitude. In fact, there exist some first derivatives of the non-equilibrium variables (critical derivatives) that are not small along all the solutions. Hence they conclude that not all the solutions are acceptable and consequently only some Cauchy or boundary data are permitted. These results suggest a criterion to choose the so-called "non-controllable data", that is the boundary data for the moments of order higher than 13.

In the present paper we show that what said above does not indicate something wrong in the previous Extended Thermodynamics theory; in fact, there are some other ways to overcome the above mentioned problems. For example:

1. The previous theory never imposed that all the non-equilibrium variables and their derivatives are of the same order of magnitude. This is a new hypothesis introduced in [1], where it is also proved that this new hypothesis implies that the critical derivatives vanish on one of the boundaries, eventually with some of their successive derivatives.

2. An hypothesis, different from the previous one, starts from the consideration that in ordinary thermodynamics the mass density, velocity and absolute temperature ρ, v_i, T, are considered variables of the equilibrium state, while their first derivatives are variables of first order with respect to equilibrium (See [3], for example). We consider reasonable that this hypothesis holds also in Extended Thermodynamics, where also some other variables Z_B are introduced and they are considered of first order with respect to equilibrium. More precisely, we retain that

(a) ρ, v_i, T are variables of the equilibrium state,

(b) Z_B and the first derivatives of ρ, v_i, T are of first order,

(c) the first derivatives of Z_B are of second order.

In other words, derivation increases of one unity the order with respect to equilibrium.

In Section 2, we will see that, with this assumption, the problem exposed in [1] is not present. However, an objection may rise to this our new hypothesis, namely that the equation (19) of [1] seems to show that $\partial_x T$ is of zero order;

see the solution (26) of such equation in [1].

But this is not true because equations (19) are not evolution equations, but represent the condition that must be satisfied, on the boundary, in order that the resulting solution be stationary!

Consequently, it is not strange that the result does not respect the requested order, because the condition on the initial manifold may be fixed arbitrarily; on the contrary, the order of equations must be chosen consistently with that on the boundary.

Moreover, it has also to be said that (26) is not the unique solution of equations (19). Another solution is

$$p = \text{constants}, \quad T = \text{constants}, \quad \sigma = 0, \quad q = 0, \quad \Delta = 0; \tag{1}$$

how much natural this solution is, it can be seen also from the following fact: How may it be possible to have a stationary solution if $gradT \neq 0$? On the contrary, we expect that T changes, depending on time, until it assumes the same value in all the positions.

3. A third way in which to proceed is that of taking the main field as independent variables and of expressing the entropy principle through the 4-potentials. The order with respect to equilibrium may be defined as in the previous way 2); when this order is fixed, one has to take the 4-potentials up to this order and to choose suitable the order for the moments in order that the entropy principle is satisfied, and also the eventual symmetries between some of the moments with some of the fluxes. This method will be explained more clearly in Section 3 and exemplified in Section 4, with the case ET^2_{14}.

We shall see that this approach may lead to difficulties if we want to draw back to the moments as independent variables, instead of the main field; but we will see that also these difficulties can be faced successfully.

2. Consequences of the Ordering Defined in Point 2) of Introduction

We want to express now, with more details, what above said in point 2) of Introduction.

The left hand sides of equations (16) in [1] can be written as

$$\left(u'_A \frac{\partial u^A}{\partial \rho} - \frac{\partial h}{\partial \rho}\right) \partial_t \rho + \left(u'_A \frac{\partial u^A}{\partial v_j} - \frac{\partial h}{\partial v_j}\right) \partial_t v_j$$

$$+\left(u'_A\frac{\partial u^A}{\partial T}-\frac{\partial h}{\partial T}\right)\partial_t T+\left(u'_A\frac{\partial u^A}{\partial Z_B}-\frac{\partial h}{\partial Z_B}\right)\partial_t Z_B;$$

$$\left(u'_A\frac{\partial u^{iA}}{\partial \rho}-\frac{\partial h^i}{\partial \rho}\right)\partial_i \rho+\left(u'_A\frac{\partial u^{iA}}{\partial v_j}-\frac{\partial h^i}{\partial v_j}\right)\partial_i v_j$$
$$+\left(u'_A\frac{\partial u^{iA}}{\partial T}-\frac{\partial h^i}{\partial T}\right)\partial_i T+\left(u'_A\frac{\partial u^{iA}}{\partial Z_B}-\frac{\partial h^i}{\partial Z_B}\right)\partial_i Z_B;$$

If we want that this expression be zero up to *the order* $\alpha+1$, so that equations (16) of [1] hold, it is necessary and sufficient that

$$u'_A\frac{\partial u^A}{\partial \rho}-\frac{\partial h}{\partial \rho},\quad u'_A\frac{\partial u^A}{\partial v_j}-\frac{\partial h}{\partial v_j},\quad u'_A\frac{\partial u^A}{\partial T}-\frac{\partial h}{\partial T}, \tag{2}$$
$$u'_A\frac{\partial u^{iA}}{\partial \rho}-\frac{\partial h^i}{\partial \rho},\quad u'_A\frac{\partial u^{iA}}{\partial v_j}-\frac{\partial h^i}{\partial v_j},\quad u'_A\frac{\partial u^{iA}}{\partial T}-\frac{\partial h^i}{\partial T},$$

be zero up to the order α, while

$$u'_A\frac{\partial u^A}{\partial Z_B}-\frac{\partial h}{\partial Z_B},\quad u'_A\frac{\partial u^{iA}}{\partial Z_B}-\frac{\partial h^i}{\partial Z_B}; \tag{3}$$

be zero up to the order $\alpha-1$.

We can rewrite these conditions by denoting $\overset{\alpha}{\psi}$ the homogeneous part of order α of the quantity ψ and taking into account that

1. The variables u^A contain only terms of order 0 and of order 1,
2. Derivation with respect to ρ, v_j and T does not lower the order,
3. Derivation with respect to Z_B lowers the order of an unity,
4. The derivatives of $\overset{1}{u}{}^A$ with respect to ρ, v_j and T are zero, because $\overset{1}{u}{}^A$ depends only on Z_B.

After that, what said in equations (2) and (3) becomes

$$\sum_{\beta=0}^{\alpha}\left(\overset{\beta}{u}{}'_A\frac{\partial \overset{0}{u}{}^A}{\partial \rho}-\frac{\partial \overset{\beta}{h}}{\partial \rho}\right)=0\,,\qquad \sum_{\beta=0}^{\alpha}\left(\overset{\beta}{u}{}'_A\frac{\partial \overset{0}{u}{}^A}{\partial v_j}-\frac{\partial \overset{\beta}{h}}{\partial v_j}\right)=0\,,$$

$$\sum_{\beta=0}^{\alpha}\left(\overset{\beta}{u}{}'_A\frac{\partial \overset{0}{u}{}^A}{\partial T}-\frac{\partial \overset{\beta}{h}}{\partial T}\right)=0\,,\qquad \sum_{\beta=1}^{\alpha}\left(\overset{\beta-1}{u}{}'_B-\frac{\partial \overset{\beta}{h}}{\partial z_B}\right)=0\,,$$

$$\sum_{\beta=0}^{\alpha}\left(\sum_{\gamma=0}^{\beta}\overset{\beta-\gamma}{u}{}'_A\frac{\partial \overset{\gamma}{u}{}^{iA}}{\partial \rho}-\frac{\partial \overset{\beta}{h}{}^i}{\partial \rho}\right)=0\,,\qquad \sum_{\beta=0}^{\alpha}\left(\sum_{\gamma=0}^{\beta}\overset{\beta-\gamma}{u}{}'_A\frac{\partial \overset{\gamma}{u}{}^{iA}}{\partial v_j}-\frac{\partial \overset{\beta}{h}{}^i}{\partial v_j}\right)=0\,,$$

$$\sum_{\beta=0}^{\alpha}\left(\sum_{\gamma=0}^{\beta}\overset{\beta-\gamma}{u}{}'_A\frac{\partial \overset{\gamma}{u}{}^{iA}}{\partial T}-\frac{\partial \overset{\beta}{h}{}^{i}}{\partial T}\right)=0\,,\qquad \sum_{\beta=1}^{\alpha}\left(\sum_{\gamma=1}^{\beta}\overset{\beta-\gamma}{u}{}'_A\frac{\partial \overset{\gamma}{u}{}^{iA}}{\partial z_B}-\frac{\partial \overset{\beta}{h}{}^{i}}{\partial z_B}\right)=0\,.$$

Therefore, we need both h and h^i up to the order α.

By applying this criterion to the case of planar unidimensional ET^1_{14}, we find that the right hand side of (25) in [1] is exactly in agreement with the entropy principle; in fact, for this case we have $\alpha = 2$. Consequently, equation (25) has to be zero up to the order 3; this is what effectively occurs because

$\left(\frac{154q^3}{75kp^2T^3}+\frac{77q\Delta^2}{900k^2p^2T^4}\right)$	is of order 3,
$\partial_x T$	is of order 1,
$\frac{11q\Delta}{450k^2p^2T^3}$	is of order 2,
Δ	is of order 1,
$\partial_x\Delta$	is of order 2.

Consequently, all the right hand side of equation (25) is of the 4-th order and, then, to be neglected with respect to the order 3 which we are considering.

3. Consequences of the Procedure Introduced in Point 3) of Introduction

We propose now another solution of the problem described in [1]. It is more evident if we express the entropy principle (see equation (7) of [1]) in terms of the 4-potentials (see equation (8) of [1]) and if we take as independent variables the components of the main field, that is

$$u^A\partial_t u'_A-\partial_t h'=0,\quad u^{iA}\partial_i u'_A-\partial_i h'^i=0,\quad u'_A P^A=\Sigma\geq 0 \qquad (4)$$

Differently from what is done in [1], in these relations we do not substitute the solution of the field equations because, from Liu ' s theorem [4], we know that the entropy principle expressed in terms of the Lagrange multipliers (or main field) must hold for whatever value of the independent variables and not only on the solutions of the field equations!

Rather than that, let us apply the counterpart, for the Lagrange multipliers, of the ordering defined in point 2) of Introduction. To this end, let us call u'_B the Lagrange multipliers different from u', u'_{ll} ; so we have that:

— u', u'_{ll} are variables of the equilibrium state,

— u'_B and the first derivatives of u', u'_{ll} are of first order,

— The first derivatives of u'_B are of second order.

In this way equations (4) become

$$\left(u-\frac{\partial h'}{\partial u'}\right)\partial_t u^{'} + \left(\frac{1}{3}u^{ll}-\frac{\partial h'}{\partial u'_{ll}}\right)\partial_t u^{'}_{ll} + \left(u^B-\frac{\partial h'}{\partial u'_B}\right)\partial_t u^{'}_B = 0\,,$$
$$\left(u^i-\frac{\partial h'^i}{\partial u'}\right)\partial_i u^{'} + \left(\frac{1}{3}u^{ill}-\frac{\partial h'^i}{\partial u'_{ll}}\right)\partial_i u^{'}_{ll} + \left(u^{iB}-\frac{\partial h'^i}{\partial u'_B}\right)\partial_i u^{'}_B = 0\,.$$

If we want to impose them only up to the order $\alpha+1$, we need that

$$\left(u-\frac{\partial h'}{\partial u'}\right)^{[\alpha]}=0\,;\ \left(\frac{1}{3}u^{ll}-\frac{\partial h'}{\partial u'_{ll}}\right)^{[\alpha]}=0\,;\ \left(u^B-\frac{\partial h'}{\partial u'_B}\right)^{[\alpha-1]}=0\,,$$
$$\left(u^i-\frac{\partial h'^i}{\partial u'}\right)^{[\alpha]}=0\,;\ \left(\frac{1}{3}u^{ill}-\frac{\partial h'^i}{\partial u'_{ll}}\right)^{[\alpha]}=0\,;\ \left(u^{iB}-\frac{\partial h'^i}{\partial u'_B}\right)^{[\alpha-1]}=0\,,$$

where $\psi^{[\alpha]}$ denotes the expression of ψ leaving out all the terms with order greater than α. They are equivalent to

$$u^{[\alpha]}-\frac{\partial h'^{[\alpha]}}{\partial u'}=0\,;\quad \frac{1}{3}u^{ll[\alpha]}-\frac{\partial h'^{[\alpha]}}{\partial u'_{ll}}=0\,;\quad u^{B[\alpha-1]}-\frac{\partial h'^{[\alpha]}}{\partial u'_B}=0\,, \tag{5}$$
$$u^{i[\alpha]}-\frac{\partial h'^{i[\alpha]}}{\partial u'}=0\,;\quad \frac{1}{3}u^{ill[\alpha]}-\frac{\partial h'^{i[\alpha]}}{\partial u'_{ll}}=0\,;\quad u^{iB[\alpha-1]}-\frac{\partial h'^{i[\alpha]}}{\partial u'_B}=0\,.$$

With this in mind now we take for $h^{'}$, $h^{'i}$ their expressions $\overset{\alpha+1}{h}{}^{'}$, $\overset{\alpha+1}{h}{}^{'i}$ truncated at the order $\alpha+1$ with respect to equilibrium; this state is defined by $u^{'}_A - u^{'}_{Aeq} = 0$ and not by $u^A - u^A_{eq} = 0$.

Now the set described by u^A and that described by u^{iA} have an intersection I which is not the empty set, for symmetries reasons.

— For each u^A or u^{iA} belonging to I we consider the expressions

$$\frac{\partial \overset{\alpha+1}{h}{}^{'}}{\partial u^{'}_A} \qquad \text{and} \qquad \frac{\partial \overset{\alpha+1}{h}{}^{'i}}{\partial u^{'}_A}; \tag{6}$$

of these, we consider that of higher order and adopt it as definition of $\overset{\alpha}{u}{}^A$ and $\overset{\alpha}{u}{}^{iA}$.

— For each u^A not belonging to I we define

$$\overset{\alpha}{u}{}^A = \frac{\partial \overset{\alpha+1}{h}{}^{'}}{\partial u^{'}_A}\,, \tag{7}$$

— while, for each u^{iA} not belonging to I we define

$$\overset{\alpha}{u}{}^{iA} = \frac{\partial \overset{\alpha+1}{h}{}'^{i}}{\partial u'_A} \,. \tag{8}$$

In such way we have

$$\begin{aligned} \overset{\alpha}{u}{}^{A} &= \frac{\partial \overset{\alpha+1}{h}{}'}{\partial u'_A} + o(\alpha+1) \\ \overset{\alpha}{u}{}^{iA} &= \frac{\partial \overset{\alpha+1}{h}{}'^{i}}{\partial u'_A} + o(\alpha+1)\,, \end{aligned} \tag{9}$$

where $o(\alpha+1)$ denotes terms of order not less than $\alpha+1$.

In fact, the order remains unchanged after derivation with respect to u' or to u'_{ll}, while decreases of an unity in the other cases.

— Consequently, if in $(9)_1$ we have $u'_A = u'$, then $(9)_1$ is satisfied with $o(\alpha+1)=0$.

— If in $(9)_1$ we have $u'_A = u'_{ll}$, then $(9)_1$ is satisfied with $o(\alpha+1)=0$, while the trace of $(9)_2$ with $u'_A = u'_j$ is

$$\frac{\partial \overset{\alpha+1}{h}{}'}{\partial u'_{ll}} = \delta_{ij} \frac{\partial \overset{\alpha+1}{h}{}'^{i}}{\partial u'_j} + o(\alpha+1)\,, \tag{10}$$

which holds because

$$\frac{\partial h'}{\partial u'_{ll}} = \delta_{ij} \frac{\partial h'^{i}}{\partial u'_j}$$

calculated at the order α gives

$$\frac{\partial \overset{\alpha}{h}{}'}{\partial u'_{ll}} = \delta_{ij} \frac{\partial \overset{\alpha+1}{h}{}'^{i}}{\partial u'_j}\,,$$

so that (10) becomes

$$\frac{\partial \overset{\alpha+1}{h}{}'}{\partial u'_{ll}} = \frac{\partial \overset{\alpha}{h}{}'}{\partial u'_{ll}} + o(\alpha+1)\,,$$

which is manifestly satisfied.

— If in $(9)_2$ we have $u'_A = u'$, then $(9)_2$ is satisfied with $o(\alpha+1)=0$; moreover, from

$$\frac{\partial h'^{i}}{\partial u'} = \frac{\partial h'}{\partial u'^{i}}$$

calculated at the order α, we obtain

$$\frac{\partial \overset{\alpha}{h}{}'^{i}}{\partial u'} = \frac{\partial \overset{\alpha+1}{h}{}'}{\partial u'^{i}}\,; \tag{11}$$

consequently, $(9)_1$ with $A = i$ and by using of (11) becomes

$$\frac{\partial \overset{\alpha+1}{h}{}'^{i}}{\partial u'} = \frac{\partial \overset{\alpha+1}{h}{}'}{\partial u'^{i}} + o(\alpha+1) = \frac{\partial \overset{\alpha}{h}{}'^{i}}{\partial u'} + o(\alpha+1)\,,$$

as we expected.

— If in $(9)_2$ we have $u'_A = u'_{ll}$, equation $(9)_2$ is satisfied with $o(\alpha+1) = 0$; moreover from

$$\frac{\partial h'^{i}}{\partial u'_{ll}} = \frac{\partial h'}{\partial u'_{ill}}$$

calculated at the order α, we have

$$\frac{\partial \overset{\alpha}{h}{}'^{i}}{\partial u'_{ll}} = \frac{\partial \overset{\alpha+1}{h}{}'}{\partial u'^{ill}}\,; \tag{12}$$

consequently, equation $(9)_1$ with $A = ill$ and by using of (12) becomes

$$\frac{\partial \overset{\alpha+1}{h}{}'^{i}}{\partial u'_{ll}} = \frac{\partial \overset{\alpha+1}{h}{}'}{\partial u'^{ill}} + o(\alpha+1) = \frac{\partial \overset{\alpha}{h}{}'^{i}}{\partial u'_{ll}} + o(\alpha+1)\,,$$

which is true.

— At last, for the elements of the set I where neither h' nor h'^{i}, are differentiated with respect to u' or to u'_{ll}, we have that

$$\frac{\partial \overset{\alpha+1}{h}{}'}{\partial u'_{Ai}} \quad \text{and} \quad \frac{\partial \overset{\alpha+1}{h}{}'^{i}}{\partial u'_{A}}\,;$$

are both of order α. Moreover, they have the same value, as it can be seen from the identity

$$\frac{\partial h'}{\partial u'_{Ai}} = \frac{\partial h'^{i}}{\partial u'_{A}}$$

calculated at the order α.

Consequently, equation $(9)_{1,2}$ are satisfied both with $o(\alpha+1) = 0$.

We emphasize that the above seen $\overset{\alpha}{u}{}^{A}$ and $\overset{\alpha}{u}{}^{iA}$ are not the expressions of u^{A} and u^{iA} up to the order α, but only the above defined quantities.

In this way, it is easy to see that equations (5) are an immediate consequence of equations (9); expressing this in an equivalent way, we see that the left hand side of equation $(4)_1$ is

$$\left(\overset{\alpha}{u}{}^{A} - \frac{\partial \overset{\alpha+1}{h}{}'}{\partial u'_A} \right) \partial_t u'^{A} = o(\alpha+1)\partial_t u'^{A} = o(\alpha+2)\,; \tag{13}$$

similarly, the left hand side of equation $(4)_2$ is

$$\left(\overset{\alpha}{u}{}^{iA} - \frac{\partial \overset{\alpha+1}{h}{}'^{i}}{\partial u'_A} \right) \partial_i u'_A = o(\alpha+1)\partial_i u'_A = o(\alpha+2)\,; \tag{14}$$

in other words, equations $(4)_{1,2}$ are satisfied up to the order $\alpha+1$, as requested.

We note that in (13), and (14), $\frac{\partial \overset{\alpha+1}{h}{}'}{\partial u'_A}$ and $\frac{\partial \overset{\alpha+1}{h}{}'^{i}}{\partial u'_A}$ are not equal to $\frac{\partial h'}{\partial u'_A}$ and $\frac{\partial h'^i}{\partial u'_A}$ truncated at the order α, nor at the order $\alpha+1$; they are simply $\frac{\partial h'}{\partial u'_A}$ and $\frac{\partial h'^i}{\partial u'_A}$ with $h' = \overset{\alpha+1}{h}{}'$ and $h'^i = \overset{\alpha+1}{h}{}'^{i}$.

Consequently, the entropy principle is so satisfied up to the requested order.

This result we have here obtained, but with the following drawback: The field equations are not all up to the same order with respect to equilibrium. This problem we can overcome with the change of independent variables from the Lagrange multipliers to u_A. Obviously, this will yield to invert some non linear functions, and this must be done exactly. Our ability to face the related calculations will not be sufficient to do this in general. We will now see how to do this in the case $\alpha = 1$ and for the 14 moments case, ET^1_{14}.

4. The 14 Moments Model with $\alpha = 1$

Let us now apply, what seen above, to the case of 14 moments and $\alpha = 1$. The field equations are

$$\begin{cases} \partial_t F + \partial_k F^k = 0\,, \\ \partial_t F^i + \partial_k (F^{<ik>} + \frac{1}{3} F^{ll}\delta^{ik}) = 0\,, \\ \partial_t F^{ll} + \partial_k F^{kll} = 0\,, \\ \partial_t F^{<ij>} + \partial_k F^{k<ij>} = P^{<ij>}\,, \\ \partial_t F^{ill} + \partial_k (F^{<ik>ll} + \frac{1}{3} F^{aall}\delta^{ik}) = P^{ill}\,, \\ \partial_t F^{iill} + \partial_k F^{kiill} = P^{ill}\,. \end{cases}$$

The expressions of h' and ϕ^k at the order 2 can be read in [5] and are

$$h' = \left(\frac{2}{3}\right)^4 \frac{1}{9!!} \tilde{K}''_2(\lambda)\lambda_{ll}^{-\frac{3}{2}} + \frac{4}{7\cdot 81} \tilde{K}'_2 \lambda_{ll}^{-\frac{7}{2}} \lambda_{aabb}$$

$$
\begin{aligned}
& -\frac{1}{2}\left(\frac{2}{3}\right)^3 \frac{1}{9!!}\tilde{K}_2^{'''}\lambda^i\lambda_i\lambda_{ll}^{-\frac{5}{2}} \\
+ \quad & \frac{4}{9}\cdot\frac{1}{9!!}\tilde{K}_2^{''}\lambda_{ll}^{-\frac{7}{2}}\lambda_{<ij>}\lambda_{<ij>} - \frac{1}{81}\tilde{K}_2^{'}\lambda_{ll}^{-\frac{9}{2}}\lambda_{ill}\lambda_{ill} \\
+ \quad & \frac{1}{2}\tilde{K}_2\lambda_{ll}^{-\frac{11}{2}}(\lambda_{aabb})^2 + 5\frac{4}{9}\cdot\frac{1}{9!!}\tilde{K}_2^{''}\lambda_i\lambda_{ill}\lambda_{ll}^{-\frac{7}{2}}.
\end{aligned}
$$

$$
\begin{aligned}
\phi'^k \quad = \quad & -\left(\frac{2}{3}\right)^3 \frac{1}{9!!}\tilde{K}_2^{''}(\lambda)^k\lambda_{ll}^{-\frac{5}{2}} \\
+ \quad & \frac{4}{7\cdot 243}\tilde{K}_2^{'}\lambda_{ll}^{-\frac{7}{2}}\lambda_{kll} + \frac{8}{7\cdot 243}\cdot\frac{1}{5}\tilde{K}_2^{''}\lambda_{ll}^{-\frac{7}{2}}\lambda_{<kr>}\lambda_r \\
- \quad & \frac{2}{81}\tilde{K}_2^{'}\lambda_{ll}^{-\frac{9}{2}}\lambda^k\lambda_{aabb} - \frac{4}{81}\cdot\frac{1}{5}\tilde{K}_2^{'}\lambda_{ll}^{-\frac{9}{2}}\lambda_{<kr>}\lambda_{rll} \\
+ \quad & \frac{1}{3}\tilde{K}_2\lambda_{ll}^{-\frac{11}{2}}\lambda_{kll}\lambda_{aabb};
\end{aligned}
$$

with $\tilde{K}_2(\lambda)$ an arbitrary function.

From what we have above said, it follows

$$
\begin{aligned}
F = \frac{\partial h'}{\partial \lambda} \quad = \quad & \left(\frac{2}{3}\right)^4 \frac{1}{9!!}\tilde{K}_2^{'''}(\lambda)\lambda_{ll}^{-\frac{3}{2}} + \frac{4}{7\cdot 81}\tilde{K}_2^{''}\lambda_{ll}^{-\frac{7}{2}}\lambda_{aabb} \\
- \quad & \frac{1}{2}\left(\frac{2}{3}\right)^3 \frac{1}{9!!}\tilde{K}_2^{IV}\lambda^i\lambda_i\lambda_{ll}^{-\frac{5}{2}} \\
+ \quad & \frac{4}{9}\cdot\frac{1}{9!!}\tilde{K}_2^{'''}\lambda_{ll}^{-\frac{7}{2}}\lambda_{<ij>}\lambda_{<ij>} - \frac{1}{81}\tilde{K}_2^{''}\lambda_{ll}^{-\frac{9}{2}}\lambda_{ill}\lambda_{ill} \\
+ \quad & \frac{1}{2}\tilde{K}_2^{\prime}\lambda_{ll}^{-\frac{11}{2}}(\lambda_{aabb})^2 + 5\frac{4}{9}\cdot\frac{1}{9!!}\tilde{K}_2^{'''}\lambda_i\lambda_{ill}\lambda_{ll}^{-\frac{7}{2}}.
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
F^k = \frac{\partial \phi'^k}{\partial \lambda} \quad = \quad & -\left(\frac{2}{3}\right)^3 \frac{1}{9!!}\tilde{K}_2^{'''}\lambda^k\lambda_{ll}^{-\frac{5}{2}} \\
+ \quad & \frac{4}{7\cdot 243}\tilde{K}_2^{''}\lambda_{ll}^{-\frac{7}{2}}\lambda_{kll} + \frac{8}{7\cdot 243}\cdot\frac{1}{5}\tilde{K}_2^{'''}\lambda_{ll}^{-\frac{7}{2}}\lambda_{<kr>}\lambda_r \\
- \quad & \frac{2}{81}\tilde{K}_2^{''}\lambda_{ll}^{-\frac{9}{2}}\lambda^k\lambda_{aabb} - \frac{4}{81}\cdot\frac{1}{5}\tilde{K}_2^{''}\lambda_{ll}^{-\frac{9}{2}}\lambda_{<kr>}\lambda_{rll} \\
+ \quad & \frac{1}{3}\tilde{K}_2^{\prime}\lambda_{ll}^{-\frac{11}{2}}\lambda_{kll}\lambda_{aabb};
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
F^{ll} = 3\frac{\partial h'}{\partial \lambda_{ll}} \quad = \quad & -3\left(\frac{2}{3}\right)^3 \frac{1}{9!!}\tilde{K}_2^{''}(\lambda)\lambda_{ll}^{-\frac{5}{2}} \\
- \quad & \frac{2}{27}\tilde{K}_2^{'}\lambda_{ll}^{-\frac{9}{2}}\lambda_{aabb} + \frac{15}{4}\left(\frac{2}{3}\right)^3 \frac{1}{9!!}\tilde{K}_2^{'''}\lambda^i\lambda_i\lambda_{ll}^{-\frac{7}{2}}
\end{aligned} \tag{17}
$$

$$- \frac{14}{3}\cdot\frac{1}{9!!}\tilde{K}_2''\lambda_{ll}^{-\frac{9}{2}}\lambda_{<ij>}\lambda_{<ij>} + \frac{1}{6}\tilde{K}_2'\lambda_{ll}^{-\frac{11}{2}}\lambda_{ill}\lambda_{ill}$$
$$- \frac{33}{4}\tilde{K}_2\lambda_{ll}^{-\frac{13}{2}}(\lambda_{aabb})^2 - \frac{70}{3}\cdot\frac{1}{9!!}\tilde{K}_2''\lambda_{ll}^{-\frac{9}{2}}\lambda_i\lambda_{ill}\,;$$

$$F^{<ij>} = \frac{\partial \phi'^{<i}}{\partial \lambda_{j>}} = \frac{8}{7\cdot 243}\cdot\frac{1}{5}\tilde{K}_2''\lambda_{ll}^{-\frac{7}{2}}\lambda^{<ij>}; \tag{18}$$

$$\begin{aligned} F^{kll} = 3\frac{\partial \phi'^k}{\partial \lambda_{ll}} &= \left(\frac{2}{3}\right)^3\frac{15}{2}\frac{1}{9!!}\tilde{K}_2''\lambda^k\lambda_{ll}^{-\frac{7}{2}} \\ &- \frac{2}{81}\tilde{K}_2'\lambda_{ll}^{-\frac{9}{2}}\lambda_{kll} - \frac{4}{81}\cdot\frac{1}{5}\tilde{K}_2''\lambda_{ll}^{-\frac{9}{2}}\lambda_{<kr>}\lambda_r \\ &+ \frac{1}{3}\tilde{K}_2'\lambda_{ll}^{-\frac{11}{2}}\lambda^k\lambda_{aabb} + \frac{2}{3}\cdot\frac{1}{5}\tilde{K}_2'\lambda_{ll}^{-\frac{11}{2}}\lambda_{<kr>}\lambda_{rll} \\ &- \frac{11}{2}\tilde{K}_2\lambda_{ll}^{-\frac{13}{2}}\lambda_{kll}\lambda_{aabb}; \end{aligned} \tag{19}$$

$$\begin{aligned} F^{k<ij>} = \frac{\partial \phi'^k}{\partial \lambda_{<rs>}}\delta_r^{<i}\delta_s^{j>} &= \frac{8}{7\cdot 243}\cdot\frac{1}{5}\tilde{K}_2''\lambda_{ll}^{-\frac{7}{2}}\delta_{k<i}\lambda_{j>} \\ &- \frac{4}{81}\cdot\frac{1}{5}\tilde{K}_2'\lambda_{ll}^{-\frac{9}{2}}\delta_{k<i}\lambda_{j>ll}; \end{aligned} \tag{20}$$

$$F^{aall} = \frac{\partial h'}{\partial \lambda_{aall}} = \frac{4}{7\cdot 81}\tilde{K}_2'\lambda_{ll}^{-\frac{7}{2}} + \tilde{K}_2\lambda_{ll}^{-\frac{11}{2}}\lambda^{aall}; \tag{21}$$

$$F^{<ki>ll} = \frac{\partial \phi'^{<k}}{\partial \lambda_{i>ll}} = -\frac{4}{81}\cdot\frac{1}{5}\tilde{K}_2'\lambda_{ll}^{-\frac{9}{2}}\lambda^{<ki>}; \tag{22}$$

$$F^{kiill} = \frac{\partial \phi'^k}{\partial \lambda_{iill}} = -\frac{2}{81}\tilde{K}_2'\lambda_{ll}^{-\frac{9}{2}}\lambda^k + \frac{1}{3}\tilde{K}_2\lambda_{ll}^{-\frac{11}{2}}\lambda^{kll}; \tag{23}$$

The equations (15), (16), (17), (18), (19), (21) are those to be inverted. Of these, (18) and (21) are linear in $\lambda^{<ij>}$ and λ^{aabb}; consequently, they allow to obtain immediately these Lagrange multipliers

$$\lambda^{<ij>} = \frac{5\cdot 7\cdot 243}{8}\frac{1}{\tilde{K}_2''}\lambda_{ll}^{\frac{7}{2}}F^{<ij>};$$

$$\lambda^{aall} = \frac{1}{\tilde{K}_2}\lambda_{ll}^2\left(\lambda_{ll}^{\frac{7}{2}}F^{aall} - \frac{4}{7\cdot 81}\tilde{K}_2'\right).$$

Equations (16) and (19) are now linear in the unknowns λ^i and λ^{ill}, i.e.,

$$F^i = (a_0\delta^{ij} + a_1F^{<ij>})\lambda_j + (b_0\delta^{ij} + b_1F^{<ij>})\lambda_{jll}\,,$$
$$F^{ill} = (c_0\delta^{ij} + c_1F^{<ij>})\lambda_j + (d_0\delta^{ij} + d_1F^{<ij>})\lambda_{jll}$$

with obvious meaning of a_r, b_r, c_r, d_r.

By multiplying these equations times $F^{<ki>}$ and $\overset{2}{F}{}^{<ki>}$ and by using the Hamilton-Kayley ' s theorem

$$\overset{3}{F}{}^{<ij>} = -s_2 F^{<ij>} + s_3 \delta^{ij}$$

(with $s_2 = -\frac{1}{2} tr \overset{2}{F}{}^{<ki>}$, $s_3 = \frac{1}{3} tr \overset{3}{F}{}^{<ki>}$), we obtain

$$\begin{aligned}
F^{<ij>} F^j = & \ (a_0 F^{<ij>} + a_1 \overset{2}{F}{}^{<ij>}) \lambda_j + (b_0 F^{<ij>} + b_1 \overset{2}{F}{}^{<ij>}) \lambda_{jll} \,, \\
F^{<ij>} F^{jll} = & \ (c_0 F^{<ij>} + c_1 \overset{2}{F}{}^{<ij>}) \lambda_j + (d_0 F^{<ij>} + d_1 \overset{2}{F}{}^{<ij>}) \lambda_{jll} \,, \\
\overset{2}{F}{}^{<ij>} F^j = & \ (a_0 \overset{2}{F}{}^{<ij>} - a_1 s_2 F^{<ij>} + a_1 s_3 \delta^{ij}) \lambda_j \\
& + (b_0 \overset{2}{F}{}^{<ij>} - b_1 s_2 F^{<ij>} + b_1 s_3 \delta^{ij}) \lambda_{jll} \,, \\
\overset{2}{F}{}^{<ij>} F^{jll} = & \ (c_0 \overset{2}{F}{}^{<ij>} - c_1 s_2 F^{<ij>} + c_1 s_3 \delta^{ij}) \lambda_j \\
& + (d_0 \overset{2}{F}{}^{<ij>} - d_1 s_2 F^{<ij>} + d_1 s_3 \delta^{ij}) \lambda_{jll} \,.
\end{aligned}$$

In this way a linear system has been obtained of 6 equations for the determination of the 6 unknowns λ^i, $F^{<ij>}\lambda^j$, $\overset{2}{F}{}^{<ij>}\lambda^j$, λ^{ill}, $F^{<ij>}\lambda^{jll}$, $\overset{2}{F}{}^{<ij>}\lambda^{jll}$; its solution is

$$\lambda_i = \frac{1}{D} \begin{vmatrix}
F^i & a_1 & 0 & b_0 & b_1 & 0 \\
F^{ill} & c_1 & 0 & d_0 & d_1 & 0 \\
F^{<ij>} F^j & a_0 & a_1 & 0 & b_0 & b_1 \\
F^{<ij>} F^{jll} & c_0 & c_1 & 0 & d_0 & d_1 \\
\overset{2}{F}{}^{<ij>} F^j & -a_1 s_2 & a_0 & b_1 s_3 & -b_1 s_2 & b_0 \\
\overset{2}{F}{}^{<ij>} F^{jll} & -c_1 s_2 & c_0 & d_1 s_3 & -d_1 s_2 & d_0
\end{vmatrix} ,$$

and

$$\lambda_{ill} = \frac{1}{D} \begin{vmatrix}
a_0 & a_1 & 0 & F^i & b_1 & 0 \\
c_0 & c_1 & 0 & F^{ill} & d_1 & 0 \\
0 & a_0 & a_1 & F^{<ij>} F^j & b_0 & b_1 \\
0 & c_0 & c_1 & F^{<ij>} F^{jll} & d_0 & d_1 \\
a_1 s_3 & -a_1 s_2 & a_0 & \overset{2}{F}{}^{<ij>} F^j & -b_1 s_2 & b_0 \\
c_1 s_3 & -c_1 s_2 & c_0 & \overset{2}{F}{}^{<ij>} F^{jll} & -d_1 s_2 & d_0
\end{vmatrix} ,$$

with

$$D = \begin{vmatrix} a_0 & a_1 & 0 & b_0 & b_1 & 0 \\ c_0 & c_1 & 0 & d_0 & d_1 & 0 \\ 0 & a_0 & a_1 & 0 & b_0 & b_1 \\ 0 & c_0 & c_1 & 0 & d_0 & d_1 \\ a_1 s_3 & -a_1 s_2 & a_0 & b_1 s_3 & -b_1 s_2 & b_0 \\ c_1 s_3 & -c_1 s_2 & c_0 & d_1 s_3 & -d_1 s_2 & d_0 \end{vmatrix}.$$

It remains now to invert equations (15) and (17) for the determination of the unknowns λ and λ_{ll}. But we know that it is not necessary to do this; in fact, as in [6] we can consider the change of variables from λ, λ_{ll} to z and T, defined by

$$\begin{cases} \tilde{K}_2'''(\lambda) = z \cdot \frac{3^7}{2^5\sqrt{2}} \cdot 35\,, \\ \lambda_{ll} = \frac{1}{2T}\,. \end{cases} \tag{24}$$

After that, the following definition can be introduced

$$\Im(z) = -\tilde{K}_2''[\lambda(z)]\frac{2^5\sqrt{2}}{3^6 \cdot 35}. \tag{25}$$

With this, equations (15) and (17) give F and F^{ll} as functions of the new variables. We note that in (15) and (17), but also in the other equations (15)-(23), the variable λ occurs only by means of $\tilde{K}_2(\lambda)$ and its derivatives; these, in turns, can be expressed in terms of $(24)_1$ and (25). In fact, from equations (24), (25) we obtain $\tilde{K}_2'''$ and $\tilde{K}_2''$. After that, by using again (25) we have

$$\Im'(z) = -3\frac{\tilde{K}_2'''}{\tilde{K}_2^{IV}} = -\frac{3^8 \cdot 35}{2^5\sqrt{2}} z \frac{1}{\tilde{K}_2^{IV}}\,,$$

from which

$$\frac{1}{\tilde{K}_2^{IV}} = -\frac{2^5\sqrt{2}}{3^8 \cdot 35}\frac{\Im'(z)}{z}\,.$$

After that we have

$$\begin{aligned} \tilde{K}_2' &= \int \tilde{K}_2'' d\lambda + c_1 = \int \tilde{K}_2'' \frac{d\lambda}{dz} dz + c_1 \\ &= \int -3\left(\frac{3^6 \cdot 35}{2^5\sqrt{2}}\right)^2 \frac{\Im(z)}{\tilde{K}_2^{IV}} dz + c_1\,, \end{aligned}$$

that is,

$$\tilde{K}_2' = \int \frac{3^5 \cdot 35}{2^5\sqrt{2}} \quad \frac{\Im(z)\Im'(z)}{z} dz + c_1\,.$$

Similarly,

$$\begin{aligned}
\tilde{K}_2 &= \int \tilde{K}_2' d\lambda + c_2 = \int \tilde{K}_2' \frac{d\lambda}{dz} dz + c_2 \\
&= \int \frac{3^7 \cdot 35}{2^5\sqrt{2}} \frac{\tilde{K}_2'}{\tilde{K}_2^{IV}} dz + c_2 = \int -\tilde{K}_2' \frac{1}{3} \frac{\Im'(z)}{z} dz + c_2 \\
&= -\frac{1}{3} \int \left(\frac{3^5 \cdot 35}{2^5\sqrt{2}} \int \frac{\Im(z)\Im'(z)}{z} dz + c_1 \right) \frac{\Im'(z)}{z} dz + c_2 \,.
\end{aligned}$$

In particular, equations (15) and (17) , calculated in

$$\lambda_r = 0, \quad \lambda_{<rs>} = 0, \quad \lambda_{rll} = 0, \quad \lambda_{aabb} = 0$$

imply that in such state we have

$$F = \left(\frac{2}{3}\right)^4 \frac{1}{9!!} \tilde{K}_2'''(\lambda) \lambda_{ll}^{-\frac{3}{2}} \,,$$

$$F_{ll} = -\frac{9}{2} \left(\frac{2}{3}\right)^4 \frac{1}{9!!} \tilde{K}_2''(\lambda) \lambda_{ll}^{-\frac{5}{2}} \,,$$

from which

$$z = T^{-\frac{3}{2}} F \,,$$

$$P = \frac{1}{3} F_{ll} = T^{\frac{5}{2}} \Im \left(\frac{F}{T^{\frac{3}{2}}} \right) .$$

as in [6]. It follows that, in such state, T is the absolute temperature.

5. Conclusions

We think that the present paper is very useful in science. In fact, we have shown that some seeming problems of Extended Thermodynamics are solved in different ways. Even if, in the general case, some calculations may be difficult to face, nobody says that all the laws of nature must adapt themselves to our potentiality in calculations! In particular cases, we have shown how to overcome these difficulties.

We esteem and appreciate very much the paper by Professor Brini and Professor Ruggeri, a paper evidencing some of these problems; also the physical hypothesis, which they suggest to overcome them, is very meaningful and interesting. However, it is surely positive to have other available options.

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