International Journal of Pure and Applied Mathematics
Volume 52 No. 1 2009, 27-36

INTERPOLATION ON $\mathbb{P}^n \times C$, $n = 2, 3$, WITH C A SMOOTH CURVE

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Abstract: Here we prove the maximal rank property for general unions in $\mathbb{P}^n \times C$, $n = 2, 3$, C a smooth curve, of fat points with multiplicity at most m with $2 \le m \le 7$ if $n = 2$ and $2 \le m \le 4$ if $n = 3$.

AMS Subject Classification: 14N05
Key Words: interpolation

1. Introduction

For any integral scheme A, any $P \in A_{reg}$ and any integer $m > 0$ let $\{mP, A\}$ denote infinitesimal neighborhood of order $m-1$ of A with P as its support, i.e. the closed subscheme of A with $(\mathcal{I}_{P,A})^m$ as ideal sheaf. Hence $\{mP, A\}$ is a zero-dimensional scheme, $\{mP, A\}_{red} = \{P\}$ and length$(A) = \binom{m+\dim(A)-1}{\dim(A)}$. The scheme $\{mP, A\}$ is called a fat point of A with multiplicity m or an m-point of A. For any finite subset S of A_{reg} set $\{mS, A\} := \cup_{P \in S}\{mP, A\}$. We often write mP or mS if there is no danger of misunderstandings (e.g. if $A = X$).

Theorem 1. *Let C be a smooth and connected projective curve of genus $g \ge 0$. Set $X := \mathbb{P}^2 \times C$ and call $\pi_1 : X \to \mathbb{P}^2$ and $\pi_2 : X \to C$ be the projections. Fix an integer m such that $2 \le m \le 7$. Set $\alpha(2) := 4$, $\alpha(3) := 8$, $\alpha(4) := 11$, $\alpha(5) := 15$, $\alpha(6) := 25$, and $\alpha(7) := 30$. Fix an integer $d \ge \alpha(m)$ and any $R \in Pic(C)$ such that $h^0(C, R) \ge \binom{m+1}{2} + m + 1$. Set $L := \pi_1^*(\mathcal{O}_{\mathbb{P}^2}(d)) \otimes \pi_2^*(R)$. Let $Z \subset X$ be a general union of fat points with multiplicities at most m. Then either $h^0(X, \mathcal{I}_Z \otimes L) = 0$ (case length$(Z) \ge h^0(C, R) \cdot \binom{d+2}{2}$) or $h^1(X, \mathcal{I}_Z \otimes L) =$*

Received: January 28, 2009

$h^1(C,R)\cdot\binom{d+2}{2}$ (case length$(Z) \le h^0(C,R)\cdot\binom{d+2}{2}$).

Theorem 2. *Let C be a smooth and connected projective curve of genus $g \ge 0$. Set $X := \mathbb{P}^3 \times C$ and call $\pi_1 : X \to \mathbb{P}^3$ and $\pi_2 : X \to C$ be the projections. Fix an integer $m \in \{2,3,4\}$. Set $\alpha'(2) := 5$, $\alpha'(3) := 36$, and $\alpha'(4) := 48$. Fix an integer $d \ge \alpha'(m)$ and any $R \in Pic(C)$ such that $h^0(C,R) \ge \binom{m+2}{3} + m + 1$. Set $L := \pi_1^*(\mathcal{O}_{\mathbb{P}^3}(d)) \otimes \pi_2^*(R)$. Let $Z \subset X$ be a general union of fat points with multiplicities at most m. Then either $h^0(X, \mathcal{I}_Z \otimes L) = 0$ (case length$(Z) \ge h^0(C,R)\cdot\binom{d+3}{2}$) or $h^1(X,\mathcal{I}_Z\otimes L) = h^1(C,R)\cdot\binom{d+3}{3}$ (case length$(Z) \le h^0(C,R)\cdot\binom{d+3}{2}$).*

We work over an algebraically closed field $\mathbb{K}$ such that char$(\mathbb{K}) = 0$.

2. The Proofs

Remark 1. Take $X = \mathbb{P}^n \times C$, $n = 2,3$, π_1 and π_2, as in Theorems 1 and 2. For every integer $d \ge -2$ and every $R \in \mathrm{Pic}(C)$ we have $h^0(X, \pi_1^*(\mathcal{O}_{\mathbb{P}^n}(d)) \otimes \pi_2^*(R)) = \binom{n+d}{n}\cdot h^0(C,R)$ and $h^1(X,\pi_1^*(\mathcal{O}_{\mathbb{P}^n}(d))\otimes\pi_2^*(R)) = \binom{n+d}{n}\cdot h^1(C,R)$ (Künneth's formula).

Remark 2. Let C be a smooth and projective curve of genus g. Fix $R \in \mathrm{Pic}(C)$. Since a general point of a projective integral curve $Y \subset \mathbb{P}^n$ is not an osculating point of Y (see [4], Theorem 14), we have $h^0(C, R(-tP)) = \max\{0, h^0(C,R) - t\}$ for a general $P \in C$ and every integer $t \ge 0$, i.e. $h^1(C, R(-tP)) = \max\{h^1(C,R), g-1+t-\deg(R)\}$ for a general $P \in C$ and every integer $t \ge 0$ (Riemann-Roch).

Remark 3. Let X be an integral projective scheme, L a line bundle on X, $Z \subset X$ a closed subscheme and $D \subset X$ an effective Cartier divisor. Let $\mathrm{Res}_H(Z)$ denote the residual scheme of Z with respect to H, i.e. the closed subsheme of X with $\mathcal{I}_Z : \mathcal{I}_H$ as its ideal sheaf. We have an exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Z)} \otimes L(-H) \to \mathcal{I}_Z \otimes L \to \mathcal{I}_{Z\cap H} \otimes (L|H) \to 0.$$

Hence

$$h^0(X, \mathcal{I}_Z \otimes L) \le h^0(X, \mathcal{I}_{\mathrm{Res}_H(Z)} \otimes L(-H)) + h^0(H, \mathcal{I}_{Z\cap H}\otimes(L|H)),$$

$$h^1(X, \mathcal{I}_Z \otimes L) \le h^1(X, \mathcal{I}_{\mathrm{Res}_H(Z)} \otimes L(-H)) + h^1(H, \mathcal{I}_{Z\cap H}\otimes(L|H)).$$

Remark 4. Here we will explain an easy version of the Differential Horace Lemma (see [1], Lemma 2.1)). Let X be an integral projective scheme, L a line bundle on X and $D \subset X$ an effective Cartier divisor of X. Fix integers

$m > 0$, $1 \le j \le m-1$, $a \ge 0$, $P \in D_{reg}$ and a zero-dimensional scheme $W \subset X$ such that $P \notin W_{red}$. To check that $h^0(X, \mathcal{I}_{W\cup\{mQ,X\}} \otimes L) \le a$ (resp. $h^1(X, \mathcal{I}_{W\cup\{mQ,X\}} \otimes L) \le a$) for a general $Q \in X_{reg}$ it is sufficient to prove $h^0(X, \mathcal{I}_{W\cup B} \otimes L) \le a$ (resp. $h^1(X, \mathcal{I}_{W\cup B} \otimes L) \le a$), where B is a virtual scheme with the following properties. Set $B_0 := B$. For all integers $i \ge 1$ define inductively the virtual scheme B_i as the virtual residue with respect to D of the virtual scheme B_{i-1}. Then $B_i = \emptyset$ for all $i \ge m$, $B \cap D = \{jP, D\}$, $B_i \cap D = \{(m-i+1)P, D\}$ for $1 \le i \le m-j-1$ and $B_i \cap D = \{(m-i)P, D\}$ for $m-j \le i \le m-1$ (scheme-theoretically). We may apply simultaneously this observation to finitely many quadruples (m, j, Q, P).

Remark 5. Fix integers m, d such that $1 \le m \le 7$ and $d \ge 3m$. Let $Z \subset \mathbf{P}^2$ be a general union of fat points, each of them with multiplicity at most m. Then either $h^1(\mathbf{P}^2, \mathcal{I}_Z(d)) = 0$ (case length$(Z) \le (d+2)(d+1)/2$) or $h^0(\mathbf{P}^2, \mathcal{I}_Z(d)) = 0$ (case length$(Z) \ge (d+2)(d+1)/2$) (see [6], Theorem 1, and [5], p. 174).

We will use several times the following lemma proved in [2] (see [2], Lemma 7) (the lemma is only stated in [2] when the ambient variety X is a projective space and H is a hyperplane).

Lemma 1. *Fix an integral projective variety X, $L \in Pic(X)$, an integral effective Cartier divisor H of X, a zero-dimensional scheme $Z \subset X$, integers $c > 0$, $\gamma \ge 0$ and a general $S \subset H$ such that $\sharp(S) = c$.*

(i) If $h^0(X, \mathcal{I}_Z \otimes L) \le \gamma + c$ and $h^0(X, \mathcal{I}_{\mathrm{Res}_H(Z)} \otimes L(-H)) \ge c$, then $h^0(X, \mathcal{I}_{Z\cup S} \otimes L) \le \gamma$.

(ii) If $h^1(X, \mathcal{I}_Z \otimes L) \le \gamma$ and $h^0(X, \mathcal{I}_{\mathrm{Res}_H(Z)} \otimes L(-H)) \ge c$, then $h^1(X, \mathcal{I}_{Z\cup S} \otimes L) \le \gamma$.

Notation 1. Fix an integer $m \ge 2$. Let $\Phi(m)$ be the set of all $(m-1)$-ples $(f_1, \dots, f_{m-1}) \in \mathbb{N}^{m-1}$ with the following properties:

(i) $\sum_{i=1}^{m-1} f_i \binom{i+1}{2} \le \binom{m+1}{2} - 1$;

(ii) if $i \ge 2$ and there are integers $x_j \ge 0$, $1 \le j \le i-1$, such that $\sum_{j=1}^{i-1} x_j \binom{j+1}{2} \ge \binom{i+1}{2}$, then $f_j < x_j$ for at least one index j.

Set $\phi_m := \max_{(f_1,\dots,f_{m-1})\in\Phi(m)} \sum_{i=1}^{m-1} f_i$. Let $\Phi'(m)$ be the set of all $(m-1)$-ples $(f_1, \dots, f_{m-1}) \in \mathbb{N}^{m-1}$ with the following properties:

(i) $\sum_{i=1}^{m-1} f_i \binom{i+2}{3} \le \binom{m+2}{3} - 1$;

(ii) if $i \ge 2$ and there are integers $x_j \ge 0$, $1 \le j \le i-1$, such that $\sum_{j=1}^{i-1} x_j \binom{j+2}{3} \ge \binom{i+2}{3}$, then $f_j < x_j$ for at least one index j.

Set $\phi'_m := \max_{(f_1,\dots,f_{m-1})\in\Phi'(m)} \sum_{i=1}^{m-1} f_i$.

Remark 6. Since $\binom{3}{2}-1=2$, $\Phi(2)=\{(0),(1),(2)\}$. Hence $\phi_2=2$. Since $\binom{4}{2}-1=5$, the set $\Phi(3)$ is formed by the pairs $(0,0)$, $(0,1)$, $(0,2)$, $(1,0)$, $(1,1)$, $(1,2)$. Hence $\phi_3=3$. The sets $\Phi(4)$ and $\Phi(5)$ are respectively listed in the statements of Lemmas 2 and 3. From these lists we get $\phi_4=\phi_5=3$.

For the case $m=3,4$ of Theorem 1 we need the following numerical lemma.

Lemma 2. *Fix non-negative integers t,a,b,c,e,f,g such that $t\ge 11$,*

$$10a+6b+3c+u+6e+3f+g \le \binom{t+2}{2} \tag{1}$$

and (e,f,g) is one of the following triples: $(0,0,0)$, $(0,0,1)$, $(0,0,2)$, $(0,1,0)$, $(0,1,1)$, $(0,1,2)$, $(1,0,0)$, $(1,0,1)$, $(1,0,2)$, $(1,1,0)$. *Then*

$$6a+3b+c+10e+10f+10g \le \binom{t+2}{2}. \tag{2}$$

Proof. Increasing if necessary u we may assume that in (1) equality holds. Thus n order to prove (4), it is sufficient to check the inequality

$$4a+3b+2c+u-4e-7f-9g \ge 0.$$

From (1) it follows that

$$10a \ge \binom{t+2}{2} - 6b-3c-u-9,$$

hence the inequality above comes from

$$\frac{2}{5}\left(\binom{t+2}{2}-9\right)-25\ge 0,$$

which is true for all $t\ge 11$. □

For the case $m=5$ of Theorem 1 we need the following lemma.

Lemma 3. *Fix non-negative integers t,a,b,c,u,v,e,f,g,h such that $t\ge 15$,*

$$15a+10b+6c+3u+v+10e+6f+3g+h \le \binom{t+2}{2} \tag{3}$$

and (e,f,g,h) is one of the following quadruples: $(0,0,0,0)$, $(0,0,0,1)$, $(0,0,0,2)$, $(0,0,1,0)$, $(0,0,1,1)$, $(0,0,1,2)$, $(0,1,0,0)$, $(0,1,0,1)$, $(0,1,0,2)$, $(0,1,1,0)$, $(1,0,0,0)$, $(1,0,0,1)$, $(1,0,0,2)$, $(1,0,1,0)$, $(1,0,1,1)$. *Then the following inequality holds:*

$$10a+6b+3c+u+15e+15f+15g+15h \le \binom{t+2}{2}. \tag{4}$$

Proof. Increasing if necessary v we may assume that in (3) equality holds. Thus order to prove (4), it is sufficient to check the inequalitiy

$$5a + 4b + 3c + 2u + v - 5e - 9f - 12g - 14h \geq 0. \tag{5}$$

From (3) it follows that

$$15a \geq \binom{t+2}{2} - 10b - 6c - 3u - v - 14. \tag{6}$$

Hence the inequality (5) comes from the inequality

$$\left(\binom{t+2}{2} - 14\right)/3 - 40 \geq 0, \tag{7}$$

which is true for all $t \geq 15$. □

Then we may continue for higher multiplicities in the following way.

Remark 7. Since $\binom{7}{2} - 1 = 20$, the set $\Phi(6)$ is formed by the following quintuples: $(0,0,0,0)$, $(0,0,0,0,1)$, $(0,0,0,0,2)$, $(0,0,0,1,0)$, $(0,0,0,1,1)$, $(0,0,0,1,2)$, $(0,0,1,0,0)$, $(0,0,1,0,1)$, $(0,0,1,0,2)$, $(0,0,1,1,0)$, $(0,1,0,0,0)$, $(0,1,0,0,1)$, $(0,1,0,0,2)$, $(0,1,0,1,0)$, $(0,1,0,1,1)$, $(1,0,0,0,0)$, $(1,0,0,0,1)$, $(1,0,0,0,2)$, $(0,0,1,0)$, $(1,0,0,1,1)$, $(1,0,0,1,2)$. Hence $\phi_6 = 4$.

Remark 8. Since $\binom{8}{2} - 1 = 27$, the set $\Phi(7)$ is formed by the following sestuples: $(0,0,0,0,0,1)$, $(0,0,0,0,0,2)$, $(0,0,0,0,1,0)$, $(0,0,0,0,1,1)$, $(0,0,0,0,1,2)$, $(0,0,0,1,0,0)$, $(0,0,0,1,0,1)$, $(0,0,0,1,0,2)$, $(0,0,0,1,1,0)$, $(0,0,1,0,0,0)$, $(0,0,1,0,0,1)$, $(0,0,1,0,0,2)$, $(0,0,1,0,1,0)$, $(0,0,1,0,1,1)$, $(0,1,0,0,0,0)$, $(0,1,0,0,0,1)$, $(0,1,0,0,0,2)$, $(0,0,0,1,0)$, $(0,1,0,0,1,1)$, $(0,1,0,0,1,2)$, $(1,0,0,0,0,1)$, $(1,0,0,0,0,2)$, $(1,0,0,0,1,0)$, $(1,0,0,0,1,1)$, $(1,0,0,0,1,2)$, $(1,0,0,1,0,0)$. Hence $\phi_7 = 4$.

Lemma 4. *Fix integers $m \geq 2$, $d \geq 3$, $e_i \geq 0$, $1 \leq i \leq m$, and $(f_1, \dots, f_{m-1}) \in \Phi(m)$ such that*

$$\sum_{i=1}^{m} e_i \binom{i+1}{2} + \sum_{i=1}^{m-1} f_{m-1} \leq \binom{d+2}{2}, \tag{8}$$

$$\binom{d+2}{2} \geq \binom{m+1}{2} - 1 + \phi_m (m+1)^2 m/4. \tag{9}$$

Then

$$\sum_{i=2}^{m} e_i \binom{i}{2} + (f_1 + \cdots + f_{m-1}) \binom{m+1}{2} \leq \binom{d+2}{2}. \tag{10}$$

Proof. Increasing if necessary e_1 we may assume that in (8) equality holds.

Thus in order to check (9) is sufficient to prove

$$\sum_{i=2}^{m} e_i i - \sum_{j=1}^{m-1} (f_j(\binom{m+1}{2} - \binom{j+1}{2})) \geq 0\,. \tag{11}$$

The inequality (21) holds if

$$\sum_{i=1}^{m} e_i i \geq \phi_m \binom{m+1}{2}. \tag{12}$$

Since the function $\phi(i) := i/(i-1)$, $i > 1$, is a decreasing function of i, to prove (20) it is sufficient to the inequality (9). □

Remark 9. By Remarks 6 and 10, and by Lemmas 2, 3 and 4 we see that all the inequalitities are satisfied if $d \geq \alpha(m)$, where $\alpha(2) = 4$, $\alpha(3) = 8$, $\alpha(4) = 11$, $\alpha(5) = 15$, $\alpha(6) = 25$ and $\alpha(7) = 30$. Notice these integers $\alpha(m)$, $2 \leq m \leq 7$, are the integers introduced in the statement of Theorem 1.

From Lemma 4 and Remarks 7 and 10 we get the following results.

Corollary 1. *Fix integers $d \geq 25$, $e_i \geq 0$, $1 \leq i \leq 6$, and $(f_1, \ldots, f_5) \in \Phi(6)$ such that*

$$\sum_{i=1}^{6} e_i \binom{i+1}{2} + \sum_{j=1}^{5} f_j \leq \binom{d+2}{2}. \tag{13}$$

Then

$$\sum_{i=2}^{6} e_i \binom{i}{2} + 21(\sum_{j=1}^{5} f_j) \leq \binom{d+2}{2}. \tag{14}$$

Corollary 2. *Fix integers $d \geq 30$, $e_i \geq 0$, $1 \leq i \leq 7$, and $(f_1, \ldots, f_6) \in \Phi(7)$ such that*

$$\sum_{i=1}^{7} e_i \binom{i+1}{2} + \sum_{j=1}^{6} f_j \leq \binom{d+2}{2}. \tag{15}$$

Then

$$\sum_{i=2}^{7} e_i \binom{i}{2} + 28(\sum_{j=1}^{6} f_j) \leq \binom{d+2}{2}. \tag{16}$$

Remark 10. We have $\Phi'(2) = \{(0), (1), (2), (3)\}$. Hence $\phi'_2 = 3$. The set Φ'_3 is formed by the following pairs: $(0,0)$, $(0,1)$, $(0,2)$, $(0,3)$, $(1,0)$, $(1,1)$, $(1,2)$, $(1,3)$, $(2,0)$, $(2,1)$. Hence $\phi'_3 = 4$. The set $\Phi'(4)$ is formed by the following triples: $(0,0,0)$, $(0,0,1)$, $(0,0,2)$, $(0,0,3)$, $(0,1,0)$, $(0,1,1)$, $(0,1,2)$, $(0,1,3)$, $(0,2,0)$, $(0,2,1)$, $(1,0,0)$, $(1,0,1)$, $(1,0,2)$, $(1,0,3)$, $(1,1,0)$, $(1,1,1)$,

$(1,1,2)$, $(1,1,3)$, $(1,2,0)$, $(1,2,1)$. Hence $\phi'_4 = 4$.

To prove Theorem 2 we only need the cases $2 \le m \le 4$ of the following numerical lemma.

Lemma 5. *Fix integers $d \ge 3$, $m \ge 2$, $e_i \ge 0$, $1 \le i \le m$, and $(f_1, \dots, f_{m-1}) \in \Phi'(m)$ such that*

$$\sum_{i=1}^{m} e_i \binom{i+1}{2} + \sum_{i=1}^{m-1} f_{m-1} \le \binom{d+3}{3}, \tag{17}$$

$$\binom{d+3}{2} \ge \binom{m+2}{3} - 1 + \phi'_m (m+2)(m+1)^2 m^2/4\,. \tag{18}$$

Then

$$\sum_{i=2}^{m} e_i \binom{i+1}{2}) + (f_1 + \cdots + f_{m-1}) \binom{m+2}{3} \le \binom{d+3}{3}. \tag{19}$$

Proof. Increasing if necessary e_1 we may assume that in (17) equality holds. Thus in order to check (18)it is sufficient to prove

$$\sum_{i=2}^{m} e_i (i+1)i/2 - \sum_{j=1}^{m-1} (f_j(\binom{m+2}{3} - \binom{j+2}{3}) \ge 0\,. \tag{20}$$

The inequality (20) holds if

$$\sum_{i=1}^{m} e_i (i+1)i/2 \ge \phi'_m \binom{m+2}{3}. \tag{21}$$

Since the function $\phi(i) := i/(i-1)$, $i > 1$, is a decreasing function of i, to prove (9) it is sufficient to use the inequality (19). □

Remark 11. Using Remark 10 we get that in Lemma 5 if $m = 2$ we may take any $d \ge 5$, if $m = 3$ we may take any any $d \ge 36$, while if $m = 4$ we may take any $d \ge 68$.

Remark 12. Fix an integer $d \ge 12$. Let $Z \subset \mathbb{P}^3$ be a general union of fat points of multiplicity at most 4. Then either $h^0(\mathbb{P}^3, \mathcal{I}_Z(d)) = 0$ or $h^1(\mathbb{P}^3, \mathcal{I}_Z(d)) = 0$ (see [2], [3]).

Proof of Theorem 1. Fix a general $P \in C$ and write $H := \mathbb{P}^2 \times \{P\}$. Since P is general in C, Remark 2 gives $h^0(C, R(-xP)) = \max\{0, h^0(C,R) - x\}$ for every integer $x \ge 0$, i.e. $h^1(C, R(-xP)) = \max\{h^1(C,R), \deg(R) + x + 1 - g\}$ for every integer $x > 0$ (Riemann-Roch). We first do the case length$(Z) \ge \binom{d+2}{2} \cdot h^0(C,R)$. We may assume length$(Z) \le h^0(X,L) + \binom{m+1}{2} - 1$ (prove that $h^0(X, \mathcal{I}_{Z'} \otimes L) = 0$ for a general union $Z' \subset Z$ of fat ponts with multiplicity at most m if the inequality is not satified). It is sufficient to prove the h^0-vanishing for a virtual specialization of Z. Let b_i, $2 \le i \le m$, be the number of fat points with multiplicity i contained in Z. Set $a_1 := \lfloor \binom{d+2}{2}/\binom{m+1}{2} \rfloor$ and $u_1 := \binom{d+2}{2} - a_1\binom{m+1}{2}$. Hence u_1 is an integer and $0 \le u_1 \le \binom{m+1}{2} - 1$. Hence there is $(f_{1,1}, \dots, f_{m-1,1}) \in \Phi(m)$ such that $u_1 = \sum_{i=1}^{m-1} f_{i,1}\binom{i+1}{2}$. Assume $b_m \ge a_1 + \sum_{i=1}^{m-1} f_{i,1}$. Notice that $0 \le \sum_{i=1}^{m-1} f_{i,1} \le \phi_m$. We specialize Z to a virtual scheme obtained taking a general union of $b_m - a_1 - \sum_{i=1}^{m-1} f_{i,1}$ fat points of X with multiplicity m, b_i fat points of X with multiplicity i, $2 \le i \le m-1$, a_1 fat points of X with multicity m and support on H and $f_{j,1}$, $1 \le j \le m-1$, schemes obtained applying Remark 4 with respect to the datum (m, j). We have $\mathrm{length}(Z_1 \cap H) = h^0(H, \mathcal{O}_H(d))$. Since $d \ge 3m$, we have $h^i(H, \mathcal{I}_{Z_1 \cap H}(d)) = 0$ (Remark 5). Hence it is sufficient to prove $h^0(X, \mathcal{I}_{\mathrm{Res}_{H(Z_1)}} \otimes L(-H)) = 0$. We have $L(-H) \cong \pi_1^*(\mathcal{O}_{\mathbb{P}^2}(d)) \otimes \pi_2^*(R(-P))$. Set $a_2 := \lfloor (h^0(H, \mathcal{O}_H(d)) - \mathrm{length}(\mathrm{Res}_H(Z_1))/rfloor/\binom{m+1}{2}$ and $u_2 := h^0(H, \mathcal{O}_H(d))) - a_2\binom{m+1}{2}$. Lemmas 2 and 3 and Corollaries 1 and 2 (respectively for $m = 3, 4, 5, 6$) gives $a_2 \ge 0$. We have $0 \le u_2 \le \binom{m+1}{2} - 1$. Hence there is $(f_{1,2}, \dots, f_{m-1,2}) \in \Phi(m)$ such that $u_2 = \sum_{i=1}^{m-1} f_{i,2}\binom{i+1}{2}$. Assume for the moment $b_m \ge a_1 + a_2 + \sum_{i=1}^{m-1}(f_{i,1} + f_{i,2})$. We specialize a_2 of the m-points of X not with support on H to general m-points of X with support on H and for all $j \in \{1, \dots, m-1\}$ apply to $f_{j,2}$ of them Remark 4 with respect to the datum (m, j). If we have a smaller number of m-points, then we use the $(m-1)$-points and then the $(m-2)$-points and so on, even to apply Remark 4 for data $(m-1, j)$, $(m-2, j)$ and so on. If at a certain step taking a virtual residue we get a scheme with some reduced connected components, then their union must be a general subset of H with the prescribed cardinality. We delete it and apply Lemma 1. We may continue doing these steps starting from this smaller scheme. And so on until either we do not have enough fat points not supported on H or we are looking at $L(-tH)$ and $h^0(X, L(-tH)) = 0$, i.e. $h^0(C, R(-tP)) = 0$, i.e. $t = h^0(C, R)$. In the latter case we get $h^0(X, \mathcal{I}_Z \otimes L) = 0$. Assume that we are in the former case, say for $L(-tH)$, and call W the associated scheme in which we (after a specialization) all the connected components of W are supported by H. Set $W_0 := W$ and define inductively the scheme W_i, $i \ge 1$, by the formula $W_i := \mathrm{Res}_H(W_{i-1})$. Hence $W_i = \emptyset$ for all $i \ge m$. Remember that in Lemma 1 we have two equivalent forms, one with an h^0-inequality and one with an h^1-inequality. In this case we use the h^1-inequality. If $t + m \le h^0(C, R)$, then we may apply at most m times he residual scheme finding $h^1(H, \mathcal{I}_W(d)) = h^1(H, \mathcal{I}_{W_1}(d)) = \cdots = h^1(H, \mathcal{I}_{W_{m-1}}(d)) = 0$. Hence $h^1(H, \mathcal{I}_W \otimes L(-tH)) = 0$. This in turn gives that either $h^0(X, \mathcal{I}_Z \otimes L) = 0$ (case $\mathrm{length}(Z) \ge (t+1-g)\binom{d+2}{2}$) or $h^1(X, \mathcal{I}_Z \otimes L) = 0$.

Hence it is sufficient to check that $t \le h^0(C, R) - m$. Notice that Z has at least length$(Z)/\binom{m+1}{2}$ connected component and that any component supported by a point of H disappears taking the residual at most m times. We need to be sure that every component of a specialization of Z disappears in at most $h^0(C, R)$ steps taking the residual. Call z_t, $t \ge 1$, the number of simple points we delete taking the residual after the step $L(-tH) \Leftrightarrow L(-(t-1))$. Hence $z_t \ge 0$ for all $t \ge 0$. If $b_m \ge \sum_{i=1}^{m-1}(a_i + e_i)$, then $z_t = 0$ for $1 \le i \le m-1$ and $z_m = a_1$.

Claim. *For all integers $t \ge m$ such that Z_t and Z_{t+1} are defined, we have* $z_1 + \cdots + z_t \ge \lceil (t+1-m)(\binom{d+2}{2} - \binom{m+1}{2} + 1)/\binom{m+1}{2} \rceil$.

Proof of Claim. The integer $w_t := z_1 + \cdots + z_t$ is at least the number of schemes (not virtual schemes, i.e. schemes for which we do not use Remark 4) forced to have support on H in the steps $L \Leftrightarrow L(-H) \Leftrightarrow \cdots \Leftrightarrow L(-(t-m+1)H)$. In the step $L(-iH) \Leftrightarrow L(-(i+1)H)$ the intersection with H of these schemes has length at least $\binom{d+2}{2} - \binom{m+1}{2} + 1$. Each of the connected components of these schemes intersects H in a scheme with length at most $\binom{m+1}{2}$, proving the claim. □

Since length$(Z) \le h^0(X, L) + \binom{m+1}{2} - 1$, after at most $h^0(C, R) - m$ steps we get the empty set. Going from h^1 to h^0 we get that the original Z gave as much independent conditions to $H^0(C, L)$ as possible, i.e. $h^0(X, \mathcal{I}_Z \otimes L) = 0$.

Now assume length$(Z) \le \binom{d+2}{2} \cdot h^0(C, R)$. Taking a general union of Z and $\binom{d+2}{2} \cdot h^0(C, R) -$ length(Z) points (i.e. fat points of X with multiplicity 1) instead of Z we reduce to the case length$(Z) = \binom{d+2}{2} \cdot h^0(C, R)$, which was just proved. □

Proof of Theorem 2. Fix a general $P \in C$ and write $H := \mathbb{P}^3 \times \{P\}$. Since P is general in C. Copy the proof of Theorem 4, just quoting Remark 12 instead of Remark 5, Lemma 5 and Remarks 11 and 10 instead of Lemmas 2, 3, 4 and Corollaries 1 and 2. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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