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A GENERALISED DYADIC NUMBER SYSTEM

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Abstract: It is defined a representation system for numbers in the unit interval, generalising the dyadic one, and two dynamical systems are given which generate it. Metric results are especially derived from the second of them. The approximative coefficient $\theta_n(x)$ is defined and studied with this second dynamical system. Moreover, it is deduced that, among other results, the Jager pair (θ_n, θ_{n-1}) has the same distribution on a set of λ-measure 1, it is concentrated on a denumerable set of segments in $[0,1]^2$, and an explicit expression is given for it.

In addition, Gauss-Kuzmin-Levy and Limit Central Theorem type results are given for some random variables in connection with this representation numbers system.

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1. Introduction

It is well known that for natural numbers $r \geq 2$, each real number x in $]0,1[$ has a series expansion (its r-base expansion) in the form $x = \sum_{n=1}^{+\infty} \frac{a_n}{r^n}$ with digits $a_n \in \{0,1,2,...,r-1\}$. Moreover, this expansion is unique, but certain

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rational numbers (that $x = \frac{p}{q}$ with $(p, q) = 1$ and q decomposes in factors all dividing r). For these rationals there are two possible series expansion, one of them is finite.

This expansions can be generalized for not integer bases. This is to say, for reals $\beta > 1$ and $x \in]0, 1[$, it is possible to establish (see [4]):

$$x = \sum_{n=1}^{+\infty} \frac{a_n}{\beta^n}$$

with $a_n \in \{0, 1, 2, ..., [\beta] - 1\}$. But now, the case is very different, because there exists a set of measure 1 such that all its points have infinite series expasions in the β-base. Hence, uniqueness for the series expansion disappears.

This handicap can be solved via the Renyi's *greedy*-representation given by the dynamic

$$T_\beta : [0, 1[\longrightarrow [0, 1[; \qquad T_\beta(x) := \beta x \ \ (mod \beta)\,,$$

obtaining digits

$$a_n = \left[\beta T_\beta^{n-1}\right].$$

This dynamic system is ergodic and there exists one, and only one, absolutely λ-continuous measure with measure preserving density h_β (see [10]). Afterwards, Gel'fond (in [5]) and Parry (in [9]) showed that

$$h_\beta(x) = \frac{\sum_{n=0}^{+\infty} \frac{1}{\beta^n} \chi_{[0,T^n(1)]}(x)}{\int_0^1 \sum\limits_{x<T^n(1)} \frac{1}{\beta^n} dx}.$$

The case $\beta < 2$ has especial interest; in this case, all digits are 0's and 1's. In case β smaller that the golden ratio, $\beta < \Phi := \frac{1+\sqrt{5}}{2}$, for each point in $]0, 1[$, the set of its series β-expansion has continuum cardinality.

If we write $a := 1/\beta$, previous expansions are in the form

$$x = \sum_{n=1}^{+\infty} a^{m_n}$$

with $m_n < m_{n+1}$ and $\frac{1}{2} < a < 1$. The proposal of this paper is to introduce a new representation for numbers in $]0, 1[$ via series expansion combining the bases a and $1 - a$; precisely, in the form

$$x = \sum_{n=1}^{+\infty} (1-a)^n \, a^{m_n}.$$

This situation is unique, very similar to that of dyadic expansions, but a denumerable set of numbers for which there are exactly two representations: one finite and the other infinite. They are obtained from a dynamical system which is ergodic and λ-preserving measure. An advantage for this system of representation is its validity for all $a \in]0,1[$, whitout the restriction $a \geq 1/2$.

In the literature we know, this generalized dyadic representation does not appear in an explicit form. It does in an implicit form, when some applications of certain singular functions are studied (see, for example, [8, p. 268] and [7, p. 227]).

With the help of the results in this paper, the authors can obtain in [1] Hausdorff dimension for fractal sets related with these functions, and an others that generalise them; particularly, the set where the Stieltjes measure associated to them concentrates its mass.

The outline of this paper is that follows. Next section is devoted to introduce the announced generalised dyadic representation system (GDRS) and we will establish its uniqueness.

The Section 3 describes a dynamical system generating this representation, but the scope of the results is limited and we modificate it in order to obtain new and better metric results, one of Loch-type among them.

In Section 4, we study the natural extension for the dynamical system and the corresponding approximative coefficients. They help us to show that the associated distribution function is singular and does not depend on x. Besides, it is proved that the correlation coefficient for the Jagger pairs is $\frac{1-a}{1+a}$.

At last, in Section 5, we show the Bernouillicity of the dynamical system; and, as applications, are proved results of Gauss-Kuzmin-Levy type. At the same time, we find applications of the iterated logarithm and central limit theorems for related random variables.

We end this first section with a little of notation and conventions for the rest of the paper.

Let $\mathbb{N}$ be the set $\mathbb{Z}^+$ of positive integers $\{1,2,...,n,n+1,...\}$. For each real x we consider the function $[x] := \max\{n \in \mathbb{Z}; n \leq x\}$. As it is usual, χ_A is the characteristic or indicator function for a set of reals $A \subseteq \mathbb{R}$. We will denote Lebesgue measure on the reals by λ.

For arbitrarily function f and natural n, $f^n := f \circ \overset{n}{\ldots} \circ f$, with the convention f^0 as the identity map.

Given two measuble spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \upsilon)$, a measuble function $f : X \to Y$ is said measure preserving if $\mu\left(f^{-1}(B)\right) = \upsilon(B)$, for each $B \in \mathcal{B}$. In case $X = Y$ and the σ-algebra $\mathcal{A} = \mathcal{B}$ is generated by a family $\mathcal{P}$ which is closed for finite intersections, a sufficient condition for f being measurable and measure preserving (see [3, p. 131]) is that

$$f^{-1}(A) \in \mathcal{A} \text{ and } \mu\left(f^{-1}(A)\right) = \mu(A) \text{ for all } A \in \mathcal{P}.$$

The system $(X, \mathcal{A}, \mu, f)$ will be said a dynamical system of representation, and the central theorem is this context (see [6]) is the so called

Theorem 1. (The Ergodic Theorem) *Let $(X, \mathcal{A}, \mu, f)$ be a dynamical system of representation. If g is λ-integrable, $g \in \mathcal{L}^1$, then the sequence*

$$\frac{1}{n} \sum_{k=0}^{n-1} g\left(f^k(x)\right)$$

converges μ-a.e., and its limit $\widehat{g}$ satisfies the following properties:

i. $\widehat{g} \in \mathcal{L}^1$.

ii. $\widehat{g}(f(x)) = f(x)$, μ-a.e.

iii. If $\mu(X) < +\infty$, then $\int_X \widehat{g} d\mu = \int_X g d\mu$.

iv. The sequence $\left(\frac{1}{n} \sum_{k=0}^{n-1} g\left(f^k(x)\right)\right)$ converges to g in the mean.

2. Generalised Dyadic Representation System

Definition 2. Let $a \in \,]0, 1[$. For each $x \in \,]0, 1]$, there exists a non negative integer n_0 such that

$$a^{n_0+1} \leq x \leq a^{n_0}.$$

Hence, $x = a^{n_0+1} + y_1$, with $0 \leq y_1 \leq a^{n_0}(1-a)$; and we can write

$$x = a^{n_0+1} + a^{n_0}(1-a)\, x_1,$$

where $x_1 \in [0, 1]$. Reasoning on x_1, we obtain

$$x = a^{n_0+1} + a^{n_0+n_1+1}(1-a) + a^{n_0+n_1}(1-a)^2\, x_2;$$

and, by induction, we have this formal equality:

$$x = \sum_{k=0}^{+\infty} (1-a)^k\, a^{1+\sum_{j=0}^{k} n_j},$$

which will be called the generalised dyadic representation for the corresponding number x.

These series converge to x, because

$$\left| x - \sum_{k=0}^{m} (1-a)^k\, a^{1+\sum_{j=0}^{k} n_j} \right| < (1-a)^{m+1},$$

and the majorization M-test of Weierstrass is applied. We summarize this situaion:

Proposition 3. *Let* $a \in \left]0,1\right[$*. If* $x \in \left]0,1\right]$*, then there exists an increasing sequence of naturals* $1 \leq m_0 \leq m_1 \leq ... \leq m_k \leq ...$*, such that*

$$x = \sum_{k=0}^{+\infty} (1-a)^k \, a^{m_k} \, .$$

Proposition 4. *The expansion in the above proposition is unique but it would be finite or estacionary (i.e.,* $m_k = m_j$ *if* $k \geq j$*).*

Proof. Because

$$1 = a + a\,(1-a) + a\,(1-a)^2 + a\,(1-a)^3 + ...,$$

in the finite or estacionary cases double expansions appear:

$$\sum_{k=0}^{n} (1-a)^k \, a^{m_k} = \sum_{k=0}^{n-1} (1-a)^k \, a^{m_k} + \sum_{k=n}^{+\infty} (1-a)^k \, a^{m_k+1}.$$

By the other hand, if the sequence (m_k) is not bounded (the expansion for x is not finite neither estacionary), let us consider infinite expansions for x and y:

$$x = \sum_{k=0}^{+\infty} (1-a)^k \, a^{m_k} \, , \quad y = \sum_{k=0}^{+\infty} (1-a)^k \, a^{m'_k} \, ,$$

where $0 \leq k \leq n-1$, implies $m_k = m'_k$, and $m_k < m'_k$ for $k \geq n$. Hence:

$$y \leq \sum_{k=0}^{n-1} (1-a)^k \, a^{m_k} + a^{m'_n} \sum_{k=n}^{+\infty} (1-a)^k = \sum_{k=0}^{n-1} (1-a)^k \, a^{m_k} + a^{m'_n - 1} \, (1-a)^n$$

$$< \sum_{k=0}^{n-1} (1-a)^k \, a^{m_k} + \sum_{k=n}^{+\infty} (1-a)^k \, a^{m_k} = x.$$

As a consequence, for an x with non bounded (m_k), x differs from y being estacionary or non estacionary. □

3. Dynamical System Associated to the GDRS

In all that follows $a \in \left]0,1\right[$.

Definition 5. $F_a(x) := \begin{cases} \frac{x}{a} ..., & x \in [0,a] \, , \\ \frac{x-a}{1-a} ..., & x \in \left]a,1\right] \, . \end{cases}$

Theorem 6. F_a *preserves* λ *and it is ergodic.*

For a proof, see [4, p. 68].

Let us consider the reciprocal for F_a. Then

$$x = \begin{cases} aF_a(x)..., & \text{if } 0 < x \leq a\,, \\ (1-a)\,F_a\,(x) + a..., & \text{if } a < x \leq 1\,, \end{cases}$$

and by iteration, we obtain:

$$\begin{aligned} x &= a^{n_0+1} + (1-a)\,a^{n_0+n_1+1} + ... \\ &\quad + (1-a)^m\,a^{n_0+n_1+...+n_m+1} + (1-a)^{m+1}\,a^{n_0+n_1+...+n_m+1} F_a^r\,(x)\,; \end{aligned}$$

and, hence, the dyadic generalized expansion is guarateed.

Definition 7. Let us consider a measure space $(X, \mathcal{A}, \mu)$ and a measurable function f which is measure preserving. The entropy of the dynamical system $(X, \mathcal{A}, \mu, f)$ is, if it exists, the number

$$h(f) := \int_X \ln \left| f'(x) \right| d\mu(x).$$

Proposition 8. *The entropy h of the system $([0,1]\,, B, \lambda, F_a)$ is*

$$\ln \frac{1}{a^a\,(1-a)^{1-a}}.$$

Proof. It is a consequence of the entropy formula:

$$h(F_a) = \int_0^1 \log \left| F_a'\,(x) \right| d\lambda\,(x)\,,$$

where F_a' denotes derivative. □

Definition 9. By a cylinder of order k we will understand a set

$$\triangle_{i_0...i_{k-1}} := A_{i_0} \cap f\,(A_{i_1}) \cap f^2\,(A_{i_2}) \cap ... \cap f^{k-1}\left(A_{i_{k-1}}\right).$$

We will simply write $\triangle_k$. The collection of all cyliders for each order k is a partition for the space X. For a cylinder of order k if we are interested in some x belonging to it, we will write $\triangle_k\,(x)$.

With the aid of Shannon-McMillan-Breiman Theorem (see [3]), it is inmediate that

Corollary 10. *There exists a set of λ-measure 1 such that*

$$\lim_n \frac{\ln \lambda\,(\triangle_n)}{n} = \ln \frac{1}{a^a\,(1-a)^{1-a}}.$$

Definition 11. We define the function H_a:

$$H_a(x) := \begin{cases} \frac{x}{1-a} - \frac{a}{1-a} ..., & a < x \leq 1\,, \\ \frac{x}{(1-a)a} - \frac{a}{1-a} ..., & a < x \leq a^2\,, \\ \frac{x}{(1-a)a^2} - \frac{a}{1-a} ..., & a^3 < x \leq a^2\,, \\ \frac{x}{(1-a)a^3} - \frac{a}{1-a} ..., & a^4 < x \leq a^3\,, \\ \vdots \end{cases}$$

By inversion, depending on where would be x:

$$x = \begin{cases} a + (1-a)\,H_a\,(x)\,..., & a < x \leq 1\,, \\ a^2 + (1-a)\,aH_a\,(x)\,..., & a^2 < x \leq a\,, \\ a^3 + (1-a)\,a^2H_a\,(x)\,..., & a^3 < x \leq a^2\,, \\ a^4 + (1-a)\,a^3H_a\,(x)\,..., & a^4 < x \leq a^3\,, \\ \vdots \end{cases}$$

This expansion yields a representation of the already studied type:

$$x = \sum_{k=0}^{+\infty} (1-a)^k \, a^{m_k}$$

with $1 \leq m_0 \leq m_1 \leq ... \leq m_k \leq ...$

Theorem 12. *H_a preserves λ and is ergodic.*

Proof. It follows as in [4, p. 68]. □

Theorem 13. *On a set of λ-measure 1,*

$$\lim_n \frac{m_n}{n} = \frac{a}{1-a}\,.$$

Proof. Let us define

$$d(x) := \begin{cases} 0..., & a < x \leq 1\,, \\ 1..., & a^2 < x \leq a\,, \\ 2..., & a^3 < x \leq a^2\,, \\ 3..., & a^4 < x \leq a^3\,, \\ \vdots \end{cases}$$

Hence:

$$\begin{aligned} m_0 &= d(x) + 1\,, \\ m_1 &= d\left(H_a\,(x)\right) + d(x) + 1\,, \\ m_2 &= d\left(H_a^2\,(x)\right) + d\left(H_a\,(x)\right) + d(x) + 1\,, \\ m_3 &= d\left(H_a^3\,(x)\right) + d\left(H_a^2\,(x)\right) + d\left(H_a\,(x)\right) + d(x) + 1\,, \end{aligned}$$

$$\vdots$$

Applying the Ergodic Theorem to H_a:

$$\lim_n \frac{m_n}{n} = \lim_n \frac{1 + d\left(H_a\left(x\right)\right) + ... + d\left(H_a^n\left(x\right)\right)}{n}$$

exists on a set of λ-measure 1 (because d is integrable) with value

$$\int_0^1 d(x)dx = \sum_{n=0}^{+\infty}\left(a^n - a^{n+1}\right)n = (1-a)\sum_{n=0}^{+\infty} a^n n = \frac{a}{1-a}. \qquad \square$$

Theorem 14. *The fractal dimension of the set of points where* $\lim_n \frac{m_n}{n} = d$ *is*

$$\frac{d\ln d - (1+d)\ln(1+d)}{d\ln a + \ln(1-a)}.$$

Proof. See [1]. □

Corollary 15. *The fractal dimension of the set of points where* $\lim_n \frac{m_n}{n} = 1$ *is*

$$\frac{-2\ln 2}{\ln a + \ln(1-a)}.$$

Proposition 16. *The entropy of the system* $([0,1], B, \lambda, H_a)$ *is*

$$\ln \frac{1}{a^{\frac{a}{1-a}}(1-a)}.$$

Proof. Entropy's formula and summing like above give the result. □

And the Shannon-McMillan-Brieman Theorem implies:

Corollary 17. *There exists a set of λ-measure 1 such that*

$$\lim_n \frac{\ln\lambda(\triangle_n)}{n} = \ln\frac{1}{a^{\frac{a}{1-a}}(1-a)}.$$

Definition 18. If $x = \sum_{k=0}^{+\infty}(1-a)^k a^{m_k}$ then let us denote

$$B_n := \sum_{k=0}^{n}(1-a)^k a^{m_k}; \qquad C_n := B_n + a^{m_{n-1}-1}(1-a)^n.$$

Lemma 19. *For the n-th cylinder,*

$$\frac{a\lambda(\triangle_n)}{1-a} \leq |x - B_{n-1}| \leq \lambda(\triangle_{n-1}).$$

Proof. The cylinder $\triangle_n$ is the interval $[B_n, C_n]$, and the following identities imply the result:

$$\lambda(\triangle_n) = a^{m_{n-1}-1}(1-a)^n, \quad B_n - B_{n-1} = a^{m_{n-1}}(1-a)^{n-1}. \qquad \square$$

Taking logarithms in the above lemma, and applying the previous corollary:

Corollary 20. *On a set of λ-measure 1,*

$$\lim_n \frac{-\ln (x - B_n)}{n} = \ln \frac{1}{(1-a)\, a^{\frac{a}{1-a}}}.$$

Corollary 21. (Loch-Type Result) *If n digits of the decimal expansion of x determine the first k among the m_j, then there exists a set of λ-measure 1, such that*

$$\lim_n \frac{n}{k} = \ln \frac{1}{(1-a)\, a^{\frac{a}{1-a}}} / \ln 10.$$

Proof. Let x be with infinite non estacionary expansion, $D_n(x)$ be the decimal cylinder $[A_n, A'_n]$ containing x. Now, let $\triangle_k$ the dyadic cylinder containing x and D_n with k as great as possible; i.e., $D_n \subset \triangle_k$ and $B_{k+1} \in D_n$ or $C_{k+1} \in D_n$. We consider two cases:

a. If $B_{k+1} \in D_n$, then $[B_{k+1}, B_{k+2}] \subset D_n$, and

$$\lambda(\triangle_{k+2}) \ll B_{k+2} - B_{k+1} = a^{m_{k+1}} (1-a)^{k+1}.$$

b. If $C_{k+1} \in D_n$, then $[C_{k+2}, C_{k+1}] \subset D_n$, and

$$C_{k+1} - C_{k+2} \geq a^{m_k} (1-a)^{k+1} \gg \lambda(\triangle_{k+1}).$$

For both cases $\lambda(\triangle_{k+j}) \ll \lambda(D_n) \leq \lambda(\triangle_k)$ $(j \in \{1,2\})$, and we obtain

$$\frac{-\ln \lambda(\triangle_k)}{k} \leq \frac{-\ln \lambda(D_n)}{k} \leq \frac{-\ln \lambda(\triangle_{k+j})}{k} + O\left(\frac{1}{k}\right),$$

with the extreme terms converging to the value

$$-\ln(1-a) - \frac{a}{1-a} \ln a;$$

hence, so it is for the central term.

Finally, with the aid of the Shannon-McMillan-Breiman Theorem, there exists the limit $\lim_n \frac{n}{k}$ with value

$$\frac{-\ln(1-a) - \frac{a}{1-a}\ln a}{\ln 10}$$

on a set of λ-measure 1. □

Proposition 22. *Let us denote by s_n the number of different elements among $\{m_0, m_1, ..., m_n\}$. Then on sets of λ-measure 1,*

$$\lim_n \frac{s_n}{n} = a.$$

Proof. Looking at H_a, we see it changes on m if $H_a^k(x) \in [0, a[$ and it does not change if $H_a^k(x) \in [a,1]$. Hence, by integration of $\chi_{[0,a]}$, and the Ergodic Theorem, we obtain

$$\lim_n \frac{s_n}{n} = a. \qquad \square$$

4. Natural Extension and Jager Pairs

Next, we follow to know the goodness of the approximation of $x \in [0,1]$ by B_{n+1}.

Definition 23. (Approximation Coefficients) For each x and n,

$$\theta_n(x) := \frac{|x - B_{n+1}|}{a^{m_n - 1}(1-a)^n}.$$

Remark 24. $\theta_n = H_a^{n+1}$; because

$$H_a^n(x) = a^{-m_{n-1}+1} \sum_{k=0}^{+\infty} a^{m_{n+k}} (1-a)^k .$$

As a consequence of Ergodic Theorem:

Theorem 25. *The sequence* (θ_n) *is uniformly distributed on a set of* λ*-measure 1 in* $[0,1]$.

Corollary 26. *On a set of* λ*-measure 1,*

$$\frac{1}{n}\sum_{k=1}^{n} \theta_k = \frac{1}{2} \text{ and } \frac{1}{n}\sum_{k=1}^{n} \theta_k^2 = \frac{1}{3}.$$

Theorem 27. *If we denote*

$$\overline{H}_a(x,y) := \left(H_a(a), a^{m_0} + (1-a)a^{m_0-1}y\right),$$

then the dynamical system $\left([0,1]^2, B, \lambda, \overline{H}_a\right)$ *is the natural extension of the given* $([0,1], B, \lambda, H_a)$, *where* m_0 *depends on* x.

Proof. The projection map

$$\pi : [0,1] \times [0,1] \to [0,1]; (x,y) \to x$$

is measurable.

$\overline{H}_a$ is λ-measure preserving: we restrict ourselves to rectangles of the form $\left[a^m, a^{m-1}\right] \times [0,1]$. Because this family generates the total σ-algebra, and $\overline{H}_a$ is λ-measure preserving on $[c,d] \times [s,t]$, hence (see [2, p. 311]) $\overline{H}_a$ is λ-measure preserving on B.

$\lambda(A\times[0,1]) = \lambda(A)$, by similar arguments as above, because this is its behaviour on the intervals $\left[a^m, a^{m-1}\right]$.

For two given cylinders, $(m_0, m_1, m_2, ..., m_n) = (u, v)$ and $(\overline{m}_0, \overline{m}_1, \overline{m}_2, ..., \overline{m}_k) = (r, s)$, let us consider the new one cylinder

$$\begin{aligned}(c,d) \quad : \quad &= (\overline{m}_n - \overline{m}_{n-1} + 1, \overline{m}_n - \overline{m}_{n-2} + 1, ..., \\ &\overline{m}_n, \overline{m}_n + \overline{m}_0 - 1, \overline{m}_n - \overline{m}_1 - 1, ..., \overline{m}_n + \overline{m}_r - 1).\end{aligned}$$

Hence, it follows that

$$\overline{H}_a^{-n}(c,d) = (u,v)\times(r,s);$$

and B is generated by reciprocal images of rectangles (because it is generated by rectangles which are product of cylinders). □

Proposition 28. $\overline{H}_a^n$ *is ergodic and mixed.*

Proof. $([0,1], B, \lambda, H_a)$ has both properties; hence its natural extension has them too. □

And, as a consequence of the Ergodic Theorem:

Corollary 29. *If* C *is a* λ*-measurable with* $\lambda(\partial C) = 0$, *then*

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} \chi_C\left(\overline{H}_a^k(x,y)\right) = \lambda(C).$$

Corollary 30. (Jager-Type Result) *There exists a set of* λ*-measure 1 in* $[0,1]$, *such that* λ *is an asintotic distribution function for* $\left\{\overline{H}_a^k(x,0)\right\}$.

Proof. Let us consider a pair (x,y) such tha the sequence $\left(\overline{H}_a^k(x,y)\right)_k$ is uniformly distributed.

For a given $\varepsilon > 0$, with n such that $(1-a)^n < \varepsilon$, if $k \geq n$, then

$$\overline{H}_a^k(x,y) = (x_k, y_k) \text{ and } \overline{H}_a^k(x,0) = (x_k, \overline{y}_k)$$

implies $|y_k - \overline{y}_k| < \varepsilon$.

Now, if $C := [a,b]\times[c,d]$, $C_{\varepsilon^-} := [a,b]\times[c+\varepsilon, d-\varepsilon]$, and $C_{\varepsilon^+} := [a,b]\times[c-\varepsilon, d+\varepsilon]$, then

$$\left.\begin{array}{l}\overline{H}_a^k(x,y) \in C_{\varepsilon^-} \implies \overline{H}_a^k(x,0) \in C \\ \overline{H}_a^k(x,0) \in C \implies \overline{H}_a^k(x,y) \in C_{\varepsilon^+}\end{array}\right\}$$

which imply these two chains of inequalities:

$$\frac{\sum_{j=1}^k \chi_{C_{\varepsilon^-}} \circ \overline{H}_a^j(x,y)}{k} \leq \frac{\sum_{j=1}^k \chi_C \circ \overline{H}_a^j(x,0)}{k} \leq \frac{\sum_{j=1}^k \chi_{C_{\varepsilon^+}} \circ \overline{H}_a^j(x,y)}{k}$$

and

$$\lambda(C) - 2\varepsilon \leq \liminf_k \frac{\sum_{j=1}^k \chi_C \circ \overline{H}_a^j(x,0)}{k} \leq \limsup_k \frac{\sum_{j=1}^k \chi_C \circ \overline{H}_a^j(x,0)}{k} \leq \lambda(C) + 2\varepsilon.$$

According with it, there exists the limit

$$\lim_k \frac{\sum_{j=1}^k \chi_C \circ \overline{H}_a^j(x,0)}{k} = \lambda(C).$$

For almost all x in $[0,1]$, there exists y (depending on x) verifying the announced property; and, hence, $\left\{\overline{H}_a^k(x,0)\right\}$ is uniformly distributed. □

We can write the identity

$$H_a^n = a^{m_n - m_{n-1}+1} + (1-a)\, a^{m_n - m_{n-1}} H_a^{n+1},$$

and rewriting it in the form

$$\theta_{n-1} = a^{m_n - m_{n-1}+1} + (1-a)\, a^{m_n - m_{n-1}} \theta_n,$$

it is usual to define

$$\Psi(x,y) := \left(x, a^{m_0(y)} + (1-a)^{m_0(y)+1}\right),$$

and hence

$$\Psi\left(\overline{H}_a^k(x,0)\right) = (\theta_n(x), \theta_{n-1}(x)).$$

Definition 31. (Jager Pairs $(\theta_n(x), \theta_{n-1}(x))$) The image of $\overline{H}_a^k(x,0)$ under the map Ψ is called a Jager pair.

Let us note that Ψ acts mapping the rectangle $[0,1] \times \left[a^r, a^{r-1}\right]$ on the segment of endpoints $\left(0, a^{r+1}\right)$ and $(a^r, 1)$. Because $\left(\overline{H}_a^k(x,0)\right)_k$ is uniformly distributed on $[0,1]^2$ for almost all x in $[0,1]$, the Jager pair $(\theta_n(x), \theta_{n-1}(x))$ admits a bidimensional distribution function corresponding to the surface of $\Psi^{-1}([0,z_1] \times [0,z_2])$. This distribution function concentrates regularly the corresponding mass $(1-a)\, a^r$ on each segment of endpoints $\left(0, a^{r+1}\right)$ and $(a^r, 1)$. Hence, it is a singular distribution. The explicit formula for $F(\theta_n \leq t_1, \theta_{n-1} \leq t_2)$ is:

$$\begin{cases} a^r t_1 ..., & a^{r+1} \leq t_2 \leq a^r, \\ & t_1 \leq \frac{t_2 - a^{r+1}}{(1-a)a^r}, \\ a^{r+1} t_1 + (1-a)\, a^r \left(t_2 - a^{r+1}\right) ..., & a^{r+1} \leq t_2 \leq a^r, \\ & t_1 > \frac{t_2 - a^{r+1}}{(1-a)a^r}. \end{cases}$$

To study Jager pairs we only need some definitions we expose as limits (if they exist):

Definition 32. $E(\theta_n) := \lim_k \frac{1}{k}\sum_{j=1}^{k}\theta_j$.

Definition 33. $E(\theta_n^2) := \lim_k \frac{1}{k}\sum_{j=1}^{k}\theta_j^2$.

Definition 34. $E(\theta_n\theta_{n-1}) := \lim_k \frac{1}{k}\sum_{j=1}^{k}\theta_j\theta_{j-1}$.

Definition 35. $\sigma^2(\theta_n) := E(\theta_n^2) - E(\theta_n)^2$.

Definition 36.

$$\rho(\theta_n,\theta_{n-1}) := \frac{E(\theta_n\theta_{n-1}) - E(\theta_{n-1})E(\theta_n)}{\sqrt{\sigma^2(\theta_n)}\sqrt{\sigma^2(\theta_{n-1})}} = \frac{E(\theta_n\theta_{n-1}) - E(\theta_n)^2}{\sigma^2(\theta_n)}.$$

Doing manipulations on $E(\theta_n\theta_{n-1})$, we have

$$\sum_{r=0}^{+\infty} a^r(1-a)\int_0^1 x\left[a^{r+1} + (1-a)a^r x\right]dx$$
$$= \sum_{r=0}^{+\infty} a^{2r}\left[\frac{a(1-a)}{2} + \frac{(1-a)^2}{3}\right] = \frac{a+2}{6(1+a)}$$

which is the one we need to assert that:

Theorem 37. $\rho(\theta_n,\theta_{n-1}) = \frac{1-a}{1+a}$.

The way Ψ acts, yields to this result:

Theorem 38.

$$\lim_n \frac{\text{Card}\{k;\theta_k<\theta_{k-1}, k=1,2,...,n\}}{n} = a(1-a)\sum_{k=0}^{+\infty}\frac{a^{2r}}{1-(1-a)a^r}.$$

The series in the above theorem has a remarkable interest. If we define a function

$$f(a) := a(1-a)\sum_{k=0}^{+\infty}\frac{a^{2r}}{1-(1-a)a^r},$$

for all a in $[0,1]$, then its graph is the given in Figure 1; hence, the average of k's for which $\theta_k > \theta_{k-1}$ is more than a half on a λ-set of measure 1.

5. Gauss-Kuzmin-Levy Type Theorems and Related Results

Theorem 39. *The system* $([0,1], B, \lambda, H_a)$ *is Bernouilli.*

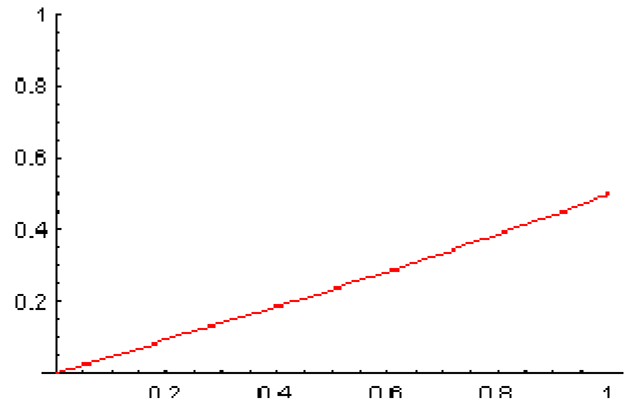

Figure 1:

Proof. Let us denote $A := \mathbb{N} \cup \{0\}$ the set of non negative integers, the σ-algebra $\mathcal{P}(A)$ of the power set of A, and the measure given by $\mu(\{n\}) := (1-a)\,a^n$. Let us now define a bijection from the denumerable product set of infinite copies of $(A, \mathcal{P}(A), \mu)$ on $[0,1]$ given by

$$H((s_1, s_2, s_3, ...)) := \sum_{k=0}^{+\infty} a^{1+\sum_{j=1}^{k+1} s_j} (1-a)^k .$$

This map induces a σ-algebra on $[0,1]$ (generated by the cylinders in dyadic generalized expansion), which is B. Because the measure μ coincides with λ on cylinders, it is actually λ.

Introducing the shift operator

$$S((s_1, s_2, s_3, ...)) := (s_2, s_3, s_4, ...),$$

it is clear that S induces H_a on $[0,1]$. □

Corollary 40. *The variables $X_j := m_j - m_{j-1}$ are independent and identically distributed, with probability*

$$P(X_j = s) = (1-a)\,a^s.$$

Corollary 41. $P(m_j = r) = P(r)\,(1-a)^j\,a^r$, *where $P(r)$ the number of times for which r is expressed by the sum of j different positive integers, taking their order into account.*

Corollary 42. (Gauss-Kuzmin-Levy) *Let m be a probability measure on the measurable space $([0,1], B)$. If $m \ll \lambda$, then*

$$\lim_n m(\{x; H_a^n(x) \leq t\}) = t.$$

The result is true, in particular case, when $dm = \frac{dx}{1+x}$ *is the Gauss measure.*

Proof. Let us consider the Radon-Nikodym derivative $h := \frac{dm}{d\lambda}$. Now, by the Ergodic Theorem:

$$\lim_n m\left(\{x; H_a^n(x) \leq t\}\right) = \lim_n \int_0^1 \left(\chi_{[0,t]} \circ H_a^n\right)(x)\, dm$$
$$= \lim_n \int_0^1 \left(\chi_{[0,t]} \circ H_a^n\right)(x)\, h(x) d\lambda$$

by mixing property,

$$= \lim_n \int_0^1 \chi_{[0,t]}(x)\, d\lambda \lim_n \int_0^1 h(x) d\lambda = t. \qquad \square$$

Definition 43. If x has not finite expansion and $\triangle i_0 i_1 ... i_k$ is its k-th cylinder, $\varsigma_k := \varsigma_k(x)$ will denote the finite expansion with k indexes given in the inverse form; i.e., if

$$x = a^{m_0} + (1-a)\, a^{m_1} + ...,$$

then

$$\varsigma_n = a^{m_k} + (1-a)\, a^{m_{k-1}} + ... + (1-a)^k\, a^{m_0}.$$

Corollary 44. *If* $H_a^n(x)$ *is uniformly distributed, then* ς_n *is too.*

Proof. Let us consider sets

$$C := C_{s_1, s_2, ..., s_n} = \{x;\ \text{its first } X_k \text{ are } s_1, s_2, ..., s_n\}$$

and

$$C' := C_{s_n, ..., s_2, s_1} = \{x;\ \text{its first } X_k \text{ are } s_n, s_{n-1}, ..., s_1\}.$$

Note that $\varsigma_n \in C \iff H_a^{k-n}(x) \in C'$; hence

$$\lim_n \frac{\operatorname{Car}\{\varsigma_n \in [0,t]\,; k = 0,1,...,n\}}{n}$$
$$= \lim_n \frac{\operatorname{Car}\left\{H_a^{k-n}(x) \in C'; k = 0,1,...,n\right\}}{n} = \lambda\left(C'\right) = \lambda(C).$$

Because C and C' are cylinders and $[0,t]$ is a reunion of cylinders, we conclude

$$\lim_n \frac{\operatorname{Car}\{\varsigma_k \in [0,t]\,; k = 0,1,...,n\}}{n} = t,$$

and the uniform distribution for the sequence is derived. $\square$

Corollary 45. (Levy) *Let* m *be a probability measure on the measurable space* $([0,1], B)$. *If* $m \ll \lambda$, *then*

$$\lim_n m\left(\{x; \varsigma_n \leq t\}\right) = t.$$

Proof. Because $\lim_n m\left(\{x; H_a^n(x) \in J\}\right) = \lambda(J)$ for arbitrary intervals J, in cases $J = C$ or $J = C'$,

$$\lim_n m\left(\{x; \varsigma_n \in C\}\right) = \lim_n m\left(\{x; H_a^n(x) \in J\}\right) = \lambda(J),$$

and in the same way as before, we conclude $\lim_n m\left(\{x; \varsigma_n \leq t\}\right) = t$. □

Theorem 46. *On a set of λ-measure 1,*

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} X_k = \frac{a}{1-a}.$$

Proof. It is a consequence of the Strong Law of Large Numbers, because $E(X_k) = \frac{a}{1-a}$ (it is a new proof of Theorem 10). □

If we add an error term, applying the Itereted Log Law (see [2, p. 154]) to the variables

$$Y_k := \frac{X_k - \frac{a}{1-a}}{\frac{\sqrt{a}}{1-a}},$$

it is inmediate that

Theorem 47. *On a set of λ-measure 1,*

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} x_k = \frac{a}{1-a} + O\left(\frac{\sqrt{\log\log n}}{\sqrt{n}}\right).$$

And, in the same way (see [2, p. 153]), we have:

Theorem 48. *If (a_n) diverges positively and $a_n = o\sqrt{n}$, then*

$$P\left(\frac{\sum_{k=1}^{n} X_k - n\frac{a}{1-a}}{\sqrt{n\frac{a}{(1-a)^2}}} \geq a_n\right) = e^{-\frac{a_n^2}{2}(1+o(1))}.$$

Theorem 49. *For random variables X_k,*

$$\lim_n \lambda\left(a < \frac{\sum_{k=1}^{n} X_k - n\frac{a}{1-a}}{\sqrt{n\frac{a}{(1-a)^2}}} < b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}}\, dx.$$

Proof. It is a jointly application of the Central Limit Theorem and the Lindeberg-Levy Theorem (see [2, p. 357]). □

Theorem 50. *With the above notation, if*

$$Z_n := \max_{1 \leq k \leq n} X_k \implies \lim_n P\left(Z_n \leq ny\right) = 1.$$

Proof. Let us note that $Z_n \leq ny \iff X_k \leq ny, k \in \{1,2,...,n\}$, and then
$$P\left(Z_n \leq ny\right) = \left(1 - a^{[ny]+1}\right)^n .$$
But this sequence on the left converges to 1. □

Theorem 51. *The sequence*
$$a_n := P\left(Z_n \leq \log_a \frac{1}{n}\right)$$
is dense in $[0, 1/e]$.

Proof. Similar arguments as above, and the density of $\{\log_a - n\left[\log_a n\right]\}$ in $[0,1]$ give the result. □

Theorem 52.
$$y < 1 \implies \lim P\left(Z_n \leq \log_a \tfrac{1}{n^y}\right) = 0\,,$$
$$y > 1 \implies \lim P\left(Z_n \leq \log_a \tfrac{1}{n^y}\right) = 1\,.$$

Theorem 53. $P\left(X_n \geq r_n \text{ infinite usually}\right) = 1 \iff \sum_{n=1}^{+\infty} a^{[r_n]+1}$ *diverges.*

Proof. Let us consider random variables
$$V_n := \begin{cases} 1..., & X_n \geq r_n\,, \\ 0..., & X_n < r_n\,. \end{cases}$$
To be $X_n \geq r_n$ infinitely many times is equivalent that $\sum_n V_n$ diverges. By the three-series theorem (see [2, p. 290]) we conclude the proof. □

Corollary 54. *On a set of λ-measure 1,*
$$\limsup_n -\frac{X_n \ln a + \ln n}{\ln\left(\ln n\right)} = 1.$$

Proof. If $r_n := -\left(\log_a n + \log_a \ln n\right)$, then
$$a^{[r_n]+1} \simeq \frac{1}{n \ln n},$$
and the series $\sum_{n\geq 1} a^{[r_n]+1}$ diverges. Hence, the set
$$\{n \in \mathbb{N}; X_n \geq -\left(\log_a n + \log_a \ln n\right)\}$$
is infinite, and it follows that
$$\limsup_n -\frac{X_n \ln a + \ln n}{\ln\left(\ln n\right)} \geq 1.$$
In fact, this is an equality. By the opposite, let us suppose that
$$\limsup_n -\frac{X_n \ln a + \ln n}{\ln\left(\ln n\right)} \geq 1 + \varepsilon$$

for some $\varepsilon > 0$. But if it is the case, then the series (using similar arguments as above) $\sum_{n\geq 1} \frac{1}{n(\ln n)^{1+\varepsilon}}$ diverges, which contradicts the theorem above. □

Proposition 55. *On a set of λ-measure 1,*

$$\liminf_{n} -\frac{X_n \ln a + \ln n}{\ln(\ln n)} = -\infty.$$

Proof. This is a consequence that on a set of λ-measure 1 $X_n = 1$ infinitely many times. □

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