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ACM AND BUCHSBAUM SPACE CURVES AND PFAFF SYSTEMS ON $\mathbb{P}^3$

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Abstract: Here, following L. Giraldo and A.J. Pan-Collantes (see *ArXiv:* math/0812.3369), we describe the parameter spaces of the codimension 1 distributions (Pfaff systems) on $\mathbb{P}^3$ with split tangent sheaf. Then we extend it to the Buchsbaum case.

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In [4], Theorem 3.3 and 3.5, and the first 2 lines of the proof of Lemma 4.2, described the zero-loci of all singular holomorphic distribution whose tangent sheaf is a split rank 2 vector bundle. our main aim to see (as a complement to [4], §4) an easy way to parametrize them. At the end of this note we study a generalization of the arithmetically Cohen-Macaulay case. Fix a singular holomorphic distribution β on $\mathbb{P}^3$ whose tangent sheaf is a decomposable vector bundle. Hence β induces an exact sequence on $\mathbb{P}^3$:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \to T\mathbb{P}^3 \to \mathcal{I}_Z(4-a-b) \to 0 \tag{1}$$

in which Z is an arithmetically Cohen-Macaulay curve in $\mathbb{P}^3$ and a, b are integers with, say, $a \geq b$. The integer $m := 3-a-b$ is the degree of the codimension 1 singular distribution β. Since $h^0(\mathbb{P}^3, T\mathbb{P}^3(-2)) = 0$, we have $a \leq 1$. We also

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know that the homogeneous polynomials F_0, F_1, F_2, F_3 inducing the distribution β generates the homogeneous ideal of Z (see [4], Theorem 3.5). Let $e(Z)$ be the maximal integer t such that $h^1(Z, \mathcal{O}_Z(t)) \neq 0$, i.e. the maximal integer t such that $h^0(Z, \omega_Z(-t)) \neq 0$ (the index of speciality of Z). Notice that $h^1(Z, \mathcal{O}_Z(t)) = h^2(\mathbb{P}^3, \mathcal{I}_Z(t))$. We distinguish 3 cases according to the values of the integers a and b.

(i) $a \leq 0$;

(ii) $a = 1$ and $b \leq 0$;

(iii) $a = b = 1$.

Case (i) is equivalent to the linear independence of the 4 polynomials F_0, F_1, F_2, F_3. Case (ii) (resp. case (iii)) arises if and only if the linear span of F_0, F_1, F_2, F_3 has dimension 3 (resp. dimension 2). Hence $a = b = 1$ if and only if Z is scheme-theoretically the complete intersection of two surfaces of degree m. Hence if $a = b = 1$ the distribution is integrable, with a degree m rational integral and everything is well-known. Hence from now on we may assume $b \leq 0$. Fix homogenenous coordinates x_0, x_1, x_2, x_3 on $\mathbb{P}^3$. We first discuss the space $A(a, b)$ of all arithmetically Cohen-Macaulay curves with the same numerical invariants (see below) as a curve Z fitting in (1). Then in (c) we will impose the condition that Z comes from a distribution, i.e. the condition $x_0F_0 + x_1F_1 + x_2F_2 + x_3F_3 = 0$ coming from the Euler's exact sequence of $T\mathbb{P}^3$ (see [5]).

(a) Here we discuss case (i). Let $B[m]$ be the set of all arithmetically Cohem-Macaulay space curves whose homogeneous ideal is generated by 4 linearly independent homogeneous polynomials of degree m. Fix any $W \in B[m]$. The Hilbert-Burch theorem (see [1], Theorem 20.15, or [8], Theorem 6.1.1) says that there are integers $c_1 \geq c_2 \geq c_3 \geq m + 1$ such that the minimal free resolution of W is of the following form:

$$0 \to \oplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^3}(-c_i) \xrightarrow{A} \mathcal{O}_{\mathbb{P}^3}(-m)^{\oplus 4} \to \mathcal{I}_W \to 0 \quad (2)$$

in which A is a 3×4 matrix of homogeneous forms. Moreover, the degree m generators of the homogeneous ideal are given by the 4 3×3 minors of A. Taking the first Chern class in (2) we get $c_3 = -c_1 - c_2 + 4m = -c_1 - c_2 + 12 - 4a - 4b$. The converse holds (Hilbert-Burch or [3]) and hence we give a partition of $B[m]$ into algebraic sets $B[m, c_1, c_2]$ coming from different integers $c_1 \geq c_2 \geq c_3 > m$ with $c_3 = -c_1 - c_2 + 12 - 4a - 4b$. For fixed m, c_1, c_2 the set $B[m, c_1, c_2]$ is parametrized by a non-empty open subset of an affine space of known dimension: the affine space with as coordinates the coefficients of the 12 homogeneous polynomials (4 of degree $c_3 - m$, 4 of degree $c_2 - m$ and 4 of degree $c_3 - m$) appearing in the matrix A of the Hilbert-Burch resolution (2). Even factoring out by the obvious redundancy, the parameter space is still smooth, irreducible and with known dimension (see [1], Theorem 2). The Euler's sequence of $T\mathbb{P}^3$, Serre duality and the cohomology of line bundles on $\mathbb{P}^3$ give $h^2(\mathbb{P}^3, T\mathbb{P}^3(t)) = 0$ for all $t \neq -4$, $h^2(\mathbb{P}^3, T\mathbb{P}^3(-4)) = 1$ and $h^3(\mathbb{P}^3, T\mathbb{P}^3(t)) = 0$ if and only if $t \geq 5$. Since $h^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(t)) = 0$ if and only if $t \geq -3$, $b \leq a$ and and $b \leq\leq 0$, from (1) we get $e(Z) = -a - 2b$. From (2) we get $e(Z) = c_3 - 4 = -c_1 - c_2 + 8 - 4a - 4b$. Hence $c_2 = -c_1 + 8 - 3a - 2b$.

(b) Here we discuss case (ii). Hence $a = 1$ and $b \leq 0$. Hence $m = 2 - b$. Let $D[m]$ be the set of all arithmetically Cohem-Macaulay space curves whose homogeneous ideal is generated by 3 linearly independent homogeneous polynomials of degree 3. Fix any $W \in D[m]$. The Hilbert-Burch Theorem (see [1], Theorem 20.15, or [8], Theorem 6.1.1) says that there are integers $c_1 \geq c_2 \geq m + 1$ such that the minimal free resolution of W is of the following form:

$$0 \to \oplus_{i=1}^{2} \mathcal{O}_{\mathbb{P}^3}(-c_i) \xrightarrow{A} \mathcal{O}_{\mathbb{P}^3}(-m)^{\oplus 3} \to \mathcal{I}_W \to 0 \tag{3}$$

in which A is a 2×3 matrix of homogeneous forms. Moreover, the degree m generators of the homogeneous ideal are given by the 3 2×2 minors of A. Taking the first Chern class in (3) we get $c_2 = -c_1 + 3m = -c_1 + 6 - 3b$. The converse holds (Hilbert-Burch or [3]) and hence we give a partition of $B[m]$ into algebraic sets $D[m, c_1]$ coming from different integers $c_1 \geq c_2 > m$ with $c_2 = -c_1 + 3m == -c_1 + 6 - 3b$. For fixed m, c_1 the set $D[m, c_1,]$ is parametrized by a non-empty open subset of an affine space of known dimension: the affine space with as coordinates the coefficients of the 6 homogeneous polynomials (3 of degree $c_2 - m$ and 3 of degree $c_1 - m$) appearing in the matrix A of the Hilbert-Burch resolution (3). Even factoring out by the obvious redundancy, the parameter space is still smooth, irreducible and with known dimension (see [1], Theorem 2). As in part (b) we get $e(Z) = -a - 2b = -2b - 1$. From (3) we get $e(Z) = c_2 - 4 = -c_1 + 2 - 3b$. Hence $c_1 = 3 - b$. If β is integrable, i.e. if β is a foliation, then this case is completely solved (see [4], Proposition 4.4): β is a pull-back of a degree m foliation of $\mathbb{P}^2$ and the convers holds.

(c) Now we add the condition that Z comes from a distribution, i.e. that $x_0F_0 + x_1F_1 + x_2F_2 + x_3F_3 = 0$. We saw that essentially it gives no restriction if $a = b = 1$. If $a \leq 0$ (resp. $a = 1$ and $b \leq 0$) then it gives a cubic (resp. quadratic) hpmogeneous equation in the parameter space of each $B[m, c_1, c_2]$ (resp. each $D[m, c_1]$), because each F_i arise as the determinant of a 3×3 (resp. 2×2) of a matrix in which the coordinates of $B[m, c_1, c_2]$ (resp. $D[m, c_1]$) appear with pure degree 1. Since this cubic equation is not identically zero, a general element of $B[m, c_1, c_2]$ does not give a distribution. Since Z is arithmetically Cohen-Macaulay, any such (F_0, F_1, F_2, F_3) induces a surjection $u : T\mathbb{P}^3 \to \mathcal{I}_Z(1+m)$. The proof of [4], Theorems 3.1 and 3.3, shows that $\mathrm{Ker}(u)$ is a split vector bundle. Since m is uniquely determined by Z, we saw in parts (a) and (b) that the degrees of the decomposable vector bundle $\mathrm{Ker}(u)$ are uniquely determined by Z.

Question 1. Which split distribution of type (i) ($a \le 0$) admits an algebraic solution?

Question 2. Take Z as in (1) or (2) or (3). What is the minimal degree of a surface S such that Z is contained in the singular locus of S?

Definition 1. Fix an integer $k \ge 1$. Let F be a coherent sheaf on $\mathbb{P}^3$. F is said to be k-Buchsbaum if for every integer t and every $f \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k))$ the linear map $\times f : H^1(\mathbb{P}^3, F(t)) \to H^1(\mathbb{P}^3, F(t+k))$ induced by the multiplication by f is zero. Acurve (i.e. a closed locally Cohen-Macaulay subscheme with pure dimension 1) $C \subset \mathbb{P}^3$ is said to be arithmetically k-Buchsbaum if its ideal shef $\mathcal{I}_C$ is k-Buchsbaum.

Theorem 1. *Let β be a singular holomirphic distibution*

$$0 \to E \to T\mathbb{P}^3 \to \mathcal{I}_C(1+m) \to 0 \tag{4}$$

with C a 1-Buchsbaum curve. Then either C is arithmetically Cohen-Macaulay or m is odd and $E \cong N((3-m)/2)$, where N is a null-correlation bundle, or E is stable and, taking $t := \lceil (3-m)/2 \rceil$, $c_2(E(-t)) \in \{1, 2\}$ (case m even) or $c_2(E(-t)) \in \{2, 4\}$.

Proof. E is locally free, because C is locally Cohen-Macaulay and with pure dimension 1 (see [4], Theorem 3.2). Fix any linear forms $f, \phi \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$. Since $h^2(\mathbb{P}^3, T\mathbb{P}^3(t)) = 0$ for all $t \ne 4$, we have an isomorphism $H^1(\mathbb{P}^3, \mathcal{I}_C(x)) \to H^2(\mathbb{P}^3, E(x-1-m))$ for all $x \ne m-3$. Hence for all $x \ne m-3$ the map $\cdot f : H^2(\mathbb{P}^3, E(x-1-m)) \to H^2(\mathbb{P}^3, E(x-m))$ induced by the multiplication by f is zero. By Sere duality we see that for all $x \ne m-3$ the multiplication map $\times f : H^1(\mathbb{P}^3, E^*(m-x-4) \to H^1(\mathbb{P}^3, E^*(m-x-3))$ induced by the multiplication by f is zero. Since E is a rank 2 vector bundle and $\det(E) \cong \mathcal{O}_{\mathbb{P}^3}(3-m)$, we have $E^* \cong E(m-3)$. Hence for all $x \ne m-3$ the map $\times f : H^1(\mathbb{P}^3, E(2m-x-7) \to H^1(\mathbb{P}^3, E(2m-x-6))$ induced by the multiplication by f is zero. Since $H^1(\mathbb{P}^3, \mathcal{I}_C(x)) \to H^2(\mathbb{P}^3, E(x-1-m))$ for all $x \ne m-3$, we also get that the map $\times f\phi : H^1(\mathbb{P}^3, E(m-x-4) \to H^1(\mathbb{P}^3, E(m-x-3))$ induced by the multiplication by $f\phi$ is zero. This is weaker that saying that E is 2-Buchsbaum, but it is enough to apply the proof [2], Corollary 8 and

Proposition 9. □

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