

EIGENVALUES OF A SELF-DIFFERENTIAL OPERATOR

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Abstract: In this paper we study the asymptotic behavior of the eigenvalues of the compact operator $Tf(x) = \int_0^1 \varphi(x, s)f(s)ds$, where $f \in C[0, 1]$ with $f(0) = 0$, and

$$\varphi(x, s) = \begin{cases} 0, & \text{if } 1 > s > 2x, \\ 1/2, & \text{if } 2x > s > 2x - 1, \\ 1, & \text{if } 2x - 1 > s > 0. \end{cases} \quad (1)$$

We show that the eigenvalues $\{A_n\}_{n=-\infty}^{\infty}$ of T satisfy

$$\lim_{n \rightarrow \pm\infty} A_n = 0, \quad \lim_{n \rightarrow \infty} \frac{A_n}{A_{-n}} = -1, \quad \lim_{n \rightarrow \infty} \frac{A_n}{A_{n-1}} = \frac{1}{4}, \quad (2)$$

and that A_n^{-1} are zeros of the power series

$$\sum_{m=0}^{\infty} \frac{B_m x^m}{2^{(m^2-m)/2} m!},$$

where B_m are the Bernoulli numbers. An asymptotic estimate of a partial theta function is involved in the proof.

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1. Introduction

A “self-differential” function is a function satisfying an equation relating the value of the derivative at a point to values of the function itself at other points. So in a sense the notion is a mixture of the conventional differential equation and the self-similar (or self-affine) function.

In previous work [5] we considered the very simple self-differential equations

$$\begin{cases} f'(x) = af(2x), & \text{if } 0 \leq x \leq \frac{1}{2}, \\ f'(x) = af(2-2x), & \text{if } \frac{1}{2} \leq x \leq 1, \\ f(0) = 2-a, f(1) = 2+a, \end{cases} \quad (1.1)$$

and

$$\begin{cases} f'(x) = af(2x), & \text{if } 0 \leq x \leq \frac{1}{2}, \\ f'(x) = af(2-2x), & \text{if } \frac{1}{2} \leq x \leq 1, \\ f(0) = 0, f(1) = 0. \end{cases} \quad (1.2)$$

The main results were: Equation (1.2) has infinitely many solutions for values of $a \in S$ where $S = \{2^{2n+1} : n = 1, 2, \dots\}$ while equation (1.1) has no solution for $a \in S$. When $a \notin S$ equation (1.1) has a unique solution and equation (1.2) does not have a solution.

In the present paper we will consider another simple functional differential equation:

$$\begin{cases} f'(x) = af(2x), & 0 \leq x \leq \frac{1}{2}, \\ f'(x) = af(2x-1), & \frac{1}{2} \leq x \leq 1, \\ f(0) = 0, f(1) = d. \end{cases} \quad (*)$$

Without loss of generality we may assume $d = 1$. This was studied in [4] Chapter 3, Case 4. An example is in Figure 1: $a = 10.1952$, the graph of f is on the left, the graph of the derivative f' is on the right, discontinuous at $1/2$. The graph of f' consists of two affine images of the graph of f .

In [4] we showed that (*) has a unique solution for values of a which are zeros of a function $G(x)$, and described $G(x)$ as a power series. The power series may be computed as $G(x) = \lim_{n \rightarrow \infty} G_n(x)$, where $G_n(x)$ is a polynomial of degree $n-1$ defined recursively: $G_1(x) = 1$ and

$$G_n(x) = G_{n-1}(x) - \sum_{k=1}^{n-1} \frac{x^k}{2^{nk-k/2-k^2/2+1} k!} G_{n-k}(x).$$

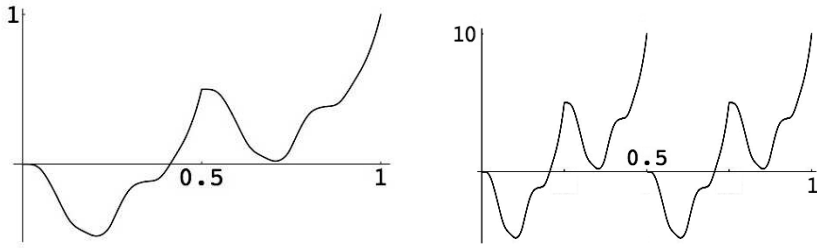


Figure 1:

In Section 3 of this paper we will show that $G(x)$ is the power series

$$\sum_{M=0}^{\infty} \frac{B_M x^M}{2^{(M^2-M)/2} M!},$$

where B_M are the Bernoulli numbers.

The fact that the values of a for which (*) has a solution are zeros of G is derived from the requirement that $f'(0) = af(0) = 0$. The smallest twelve zeros of G are:

$$\begin{aligned} a_1 &= 10.1952, & a_{-1} &= 2.53281, \\ a_2 &= 37.9638, & a_{-2} &= -52.1525, \\ a_3 &= 145.083, & a_{-3} &= -143.617, \\ a_4 &= 571.596, & a_{-4} &= -571.602, \\ a_5 &= 2277.598, & a_{-5} &= -2277.598, \\ a_6 &= 9101.829, & a_{-6} &= -9101.829. \end{aligned}$$

These values of a_n suggest that the zeros of $G(x)$ have the following pattern: $a_n \approx -a_{-n}$ and $a_{n+1} \approx 4a_n$ for large values of n . In Section 2 of this paper we will show that the solutions of (*) are the eigenfunctions of the operator T , defined in the abstract, with eigenvalues $A_n = 1/a_n$. The properties of a_n given above will follow from Theorem 2.1.

2. Eigenvalue Formulation

In this section we explain the relation between the eigenvalues of the operator T and the solutions of equation (*).

Let f be a solution of (*). The function f satisfies

$$f(x) = f(0) + \int_0^x f'(t)dt = \int_0^x af(2t)dt = \frac{a}{2} \int_0^{2x} f(s)ds \quad (2.1)$$

for $0 \leq x \leq \frac{1}{2}$. From (2.1) with $x = \frac{1}{2}$ we obtain

$$f\left(\frac{1}{2}\right) = \frac{a}{2} \int_0^1 f(s) ds.$$

When $\frac{1}{2} \leq x \leq 1$ we have

$$\begin{aligned} f(x) &= f\left(\frac{1}{2}\right) + \int_{1/2}^x f'(t) dt = f\left(\frac{1}{2}\right) + \int_{1/2}^x af(2t-1) dt \\ &= \frac{a}{2} \int_0^1 f(s) ds + \frac{a}{2} \int_0^{2x-1} f(s) ds. \end{aligned} \quad (2.2)$$

From equations (2.1) and (2.2) the function f satisfies

$$f(x) = a \int_0^1 \varphi(x, s) f(s) ds,$$

where $\varphi : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is defined by (1) as in the abstract. Define the operator T by

$$Tf(x) = \int_0^1 \varphi(x, s) f(s) ds.$$

The eigenfunctions of T are differentiable on the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ and satisfy equations (2.1) and (2.2). Their derivative is discontinuous at $x = \frac{1}{2}$, where the derivative from the left is ad and the derivative from the right is 0. The solutions of equation (*) satisfy $Tf(x) = \frac{1}{a}f(x)$ and the eigenfunctions of T are solutions of (*).

Then by [4], the operator T has eigenvalues $\{A_n\}_{n=-\infty}^{\infty}$, where $A_n = 1/a_n$ and a_n are the zeros of G .

Theorem 2.1. *The eigenvalues of T satisfy:*

- (i) $\left|A_n - \frac{\sqrt{2}}{\pi 4^n}\right| < \frac{1}{8^{n+1}}$, for n sufficiently large positive integer.
- (ii) $\left|A_{-n} + \frac{\sqrt{2}}{\pi 4^{-n}}\right| < \frac{1}{8^{1-n}}$, for $n < 0$ and $|n|$ sufficiently large.

The proof is in Section 5. Properties (2) of the eigenvalues of T follow from Theorem 2.1.

3. The Coefficients of G

We now evaluate the coefficients b_n of the power series $G(x) = \sum_{n=0}^{\infty} b_n x^n$. From [4], $G(x)$ is the limit of the sequence of polynomials $G_n(x)$, where

$$G_1(x) = 1 \quad \text{and} \quad G_n(x) = G_{n-1}(x) - \sum_{k=1}^{n-1} \frac{x^k}{2^{nk-k/2-k^2/2+1} k!} G_{n-k}(x).$$

Write this as

$$G_n(x) = G_{n-1}(x) - \sum_{k=1}^{n-1} b(n, k) x^k G_{n-k}(x), \tag{3.1}$$

where

$$b(n, k) = \frac{1}{2^{nk-k/2-k^2/2+1} k!}$$

for $1 \leq k < n$. Writing $G_n(x) = c_{n,0} + c_{n,1}x + c_{n,2}x^2 + \dots + c_{n,n-1}x^{n-1}$ and comparing the coefficients of x^M on both sides of (3.1) we obtain

$$c_{n,M} = c_{n-1,M} - \sum_{k=1}^M b(n, k) c_{n-k, M-k}. \tag{3.2}$$

Also $c_{n,0} = 1$ because the constant term of the polynomial G_n is 1. Now

$$\begin{aligned} c_{n,1} &= c_{n-1,1} - b(n, 1) c_{n-1,0} = c_{n-1,1} - b(n, 1) \\ &= c_{1,1} - \sum_{k=2}^n b(k, 1) = - \sum_{k=2}^n \frac{1}{2^k} = -\frac{1}{2} \left(1 - \frac{1}{2^{n-1}} \right) \end{aligned}$$

because $c_{1,1} = 0$. Then $b_0 = \lim_{n \rightarrow \infty} c_{n,0} = 1$ and $b_1 = \lim_{n \rightarrow \infty} c_{n,1} = -\frac{1}{2}$. Using (3.2) and letting $n \rightarrow \infty$ we obtain for the coefficients b_M of $G(x)$:

$$b_M = \sum_{n_i, k_i, t} (-1)^t b(n_1, k_1) b(n_2, k_2) b(n_3, k_3) \dots b(n_t, k_t), \tag{3.3}$$

where the sum is over all integers n_i, k_i and t satisfying

$$\left\{ \begin{array}{l} 1 \leq t \leq M, \\ k_1 + \dots + k_t = M, \\ k_i \geq 1, \quad \text{for } i = 1, \dots, t, \\ n_t \geq k_t + 1, \\ n_i \geq n_{i+1} + k_i, \quad \text{for } i = 1, \dots, t-1. \end{array} \right. \tag{3.4}$$

Let

$$h(t, M) = \sum_{n_i, k_i} b(n_1, k_1) b(n_2, k_2) b(n_3, k_3) \dots b(n_t, k_t),$$

where n_i and k_i satisfy (3.4) with a fixed value of t . From equation (3.3)

$$b_M = \sum_{t=1}^M (-1)^t h(t, M). \quad (3.5)$$

Next we prove some formulas.

Claim 3.1. Let $s_m = \sum_{i=1}^m k_i$, $\sigma_m = \sum_{i=1}^m s_i$, and

$$u_m = \sigma_m + \frac{s_m}{2} - \frac{s_m^2}{2} - m.$$

Then

$$\sum_{n_m=n_{m+1}+k_m}^{\infty} \sum_{n_{m-1}=n_m+k_{m-1}}^{\infty} \cdots \sum_{n_2=n_3+k_2}^{\infty} \sum_{n_1=n_2+k_1}^{\infty} \left(\prod_{i=1}^m b(n_i, k_i) \right) = \frac{N_m}{D_m}, \quad (3.6)$$

where

$$N_m = 2^{u_m - n_{m+1}s_m}, \quad \text{and} \quad D_m = \prod_{i=1}^m (2^{s_i} - 1)k_i!.$$

Proof. We use induction on m . When $m = 1$ we must show that

$$\sum_{n=n_2+k_1}^{\infty} b(n, k) = \frac{2^{u_1 - n_2 s_1}}{(2^{s_1} - 1)k_1!} = \frac{2^{(3/2 - n_2)k_1 - k_1^2/2 - 1}}{k_1!(2^{k_1} - 1)}$$

because $s_1 = \sigma_1 = k_1$ and $u_1 = (3/2)k_1 - k_1^2/2 - 1$. But

$$\begin{aligned} \sum_{n=n_2+k_1}^{\infty} b(n, k) &= \sum_{n=n_2+k_1}^{\infty} \frac{1}{2^{n_1 k_1 - k_1^2/2 - k_1/2 + 1} k_1!} \\ &= \frac{2^{k_1^2/2 + k_1/2 - 1}}{k_1!} \sum_{n=n_2+k_1}^{\infty} \frac{1}{2^{n_1 k_1}} \\ &= \frac{2^{k_1^2/2 + k_1/2 - 1}}{k_1!} \frac{1}{2^{k_1(n_2+k_1)}} \frac{2^{k_1}}{2^{k_1} - 1} \\ &= \frac{2^{(3/2 - n_2)k_1 - k_1^2/2 - 1}}{k_1!(2^{k_1} - 1)}, \end{aligned}$$

as required. Now suppose that (3.6) holds for $m - 1$:

$$\sum_{n_{m-1}=n_m+k_{m-1}}^{\infty} \cdots \sum_{n_2=n_3+k_2}^{\infty} \sum_{n_1=n_2+k_1}^{\infty} \left(\prod_{i=1}^{m-1} b(n_i, k_i) \right) = \frac{N_{m-1}}{D_{m-1}}.$$

Then for m we have

$$\sum_{n_m=n_{m+1}+k_m}^{\infty} \sum_{n_{m-1}=n_m+k_{m-1}}^{\infty} \cdots \sum_{n_2=n_3+k_2}^{\infty} \sum_{n_1=n_2+k_1}^{\infty} \left(\prod_{i=1}^m b(n_i, k_i) \right)$$

$$\begin{aligned}
 &= \sum_{n_m=n_{m+1}+k_m}^{\infty} b(n_m, k_m) \left[\sum_{n_{m-1}=n_m+k_{m-1}}^{\infty} \cdots \sum_{n_1=n_2+k_1}^{\infty} \left(\prod_{i=1}^m b(n_i, k_i) \right) \right] \\
 &= \sum_{n_m=n_{m+1}+k_m}^{\infty} b(n_m, k_m) \frac{N_{m-1}}{D_{m-1}} \\
 &= \sum_{n_m=n_{m+1}+k_m}^{\infty} \frac{2^{u_{m-1}-n_m s_{m-1}}}{2^{n_m k_m - k_m^2/2 - k_m/2 + 1} k_m! D_{m-1}} \\
 &= \frac{2^{u_{m-1}} 2^{k_m^2/2 + k_m/2 - 1}}{k_m! D_{m-1}} \sum_{n_m=n_{m+1}+k_m}^{\infty} \frac{1}{2^{n_m(k_m + s_{m-1})}} \\
 &= \frac{2^{u_{m-1} + k_m^2/2 + k_m/2 - 1}}{k_m! D_{m-1}} \sum_{n_m=n_{m+1}+k_m}^{\infty} \frac{1}{2^{n_m s_m}} \\
 &= \frac{2^{\sigma_{m-1} + s_{m-1}/2 - s_{m-1}^2/2 - m + 1 + k_m^2/2 + k_m/2 - 1}}{k_m! D_{m-1}} \frac{1}{2^{(n_{m+1} + k_m) s_m}} \frac{2^{s_m}}{2^{s_m} - 1} \\
 &= \frac{2^{\sigma_{m-1} + s_m/2 - s_{m-1}^2/2 - m + \frac{k_m^2}{2} + s_m - k_m s_m - n_{m+1} s_m}}{(2^{s_m} - 1) k_m! D_{m-1}} \\
 &= \frac{2^{\sigma_m + s_m/2 - m - s_m^2/2 - n_{m+1} s_m}}{D_m} = \frac{2^{u_m - n_{m+1} s_m}}{D_m} = \frac{N_m}{D_m},
 \end{aligned}$$

as required. This completes the induction. \square

Claim 3.2.

$$\begin{aligned}
 &\sum_{n_m=k_m+1}^{\infty} \sum_{n_{m-1}=m_m+k_{m-1}}^{\infty} \cdots \sum_{n_2=m_3+k_2}^{\infty} \sum_{n_1=n_2+k_1}^{\infty} \left(\prod_{i=1}^m b(n_i, k_i) \right) \\
 &= \frac{2^{\sigma_m - s_m/2 - m - s_m^2/2}}{\prod_{i=1}^m (2^{s_i} - 1) k_i!}.
 \end{aligned}$$

Proof.

$$\begin{aligned}
 &\sum_{n_m=k_m+1}^{\infty} \sum_{n_{m-1}=m_m+k_{m-1}}^{\infty} \cdots \sum_{n_2=m_3+k_2}^{\infty} \sum_{n_1=n_2+k_1}^{\infty} \left(\prod_{i=1}^m b(n_i, k_i) \right) \\
 &= \sum_{n_m=k_m+1}^{\infty} b(n_m, k_m) \left[\sum_{n_{m-1}=m_m+k_{m-1}}^{\infty} \cdots \sum_{n_1=n_2+k_1}^{\infty} \left(\prod_{i=1}^m b(n_i, k_i) \right) \right] \\
 &= \sum_{n_m=k_m+1}^{\infty} b(n_m, k_m) \frac{2^{u_{m-1} - n_m s_{m-1}}}{\prod_{i=1}^{m-1} (2^{s_i} - 1) k_i!}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{2^{u_{m-1}}}{\prod_{i=1}^{m-1} (2^{s_i} - 1) k_i!} \sum_{n_m=k_m+1}^{\infty} \frac{2^{-n_m s_{m-1}}}{k_m! 2^{n_m k_m - k_m^2/2 - k_m/2 + 1}} \\
&= \frac{2^{\sigma_{m-1} + s_{m-1}/2 - s_{m-1}^2/2 - (m-1) + k_m^2/2 + k_m/2 - 1}}{k_m! \prod_{i=1}^{m-1} (2^{s_i} - 1) k_i!} \sum_{n_m=k_m+1}^{\infty} \frac{1}{2^{n_m s_m}} \\
&= \frac{2^{\sigma_{m-1} + s_m/2 - m - s_{m-1}^2/2 + k_m^2/2}}{k_m! \prod_{i=1}^{m-1} (2^{s_i} - 1) k_i!} \frac{1}{2^{(k_m+1)s_m}} \frac{2^{s_m}}{2^{s_m} - 1} = \frac{2^{\sigma_{m-1} - s_m/2 - m - s_m^2/2}}{\prod_{i=1}^m (2^{s_i} - 1) k_i!}. \quad \square
\end{aligned}$$

Claim 3.3.

$$h(t, M) = \frac{1}{P(M)} \sum_{k_1=1}^{M-t+1} \sum_{k_2=1}^{M-s_1-t+2} \cdots \sum_{k_{t-1}=1}^{M-s_{t-2}-1} \left[\binom{M}{k_1 \dots k_t} \prod_{i=1}^{t-1} \frac{2^{s_i-1}}{2^{s_i}-1} \right],$$

where $P(M) = M!(2^M - 1)2^{M^2/2 - M/2 + 1}$.

Proof. Recall the definition

$$h(t, M) = \sum_{n_i, k_i} b(n_1, k_1) b(n_2, k_2) b(n_3, k_3) \cdots b(n_t, k_t)$$

where the numbers n_i and k_i satisfy

$$\begin{cases} k_1 + \cdots + k_t = M, \\ k_i \geq 1, \end{cases} \quad \text{for } i = 1, \dots, t, \quad (3.7)$$

and

$$\begin{cases} n_t \geq k_t + 1, \\ n_i \geq n_{i+1} + k_i, \end{cases} \quad \text{for } i = 1, \dots, t-1. \quad (3.8)$$

So

$$h(t, M) = \sum_{k_i} \left[\sum_{n_t=k_t+1}^{\infty} \sum_{n_{m-1}=n_t+k_{t-1}}^{\infty} \cdots \sum_{n_2=n_3+k_2}^{\infty} \sum_{n_1=n_2+k_1}^{\infty} \left(\prod_{i=1}^t b(n_i, k_i) \right) \right].$$

The first sum is over all $\{k_i\}_{i=1}^t$ which satisfy (3.7). From Claim 3.2:

$$\begin{aligned}
h(t, M) &= \sum_{k_i} \left[\frac{2^{\sigma_t - s_t/2 - t - s_t^2/2}}{\prod_{i=1}^t (2^{s_i} - 1) k_i!} \right] = \sum_{k_i} \left[\frac{1}{\prod_{i=1}^t k_i!} \frac{2^{\sigma_{t-1} + s_t/2 - t - s_t^2/2}}{(2^{s_t} - 1) \prod_{i=1}^{t-1} (2^{s_i} - 1)} \right] \\
&= \sum_{k_i} \left[\frac{M!}{\prod_{i=1}^t k_i!} \frac{2^{s_t/2 - 1 - s_t^2/2}}{M!(2^{s_t} - 1)} \frac{2^{\sigma_{t-1} - t + 1}}{\prod_{i=1}^{t-1} (2^{s_i} - 1)} \right]
\end{aligned}$$

We have that $s_t = M$. Then

$$h(t, M) = \sum_{k_i} \left[\binom{M}{k_1 \ k_2 \ \dots \ k_t} \frac{1}{M!(2^M - 1)} \frac{2^{M^2/2 - M/2 + 1}}{(2^{M^2/2 - M/2 + 1})} \frac{\prod_{i=1}^{t-1} 2^{s_i-1}}{\prod_{i=1}^{t-1} (2^{s_i} - 1)} \right]$$

$$= \frac{1}{P(M)} \sum_{k_1=1}^{M-t+1} \sum_{k_2=1}^{M-s_1-t+2} \cdots \sum_{k_{t-1}=1}^{M-s_{t-2}-1} \left[\binom{M}{k_1 \dots k_t} \prod_{i=1}^{t-1} \frac{2^{s_i-1}}{2^{s_i}-1} \right]. \quad \square$$

Denote

$$c_M = M!(2^M - 1)2^{M^2/2-M/2+1}b_M,$$

$$H(t, M) = M!(2^m - 1)2^{M^2/2-M/2+1}h(t, M).$$

Then $c_0 = 1$ and $c_1 = -1$. From Claim 3.3 we have

$$H(t, M) = \sum_{k_1=1}^{M-t+1} \sum_{k_2=1}^{M-s_1-t+2} \cdots \sum_{k_{t-1}=1}^{M-s_{t-2}-1} \left[\binom{M}{k_1 k_2 \dots k_t} \prod_{i=1}^{t-1} \frac{2^{s_i-1}}{2^{s_i}-1} \right],$$

so equation (3.5) becomes

$$c_M = \sum_{i=1}^M (-1)^i H(i, M). \quad (3.9)$$

Claim 3.4. A recursive formula for $H(t, M)$:

$$H(t, M) = \sum_{k=t-1}^{M-1} \binom{M}{k} \frac{2^{k-1}}{2^k-1} H(t-1, k).$$

Proof. Substitute $k_1 = s_1$ and $k_i = s_i - s_{i-1}$ in the above formula for H .

$$H(t, M) = \sum_{s_1=1}^{M-t+1} \sum_{s_2=s_1+1}^{M-t+2} \cdots \sum_{s_{t-1}=s_{t-2}+1}^{M-1} \left[\binom{M}{k_1 k_2 \dots k_t} \prod_{i=1}^{t-1} \frac{2^{s_i-1}}{2^{s_i}-1} \right]$$

$$= \sum_{s_1=1}^{M-t+1} \sum_{s_2=2}^{M-t+2} \sum_{s_3=3}^{M-t+3} \cdots \sum_{s_{t-1}=t-1}^{M-1} \left[\binom{M}{k_1 k_2 \dots k_t} \prod_{i=1}^{t-1} \frac{2^{s_i-1}}{2^{s_i}-1} \right],$$

where the numbers s_i satisfy $1 \leq s_1 < s_2 < s_3 < \cdots < s_{t-1} < M$. Then $H(t, M) =$

$$\sum_{s_1=1}^{M-t+1} \sum_{s_2=2}^{M-t+2} \cdots \sum_{s_{t-2}=t-2}^{M-2} \left(\sum_{s_{t-1}=t-1}^{M-1} \left[\frac{M!}{s_{t-1}!k_t!k_1!k_2!\cdots k_{t-1}!} \prod_{i=1}^{t-1} \frac{2^{s_i-1}}{2^{s_i}-1} \right] \right).$$

We have that $k_t + s_{t-1} = s_t = M$. So $H(t, M) =$

$$\sum_{s_{t-1}=t-1}^{M-1} \binom{M}{s_{t-1}} \frac{2^{s_{t-1}-1}}{2^{s_{t-1}}-1} \left(\sum_{s_1=1}^{M-t+1} \cdots \sum_{s_{t-2}=t-2}^{M-2} \binom{s_{t-1}}{k_1 k_2 \cdots k_{t-1}} \prod_{i=1}^{t-2} \frac{2^{s_i-1}}{2^{s_i}-1} \right),$$

where $1 \leq s_1 < s_2 < s_3 < \dots < s_{t-2} < s_{t-1}$. Hence

$$H(t, M) = \sum_{s_{t-1}=t-1}^{M-1} \binom{M}{s_{t-1}} \frac{2^{s_{t-1}-1}}{2^{s_{t-1}} - 1} H(t-1, s_{t-1}).$$

By substituting $k = s_{t-1}$ we obtain

$$H(t, M) = \sum_{k=t-1}^{M-1} \binom{M}{k} \frac{2^{k-1}}{2^k - 1} H(t-1, k) \quad (3.10)$$

as claimed. \square

Claim 3.5. A recursive formula for c_n :

$$c_n = -1 - \sum_{k=1}^{n-1} \frac{2^{k-1}}{2^k - 1} \binom{n}{k} c_k.$$

Proof. We have that $H(1, n) = P(n)h(1, n) = n!(2^n - 1)2^{n^2/2-n/2+1}h(1, n)$. Then

$$\begin{aligned} H(1, n) &= P(n) \sum_{i=n+1}^{\infty} b(i, n) = n!(2^n - 1)2^{n^2/2-n/2+1} \sum_{i=n+1}^{\infty} \frac{1}{2^{in-n^2/2-n/2+1}n!} \\ &= (2^n - 1)2^{n^2/2} \sum_{i=n+1}^{\infty} \frac{1}{2^{in}} = (2^n - 1)2^{n^2/2} \frac{1}{2^{n(n+1)}} \frac{2^n}{2^n - 1} = 1. \end{aligned}$$

From (3.9)

$$c_n = -1 + \sum_{t=2}^n (-1)^t H(t, n) = -1 + \sum_{t=2}^n \sum_{k=t-1}^{n-1} (-1)^t \binom{n}{k} \frac{2^{k-1}}{2^k - 1} H(t-1, k).$$

By substituting $s = t - 1$ then changing the order of summation we obtain

$$\begin{aligned} c_n &= -1 + \sum_{s=1}^n \sum_{k=s}^{n-1} (-1)^{s-1} \binom{n}{k} \frac{2^{k-1}}{2^k - 1} H(s, k) \\ &= -1 + \sum_{s=1}^n (-1) \binom{n}{k} \frac{2^{k-1}}{2^k - 1} \sum_{k=s}^{n-1} (-1)^s H(s, k) \\ &= -1 - \sum_{k=1}^{n-1} \binom{n}{k} \frac{2^{k-1}}{2^k - 1} \sum_{s=1}^k (-1)^s H(s, k) = -1 - \sum_{k=1}^{n-1} \frac{2^{k-1}}{2^k - 1} \binom{n}{k} c_k. \quad \square \end{aligned}$$

Bernoulli Numbers. The sequence c_n satisfies:

$$c_0 = 1; \quad c_1 = -1; \quad c_n = -1 - \sum_{k=1}^{n-1} \frac{2^{k-1}}{2^k - 1} \binom{n}{k} c_k \quad (3.11)$$

The first few terms:

$$c_2 = 1, \quad c_3 = 0, \quad c_4 = -1, \quad c_5 = 0, \quad c_6 = 3, \quad c_7 = 0, \quad \dots$$

Recall the Bernoulli numbers B_n satisfy $B_0 = 1$,

$$\sum_{k=0}^n \binom{n}{k} B_k = B_n, \tag{3.12}$$

$$\sum_{k=0}^n \binom{n}{k} (2^k - 1) B_k = -(2^n - 1) B_n, \quad n \geq 2. \tag{3.13}$$

Proposition 3.6. *Let B_n be the Bernoulli numbers. Then $c_0 = 1$,*

$$c_n = 2(2^n - 1) B_n, \quad n \geq 1 \tag{3.14}$$

is the sequence that satisfies (3.11).

Proof. First note that $c_1 = -1$ in (3.14). In (3.12), cancel B_n and use $B_0 = 1$ to get

$$\sum_{k=1}^{n-1} \binom{n}{k} B_k = -1$$

for $n \geq 2$. Then substitute (3.14) to get

$$\sum_{k=1}^{n-1} \frac{1}{2^k - 1} \binom{n}{k} c_k = -2.$$

Similarly from (3.13) we get

$$\sum_{k=1}^{n-1} \binom{n}{k} (2^k - 1) B_k = -2(2^n - 1) B_n, \quad \sum_{k=1}^{n-1} \binom{n}{k} c_k = -2c_n.$$

Therefore

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{2^{k-1}}{2^k - 1} \binom{n}{k} c_k &= \sum_{k=1}^{n-1} \frac{1}{2} \left(1 + \frac{1}{2^k - 1} \right) \binom{n}{k} c_k \\ &= \frac{1}{2} \sum_{k=1}^{n-1} \binom{n}{k} c_k + \frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{2^k - 1} \binom{n}{k} c_k = -c_n - 1. \end{aligned}$$

Therefore c_n defined by (3.14) is the solution to (3.11). □

Thus we see our problem is related to the Bernoulli numbers in this way:

$$\begin{aligned} c_n &= 2(2^n - 1) B_n, \\ b_n &= \frac{c_n}{n!(2^n - 1)2^{n^2/2-n/2+1}} = \frac{B_n}{n!2^{n^2/2-n/2}}, \end{aligned}$$

$$G(x) = \sum_{m=0}^{\infty} b_m x^m = \sum_{m=0}^{\infty} \frac{B_m x^m}{2^{(m^2-m)/2} m!}.$$

4. Asymptotic Estimates for the Zeros of G

In Section 3 we determined the coefficients of the power series G :

$$G(x) = \sum_{M=0}^{\infty} \frac{B_M x^M}{2^{M^2/2-M/2} M!}.$$

The Bernoulli numbers B_M satisfy: $B_0 = 1, B_1 = -\frac{1}{2}$ and

$$B_{2n+1} = 0, \quad \text{for } n \geq 1,$$

$$B_{2n} = \frac{2(2n)!}{\pi^{2n} 2^{2n}} (-1)^n \zeta(2n),$$

where $\zeta(2n) = \sum_{k=1}^{\infty} 1/k^{2n}$ is the Riemann zeta function. Then

$$\begin{aligned} G(x) &= 1 - \frac{x}{2} + \sum_{n=1}^{\infty} \frac{B_{2n} x^{2n}}{2^{2n^2-n} (2n)!} = 1 - \frac{x}{2} + \sum_{n=1}^{\infty} \frac{2(2n)! (-1)^n \zeta(2n) x^{2n}}{\pi^{2n} 2^{2n} 2^{2n^2-n} (2n)!} \\ &= 1 - \frac{x}{2} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n \zeta(2n) x^{2n}}{\pi^{2n} 2^{2n^2+n}} = 1 - \frac{x}{2} + 2 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{4^{n^2} 2^n \pi^{2n} k^{2n}}. \end{aligned}$$

Define f, g and F by

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{4^{n^2}}, \\ g(x) &= \sum_{k=1}^{\infty} f\left(\frac{x}{k^2}\right), \\ F(x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{x^n 4^{n^2}}. \end{aligned} \tag{4.1}$$

Then writing $y = y(x) = x^2/(2\pi^2)$ we get

$$\begin{aligned} G(x) &= 1 - \frac{x}{2} + 2 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^n}{4^{n^2}} \left(\frac{y}{k^2}\right)^n \\ &= 1 - \frac{x}{2} - 2 \sum_{k=1}^{\infty} f\left(\frac{y}{k^2}\right) = 1 - \frac{x}{2} - 2g(y). \end{aligned}$$

We have $y > 0$ and $x = \pm\pi\sqrt{2y}$. Define the functions G_1 and G_2 by

$$G_i(y) = 1 + (-1)^i \pi \sqrt{\frac{y}{2}} - 2g(y) \tag{4.2}$$

for $i = 1, 2$, so that $G(x) = G_1(y(x))$ when $x > 0$ and $G(x) = G_2(y(x))$ when $x < 0$.

Equation (*) has a solution if $a = a_n$, where $G(a_n) = 0$. We number them so that $a_n > 0$ for $n > 0$ and $a_n < 0$ for $n < 0$ (except a_{-1}). Then

$$\begin{aligned} G_1\left(\frac{a_n^2}{2\pi^2}\right) &= 0 && \text{if } n > 0 \text{ and} \\ G_2\left(\frac{a_n^2}{2\pi^2}\right) &= 0 && \text{if } n < -1. \end{aligned}$$

Technical Estimates.

Claim 4.1. *The functions $f(x)$ and $F(x)$ satisfy:*

- (i) $0 < f(x) < \frac{x}{4}$ for $0 < x < 1$.
- (ii) $0 < F(x) < \frac{1}{4x}$ for $x > 1$.

Proof. (i) First,

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{4n^2} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{4(2n-1)^2} \left(1 - \frac{x}{4^{4n-1}}\right).$$

Note $1 - x/4^{4n-1} > 0$ and so $f(x) > 0$. Next

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{4n^2} = \frac{x}{4} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{x^n}{4n^2} = \frac{x}{4} - \sum_{n=1}^{\infty} \frac{x^{2n}}{4^{4n^2}} \left(1 - \frac{x}{4^{4n+1}}\right).$$

Then $f(x) < x/4$ because $1 - x/4^{4n+1} > 0$.

(ii) follows from (i) since $F(x) = f(1/x)$. □

Claim 4.2. *The function $f(x)$ satisfies the functional equations:*

- (i) $f(16x) = 4x - 4xf(x)$.
- (ii) $f'(16x) = \frac{1}{4} - \frac{f(x)}{4} - \frac{xf'(x)}{4}$.

Proof. (i) $f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} x^n / 4n^2$ so

$$f(16x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{16^n x^n}{4n^2} = x \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n-1}}{4n^2 - 2n}$$

$$\begin{aligned}
&= 4x - 4x \sum_{n=2}^{\infty} (-1)^{n-2} \frac{x^{n-1}}{4^{n^2-2n+1}} = 4x - 4x \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{4^{n^2}} \\
&= 4x - 4xf(x).
\end{aligned}$$

(ii) By differentiating (i) we obtain: $16f'(16x) = 4 - 4f(x) - 4xf'(x)$. \square

Define intervals I_n and J_n :

$$I_n = [4 \cdot 16^n - 8^n, 4 \cdot 16^n + 8^n], \quad J_n = [4 \cdot 16^n + 8^n, 4 \cdot 16^{n+1} - 8^{n+1}].$$

Claim 4.3. (i) $1 - x > e^{-2x}$ for $0 < x < 1/3$.

(ii) $\frac{x^s}{4^{s^2}} < \frac{x^t}{4^{t^2}}$ for $x \in J_n$ and $1 \leq s < t \leq n+1$.

Proof. (i) Let $h(x) = 1 - x - e^{-2x}$. Then $h'(x) = 2e^{-2x} - 1 = 0$ for $x = \frac{\ln 2}{2}$. The function $h(x)$ is increasing because $\frac{\ln 2}{2} > \frac{1}{3}$ so $h(x) > h(0) = 0$.

(ii) $x^{t-s} > (4 \cdot 16^n)^{t-s} = (4^{2n+1})^{t-s} \geq 4^{(t+s)(t-s)} = 4^{t^2-s^2}$. \square

Theta Function. We will use a theta function and the corresponding partial theta function with the notation:

$$\begin{aligned}
\tau(p, q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} p^n, \\
\rho(p, q) &= \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2} p^n.
\end{aligned}$$

The theta function satisfies the Jacobi triple product identity [8, Theorem 352]:

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} p^n = (q)_{\infty} (p)_{\infty} \left(\frac{q}{p} \right)_{\infty},$$

where

$$(q)_{\infty} = \prod_{n=1}^{\infty} (1 - q^n), \quad (p)_{\infty} = \prod_{n=1}^{\infty} (1 - pq^{n-1}), \quad \text{and} \quad \left(\frac{q}{p} \right)_{\infty} = \prod_{n=1}^{\infty} \left(1 - \frac{q^n}{p} \right).$$

When $q = 1/16$ and $p = x/4$ the theta function becomes

$$\tau\left(\frac{x}{4}, \frac{1}{16}\right) = \sum_{n=-\infty}^{\infty} (-1)^n \frac{x^n}{16^{n(n-1)/2} 4^n} = \sum_{n=-\infty}^{\infty} (-1)^n \frac{x^n}{4^{n^2}}$$

and

$$(q)_{\infty} = \prod_{n=1}^{\infty} (1 - q^n) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{16^n} \right),$$

$$(p)_\infty = \prod_{n=1}^{\infty} (1 - pq^{n-1}) = \prod_{n=1}^{\infty} \left(1 - \frac{x}{4 \cdot 16^{n-1}}\right),$$

$$\left(\frac{q}{p}\right)_\infty = \prod_{n=1}^{\infty} \left(1 - \frac{q^n}{p}\right) = \prod_{n=1}^{\infty} \left(1 - \frac{4}{x \cdot 16^{n-1}}\right).$$

So the Jacobi triple product identity for $\tau(x/4, 1/16)$ yields

$$\sum_{n=-\infty}^{\infty} (-1)^n \frac{x^n}{4n^2} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{16^n}\right) \prod_{n=1}^{\infty} \left(1 - \frac{x}{4 \cdot 16^{n-1}}\right) \prod_{n=1}^{\infty} \left(1 - \frac{4}{x \cdot 16^{n-1}}\right).$$

Now we find a convenient expression for f . Note that $f(x) = \rho(x/4, 1/16) - 1$ where ρ is the partial theta function. Using our previous notation,

$$\sum_{n=-\infty}^{\infty} (-1)^n \frac{x^n}{4n^2} = \sum_{n=-\infty}^{-1} (-1)^n \frac{x^n}{4n^2} + 1 + \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{4n^2} = F(x) + 1 + f(x),$$

$$f(x) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{16^n}\right) \prod_{n=1}^{\infty} \left(1 - \frac{x}{4 \cdot 16^{n-1}}\right) \prod_{n=1}^{\infty} \left(1 - \frac{4}{x \cdot 16^{n-1}}\right) - 1 - F(x).$$

From the triangle inequality we have the following estimates for $|f|$.

$$|f(x)| < \prod_{n=1}^{\infty} \left(1 - \frac{1}{16^n}\right) \prod_{n=1}^{\infty} \left(1 - \frac{x}{4 \cdot 16^{n-1}}\right) \prod_{n=1}^{\infty} \left(1 - \frac{4}{x \cdot 16^{n-1}}\right) + |F(x)| + 1,$$

$$|f(x)| > \prod_{n=1}^{\infty} \left(1 - \frac{1}{16^n}\right) \prod_{n=1}^{\infty} \left(1 - \frac{x}{4 \cdot 16^{n-1}}\right) \prod_{n=1}^{\infty} \left(1 - \frac{4}{x \cdot 16^{n-1}}\right) - |F(x)| - 1.$$

From Claim 4.1: $|F(x)| < 1/(4x) < 1$ for $x > 1$. Then

$$|f(x)| < \prod_{n=1}^{\infty} \left(1 - \frac{1}{16^n}\right) \prod_{n=1}^{\infty} \left(1 - \frac{x}{4 \cdot 16^{n-1}}\right) \prod_{n=1}^{\infty} \left(1 - \frac{4}{x \cdot 16^{n-1}}\right) + 2, \quad (4.3)$$

$$|f(x)| > \prod_{n=1}^{\infty} \left(1 - \frac{1}{16^n}\right) \prod_{n=1}^{\infty} \left(1 - \frac{x}{4 \cdot 16^{n-1}}\right) \prod_{n=1}^{\infty} \left(1 - \frac{4}{x \cdot 16^{n-1}}\right) - 2. \quad (4.4)$$

We will also use the following inequalities

$$|f(16x)| \geq 4|x|(|f(x)| - 1), \quad |f'(16x)| \geq \frac{1}{4}|x| |f'(x)| - \frac{1}{4}|f(x)| - \frac{1}{4}, \quad (4.5)$$

$$|g(x)| \geq |f(x)| - \sum_{k=2}^{\infty} \left|f\left(\frac{x}{k^2}\right)\right|, \quad |g'(x)| \geq |f'(x)| - \sum_{k=2}^{\infty} \frac{1}{k^2} \left|f'\left(\frac{x}{k^2}\right)\right|, \quad (4.6)$$

$$|G_i(x)| \geq 2|g(x)| - \pi\sqrt{\frac{x}{2}} - 1, \quad |G'_i(x)| \geq 2|g'(x)| - \frac{\pi}{2\sqrt{2x}}. \quad (4.7)$$

Inequalities (4.5) follow from Claim 4.2 and the triangle inequality and (4.6) and (4.7) follow from the definitions (4.1), (4.2) of g and G_i . The main result of Section 4 is the following theorem.

Theorem 4.4. *The functions $G_i(x)$ have a unique zero in each interval I_n and have no zero on the intervals J_n , for $i = 1, 2$ and n sufficiently large.*

We prove Theorem 4.4 in Sections 4.1 and 4.2.

4.1. The Intervals J_n

In Section 4.1 we show that the functions $G_1(x)$ and $G_2(x)$ have no zeros on the intervals J_n , for sufficiently large values of n .

Lemma 4.5. *$|f(x)| > \frac{1}{100} \frac{x^n}{2^n 4^{n^2}}$ for $x \in J_n$ and $n \geq 2$.*

Proof. First,

$$|f(x)| > \prod_{n=1}^{\infty} \left| 1 - \frac{1}{16^n} \right| \prod_{n=1}^{\infty} \left| 1 - \frac{x}{4 \cdot 16^{n-1}} \right| \prod_{n=1}^{\infty} \left| 1 - \frac{4}{x \cdot 16^{n-1}} \right| - 2.$$

From Claim 4.3(i) we have $1 - 1/16^n > e^{-2/16^n}$ and $1 - 4/(x \cdot 16^n) > e^{-8/(x \cdot 16^n)}$. So

$$\prod_{n=1}^{\infty} \left(1 - \frac{1}{16^n} \right) > \prod_{n=1}^{\infty} e^{-2/16^n} = \exp \left(-\frac{2}{16} \sum_{n=0}^{\infty} \frac{1}{16^n} \right) = e^{-2/15}, \quad (4.8)$$

$$\prod_{n=1}^{\infty} \left(1 - \frac{4}{x \cdot 16^n} \right) > \prod_{n=1}^{\infty} e^{-8/(x \cdot 16^n)} = \exp \left(-\frac{8}{16x} \sum_{n=0}^{\infty} \frac{1}{16^n} \right) = e^{-8/(15x)}. \quad (4.9)$$

Now we estimate

$$\prod_{k=0}^{\infty} \left| 1 - \frac{x}{4 \cdot 16^k} \right| = \left[\prod_{k=0}^{n-1} \left| 1 - \frac{x}{4 \cdot 16^k} \right| \right] |h(x)| \left[\prod_{k=n+2}^{\infty} \left| 1 - \frac{x}{4 \cdot 16^k} \right| \right],$$

where

$$h(x) = \left(1 - \frac{x}{4 \cdot 16^n} \right) \left(1 - \frac{x}{4 \cdot 16^{n+1}} \right).$$

The function $|h(x)|$ has minimum on the interval J_n at $x = 4 \cdot 16^n + 8^n$:

$$h(4 \cdot 16^n + 8^n) = \left(1 - \frac{4 \cdot 16^n + 8^n}{4 \cdot 16^n} \right) \left(1 - \frac{4 \cdot 16^n + 8^n}{4 \cdot 16^{n+1}} \right) = \frac{7}{32 \cdot 2^n}.$$

Hence $|h(x)| \geq 7/(32 \cdot 2^n)$ for $x \in J_n$

$$\prod_{k=0}^{\infty} \left| 1 - \frac{x}{4 \cdot 16^k} \right| = \left[\prod_{k=0}^{n-1} \frac{x}{4 \cdot 16^k} \prod_{k=0}^{n-1} \left| \frac{4 \cdot 16^k}{x} - 1 \right| \right] |h(x)| \prod_{k=n+2}^{\infty} \left| 1 - \frac{x}{4 \cdot 16^k} \right|$$

and

$$\prod_{k=0}^{n-1} \frac{x}{4 \cdot 16^k} = \frac{x^n}{4^n 16^{(n-1)n/2}} = \frac{x^n}{4^n 4^{n(n-1)}} = \frac{x^n}{4^{n^2}}.$$

Then

$$\prod_{k=0}^{\infty} \left| 1 - \frac{x}{4 \cdot 16^k} \right| > \frac{x^n}{4^{n^2}} \frac{7}{32 \cdot 2^n} \prod_{k=0}^{n-1} \left| 1 - \frac{4 \cdot 16^k}{x} \right| \prod_{k=n+2}^{\infty} \left| 1 - \frac{x}{4 \cdot 16^k} \right|. \quad (4.10)$$

We have

$$\frac{4 \cdot 16^k}{x} < \frac{4 \cdot 16^{n-1}}{4 \cdot 16^n + 8^n} < \frac{1}{16} < \frac{1}{3}$$

for $0 \leq k \leq n - 1$, and

$$\frac{x}{4 \cdot 16^k} < \frac{4 \cdot 16^{n+1} - 8^{n+1}}{4 \cdot 16^{n+2}} < \frac{1}{16} < \frac{1}{3}$$

for $k \geq n + 2$. From Claim 4.3(i) and (4.10)

$$\begin{aligned} \prod_{k=0}^{\infty} \left| 1 - \frac{x}{4 \cdot 16^k} \right| &> \frac{7}{32} \frac{x^n}{2^n 4^{n^2}} \prod_{k=0}^{n-1} e^{-8 \cdot 16^k/x} \prod_{k=n+2}^{\infty} e^{-2x/(4 \cdot 16^k)}, \\ \prod_{k=0}^{\infty} \left| 1 - \frac{x}{4 \cdot 16^k} \right| &> \frac{7}{32} \frac{x^n}{2^n 4^{n^2}} \exp \left(-\frac{8(16^n - 1)}{15x} - \frac{x}{30 \cdot 16^{n+1}} \right). \end{aligned}$$

Next: for $x \in J_n$,

$$\begin{aligned} \frac{8(16^n - 1)}{15x} + \frac{x}{30 \cdot 16^{n+1}} &< \frac{8}{15} \frac{16^n - 1}{4 \cdot 16^n + 8^n} + \frac{1}{30} \frac{4 \cdot 16^{n+1} - 8^{n+1}}{16^{n+1}} \\ &< \frac{8}{15} \frac{16^n}{4 \cdot 16^n} + \frac{1}{30} \frac{4 \cdot 16^{n+1}}{16^{n+1}} < \frac{8}{15} \frac{1}{4} + \frac{4}{30} = \frac{4}{15}. \end{aligned}$$

Then

$$\prod_{k=0}^{\infty} \left| 1 - \frac{x}{4 \cdot 16^k} \right| > \frac{7e^{-4/15}}{32} \frac{x^n}{2^n 4^{n^2}}. \quad (4.11)$$

From (4.8), (4.9), and (4.11):

$$\begin{aligned} |f(x)| &> \prod_{n=1}^{\infty} \left| 1 - \frac{1}{16^n} \right| \prod_{n=1}^{\infty} \left| 1 - \frac{x}{4 \cdot 16^{n-1}} \right| \prod_{n=1}^{\infty} \left| 1 - \frac{4}{x \cdot 16^{n-1}} \right| - 2 \\ &> \frac{7e^{-4/15}}{32} \frac{x^n}{2^n 4^{n^2}} e^{-2/15} e^{-8/(15x)} - 2 > \frac{7e^{-14/15}}{32} \frac{x^n}{2^n 4^{n^2}} - 2. \end{aligned}$$

So $|f(x)| - \frac{1}{100} \frac{x^n}{2^n 4^{n^2}} > \left(\frac{7e^{-14/15}}{32} - \frac{1}{100} \right) \frac{x^n}{2^n 4^{n^2}} - 2 > \frac{19}{250} 2^2 4^4 - 2 > 0. \quad \square$

Lemma 4.6. $|f(x)| < \frac{4x^n}{4^{n^2}}$ for $x \in [4 \cdot 16^n, 4 \cdot 16^{n+1}]$ and $n \geq 1$.

Proof. Since

$$\prod_{n=1}^{\infty} \left(1 - \frac{1}{16^n}\right) < 1 \quad \text{and} \quad \prod_{n=1}^{\infty} \left(1 - \frac{4}{x \cdot 16^n}\right) < 1,$$

we have

$$\prod_{k=0}^{\infty} \left|1 - \frac{x}{4 \cdot 16^k}\right| = \left[\prod_{k=0}^{n-1} \left|1 - \frac{x}{4 \cdot 16^k}\right| \right] |h(x)| \prod_{k=n+2}^{\infty} \left|1 - \frac{x}{4 \cdot 16^k}\right|,$$

where

$$h(x) = \left(1 - \frac{x}{4 \cdot 16^n}\right) \left(1 - \frac{x}{4 \cdot 16^{n+1}}\right).$$

The maximum of $|h(x)|$ on J_n is at $x = 34 \cdot 16^n$:

$$|h(x)| \leq \frac{1}{16^{2n+1}} (30 \cdot 16^n)^2 = \frac{900 \cdot 16^{2n}}{16^{2n+2}} = \frac{225}{64}.$$

We have $\prod_{k=n+2}^{\infty} (1 - x/(4 \cdot 16^k)) < 1$ because $x < 4 \cdot 16^{n+1}$. So

$$\left| \prod_{k=0}^{n-1} \left(1 - \frac{x}{4 \cdot 16^k}\right) \right| < \prod_{k=0}^{n-1} \frac{x}{4 \cdot 16^k} = \frac{x^n}{4^n 16^{n(n-1)/2}} = \frac{x^n}{4^{n^2}}.$$

Therefore

$$\prod_{k=0}^{\infty} \left|1 - \frac{x}{4 \cdot 16^k}\right| < \left[\prod_{k=0}^{n-1} \left|1 - \frac{x}{4 \cdot 16^k}\right| \right] |h(x)| < \frac{225}{64} \frac{x^n}{4^{n^2}}$$

and

$$\begin{aligned} |f(x)| &< \prod_{n=1}^{\infty} \left(1 - \frac{1}{16^n}\right) \prod_{n=1}^{\infty} \left(1 - \frac{x}{4 \cdot 16^{n-1}}\right) \prod_{n=1}^{\infty} \left(1 - \frac{4}{x \cdot 16^{n-1}}\right) + 2 \\ &< \frac{225}{64} \frac{x^n}{4^{n^2}} + 2 \end{aligned}$$

Then $\frac{4x^n}{4^{n^2}} - |f(x)| > \frac{31}{64} \frac{x^n}{4^{n^2}} - 2 > \frac{31}{64} \frac{(4 \cdot 16^n)^n}{4^{n^2}} - 2 > \frac{31 \cdot 4^2}{64} - 2 = 5 > 0$. \square

Claim 4.7. *The following inequalities hold for n sufficiently large.*

$$(i) \left|f\left(\frac{x}{k^2}\right)\right| \leq \frac{4x^n}{k^{2m} 4^{n^2}} \quad \text{for } 4^{n-m-1} < k \leq 4^{n-m}.$$

$$(ii) \sum_{k=4^{n-m-1}+1}^{4^{n-m}} \left|f\left(\frac{x}{k^2}\right)\right| < \frac{192x^n}{4^n 4^{n^2}} \quad \text{for } 1 \leq m \leq n-2.$$

$$(iii) \sum_{k=5}^{4^{n-1}} \left|f\left(\frac{x}{k^2}\right)\right| < \frac{192(n-2)x^n}{4^n 4^{n^2}}.$$

Proof. (i) Let $4 \cdot 16^l \leq x/k^2 \leq 4 \cdot 16^{l+1}$. We have that

$$\frac{x}{k^2} \geq \frac{4 \cdot 16^n + 8^n}{(4^{n-m})^2} \geq \frac{4 \cdot 16^n}{16^{n-m}} = 4 \cdot 16^m.$$

Then $l \geq m$. From Lemma 4.6

$$\left| f\left(\frac{x}{k^2}\right) \right| \leq \frac{4 \cdot x^l}{k^{2l} 4^{l^2}}.$$

But $x^l/4^{l^2} < x^n/4^{n^2}$ and $1/k^{2l} < 1/k^{2m}$, so

$$\left| f\left(\frac{x}{k^2}\right) \right| \leq \frac{4 \cdot x^n}{k^{2m} 4^{n^2}}.$$

$$\begin{aligned} \text{(ii)} \quad \sum_{k=4^{n-m-1}+1}^{4^{n-m}} \left| f\left(\frac{x}{k^2}\right) \right| &\leq \sum_{k=4^{n-m-1}+1}^{4^{n-m}} \frac{4x^n}{4^{n^2} k^{2m}} \\ &< \frac{4x^n}{4^{n^2}} \sum_{k=4^{n-m-1}+1}^{4^{n-m}} \frac{1}{4^{2m(n-m-1)}} \leq \frac{4x^n}{4^{n^2}} \frac{4^{n-m} - 4^{n-m-1}}{4^{2m(n-m-1)}} \\ &= \frac{4x^n}{4^{n^2}} \frac{3 \cdot 4^{n-m-1}}{4^{2m(n-m-1)}} = \frac{12x^n}{4^{n^2}} \frac{1}{4^{(2m-1)(n-m-1)}}. \end{aligned}$$

The minimum value of $h(m) = (2m - 1)(n - m - 1)$ is $h(1) = n - 2$. So

$$\sum_{k=4^{n-m-1}+1}^{4^{n-m}} \left| f\left(\frac{x}{k^2}\right) \right| < \frac{12x^n}{4^{n^2}} \frac{1}{4^{n-2}} = \frac{192x^n}{4^n 4^{n^2}}.$$

$$\text{(iii)} \quad \sum_{k=5}^{4^{n-1}} \left| f\left(\frac{x}{k^2}\right) \right| = \sum_{m=1}^{n-2} \sum_{k=4^{n-m-1}+1}^{4^{n-m}} \left| f\left(\frac{x}{k^2}\right) \right| < \frac{192(n-2)x^n}{4^n 4^{n^2}}. \quad \square$$

Write $\hat{f} = \max\{8|f(x)| : 0 \leq x \leq 1024\}$ and $\hat{f}_2 = \max\{|f'(x)| : 0 \leq x \leq 2\}$.

Claim 4.8. *The following inequalities hold for n sufficiently large.*

$$\text{(i)} \quad \sum_{k=2}^4 \left| f\left(\frac{x}{k^2}\right) \right| \leq \frac{48x^n}{4^n 4^{n^2}}.$$

$$\text{(ii)} \quad \sum_{k=4^{n-1}+1}^{8 \cdot 4^{n-1}} \left| f\left(\frac{x}{k^2}\right) \right| < 4^n \hat{f}.$$

$$\text{(iii)} \quad \sum_{k=8 \cdot 4^n}^{\infty} \left| f\left(\frac{x}{k^2}\right) \right| \leq x.$$

Proof. (i) We have $x/k^2 \geq x/16 \geq (4 \cdot 16^n + 8^n)/16 \geq 4 \cdot 16^{n-1}$ and

$$\frac{x}{k^2} \leq \frac{x}{4} \leq \frac{4 \cdot 16^{n+1} - 8^{n+1}}{4} \leq 16^{n+1} < 4 \cdot 16^{n+1}.$$

Therefore x/k^2 satisfies the conditions of Lemma 4.6 with $n-1$ or n and so

$$f\left(\frac{x}{k^2}\right) < \frac{4x^{n-1}}{k^{2(n-1)}4^{(n-1)^2}} \quad \text{or} \quad f\left(\frac{x}{k^2}\right) < \frac{4x^n}{k^{2n}4^{n^2}}.$$

In both cases

$$f\left(\frac{x}{k^2}\right) < \frac{4x^n}{k^{2(n-1)}4^{n^2}}.$$

because $1/k^{2n} < 1/k^{n-1}$ and $x^{n-1}/4^{(n-1)^2} < x^n/4^{n^2}$ for $x \in J_n$. Hence

$$\sum_{k=2}^4 \left| f\left(\frac{x}{k^2}\right) \right| \leq \sum_{k=2}^4 \frac{4 \cdot x^n}{k^{2(n-1)}4^{n^2}} \leq \frac{12 \cdot x^n}{2^{2(n-1)}4^{n^2}} = \frac{48x^n}{4^n 4^{n^2}}.$$

(ii) We have $0 < x/k^2 < 4 \cdot 16^{n+1}/16^{n-1} = 1024$. So

$$\sum_{k=4^{n-1}+1}^{8 \cdot 4^{n-1}} \left| f\left(\frac{x}{k^2}\right) \right| \leq (8 \cdot 4^n - 4^{n-1} - 1) \frac{\widehat{f}}{8} < \frac{8 \cdot 4^n \widehat{f}}{8} = 4^n \widehat{f}$$

(iii) We have $0 < x/k^2 < (4 \cdot 16^{n+1} - 8^{n+1})/(64 \cdot 16^n) < 4 \cdot 16^{n+1}/(64 \cdot 16^n) = 1$, so by Claim 4.1(i) we have $|f(x/k^2)| \leq x/(4k^2)$. Therefore

$$\sum_{k=8 \cdot 4^n}^{\infty} \left| f\left(\frac{x}{k^2}\right) \right| \leq \sum_{k=8 \cdot 4^n}^{\infty} \frac{x}{4k^2} < \frac{x}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2 x}{4 \cdot 6} < x. \quad \square$$

Lemma 4.9. $|g(x)| > \frac{1}{200} \frac{x^n}{2^n 4^{n^2}}$ for $x \in J_n$ and n sufficiently large.

Proof. By Lemma 4.5 we have $|f(x)| > (1/100)x^n/(2^n 4^{n^2})$. By Claims 4.7(iii) and 4.8

$$\sum_{k=2}^{\infty} \left| f\left(\frac{x}{k^2}\right) \right| < \frac{192(n-2)x^n}{4^n 4^{n^2}} + \frac{48x^n}{4^n 4^{n^2}} + 4^n \widehat{f} + x.$$

Then

$$\begin{aligned} |g(x)| &\geq |f(x)| - \sum_{k=2}^{\infty} \left| f\left(\frac{x}{k^2}\right) \right| \\ &> \frac{1}{100} \frac{x^n}{2^n 4^{n^2}} - \frac{192(n-2)x^n}{4^n 4^{n^2}} - \frac{48x^n}{4^n 4^{n^2}} - 4^n \widehat{f} - x. \end{aligned}$$

Therefore for large enough n we have $|g(x)| > \frac{1}{200} \frac{x^n}{2^n 4^{n^2}}$ and therefore

$$\frac{1}{200} \frac{x^n}{2^n 4^{n^2}} - \frac{48x^n}{4^n 4^{n^2}} - \frac{192(n-2)x^n}{4^n 4^{n^2}} - 4^n \widehat{f} - x > 0. \quad \square$$

Corollary 4.10. $|G_i(x)| > \frac{1}{200} \frac{x^n}{2^n 4^{n^2}}$ for $y \in J_n$ and $i = 1, 2$.

Proof. $|G_i(x)| \geq 2|g(x)| - \pi\sqrt{x/2} - 1 > (1/100)x^n/(2^n 4^{n^2}) - \pi\sqrt{x/2} - 1$.

Therefore $|G_i(x)| > (1/200)x^n/(2^n 4^{n^2})$ for $i = 1, 2$ and n sufficiently large. \square

Corollary 4.11. *For sufficiently large n , the functions $G_1(x)$ and $G_2(x)$ have no zeros in the interval J_n .*

Proof. If $x \in J_n$ we have

$$|G_i(x)| > \frac{1}{200} \frac{x^n}{2^n 4^{n^2}} > \frac{1}{200} \frac{(4 \cdot 16^n)^n}{2^n 4^{n^2}} = \frac{1}{200} 2^n 4^{n^2}.$$

Therefore $G_i(x) \neq 0$. \square

4.2. The Intervals I_n

In this section we show that the functions G_1 and G_2 have unique zeros on the intervals I_n . Let $\text{sgn}[x] = 1$ if $x > 0$ and $\text{sgn}[x] = -1$ if $x < 0$.

Lemma 4.12. $\text{sgn}[G_i(4 \cdot 16^n - 8^n)] = (-1)^n$ and $\text{sgn}[G_i(4 \cdot 16^n + 8^n)] = (-1)^{n+1}$ for $i = 1, 2$ and n sufficiently large.

Proof. The numbers $4 \cdot 16^n - 8^n$ and $4 \cdot 16^n + 8^n$ are endpoints of the intervals I_n and J_n . By the proofs of Lemma 4.9 and Corollary 4.10, the signs of $g(x)$ and $G_i(x)$ are the same as the sign of $f(x)$ for sufficiently large values of n . By Lemma 4.5,

$$\text{sgn}[f(x)] = \text{sgn} \left[\prod_{n=1}^{\infty} \left(1 - \frac{1}{16^n} \right) \prod_{n=1}^{\infty} \left(1 - \frac{x}{4 \cdot 16^{n-1}} \right) \prod_{n=1}^{\infty} \left(1 - \frac{4}{x \cdot 16^{n-1}} \right) \right].$$

But

$$\prod_{n=1}^{\infty} \left(1 - \frac{1}{16^n} \right) > 0, \quad \prod_{n=1}^{\infty} \left(1 - \frac{4}{x \cdot 16^n} \right) > 0,$$

$$1 - \frac{x}{4 \cdot 16^{k-1}} > 0 \quad \text{for } k \geq n + 2, \text{ and } 1 - \frac{x}{4 \cdot 16^{k-1}} < 0 \quad \text{for } 1 \leq k \leq n.$$

So

$$\begin{aligned} \text{sgn}[f(x)] &= \text{sgn} \left[\prod_{k=1}^n \left(1 - \frac{x}{4 \cdot 16^{k-1}} \right) \right] \text{sgn} \left[1 - \frac{x}{4 \cdot 16^n} \right] \\ &= (-1)^n \text{sgn} \left[1 - \frac{x}{4 \cdot 16^n} \right] \end{aligned}$$

for $x \in J_n$. Hence

$$\text{sgn}[f(4 \cdot 16^n + 8^n)] = (-1)^n \text{sgn} \left[1 - \frac{4 \cdot 16^n + 8^n}{4 \cdot 16^n} \right] = (-1)^{n+1},$$

$$\text{sgn}[f(4 \cdot 16^n - 8^n)] = (-1)^n \text{sgn} \left[1 - \frac{4 \cdot 16^n - 8^n}{4 \cdot 16^n} \right] = (-1)^n. \quad \square$$

Lemma 4.12 establishes the existence of zeros of the functions G_1 and G_2 on the intervals I_n . It remains to show that these zeros are unique.

Claim 4.13. $|f(x)| < 2 \cdot 4^{n^2}$ for $x \in [1, 16^n + 8^n]$ and n sufficiently large.

Proof. Let n be large enough such that $2 \cdot 4^{n^2} > \max\{f(x) \mid 1 \leq x \leq 16^3 + 8^3\}$. Let $16^{k-1} + 8^{k-1} \leq x \leq 16^k + 8^k$ where $4 \leq k \leq n$. Then

$$4 \cdot 16^{k-2} < 16^{k-1} + 8^{k-1} < x < 16^k + 8^k < 4 \cdot 16^k$$

The conditions of Lemma 4.6 are satisfied with $n = k - 2$ or $n = k - 1$. Now $4 \cdot x^{k-2}/4^{(k-2)^2} < 4 \cdot x^{k-1}/4^{(k-1)^2}$ for $x > 16^{k-1}$. Therefore

$$\begin{aligned} |f(x)| &< \frac{4 \cdot x^{k-1}}{4^{(k-1)^2}} < \frac{4 \cdot x^{k-1}}{4^{(k-1)^2}} < \frac{4 \cdot (16^k + 8^k)^{k-1}}{4^{k^2-2k+1}} \\ &= \frac{16^{k(k-1)} (1 + 1/2^k)^{k-1}}{4^{k^2-2k}} < 4^{k^2} \left(1 + \frac{1}{2^k}\right)^{k-1} \\ &< 4^{k^2} \left(1 + \frac{1}{2(k-1)}\right)^{k-1} < e^{1/2} 4^{k^2} < 2 \cdot 4^{k^2} \end{aligned}$$

because $2^{k-1} > k - 1$. Since $k \leq n$, the claim is established. \square

Lemma 4.14. $|f'(x)| < 4^{(n-1)^2+2(n-1)/3-1}$ for $x \in [1, 16^n + 8^n]$ and $n \geq 2$.

Proof. We use induction on n . When $n = 2$ we must show that $|f'(x)| < 4^{2/3}$ for $x \in [1, 320]$

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{nx^{n-1}}{4^{n^2}} = \frac{1}{4} - \frac{x}{128} + \frac{3x^2}{4^9} - \frac{4x^3}{4^{16}} + \sum_{n=4}^{\infty} (-1)^{n-1} \frac{nx^{n-1}}{4^{n^2}} \\ &\leq \left| \frac{1}{4} - \frac{x}{128} + \frac{3x^2}{4^9} - \frac{4x^3}{4^{16}} \right| + \sum_{n=5}^{\infty} \frac{nx^{n-1}}{4^{n^2}}. \end{aligned}$$

The maximum value of

$$h(x) = \left| \frac{1}{4} - \frac{x}{128} + \frac{3x^2}{4^9} - \frac{4x^3}{4^{16}} \right|$$

for $x \in [1, 320]$ is $h(320) = 4541/4096 < 1.11$. We have $(n+1)x^n/4^{(n+1)^2} < (1/16)nx^{n-1}/4^{n^2}$ for $n \geq 5$. Then

$$\sum_{n=5}^{\infty} \frac{nx^{n-1}}{4^{n^2}} = \sum_{n=0}^{\infty} \frac{(n+5)x^{n+4}}{4^{(n+5)^2}} < \frac{5x^4}{4^{25}} \sum_{n=0}^{\infty} \frac{1}{16^n} < \frac{5 \cdot 320^4}{4^{25}} \frac{16}{15} < 0.0001$$

because $(n+5)x^{n+4}/4^{(n+5)^2} < 5x^4/(16^n 4^{25})$. Therefore

$$|f'(x)| < 1.11 + 0.0001 = 1.1101 < 4^{2/3}.$$

Now suppose $n \geq 2$ and that $|f'(x)| < 4^{(n-1)^2+2(n-1)/3-1}$ for $x \in [1, 16^n + 8^n]$.

Now using Claim 4.13,

$$\begin{aligned} |f'(16x)| &\leq \frac{1}{4} + \frac{|f(x)|}{4} + \frac{|xf'(x)|}{4} \\ &< \frac{1}{4} + \frac{1}{2}4^{n^2} + \frac{1}{4}(16^n + 8^n)\frac{1}{4}4^{n^2-2n+1+2n/3-2/3} \\ &< \frac{1}{4} + \frac{1}{2}4^{n^2} + \frac{16^n + 8^n}{16^n} \frac{4^{1/3}}{4} \frac{1}{4}4^{n^2+2n/3} \\ &= \frac{1}{4} + \frac{1}{2}4^{n^2} + \left(1 + \frac{1}{2^n}\right) \frac{4^{1/3}}{4} \frac{1}{4}4^{n^2+2n/3}. \end{aligned}$$

But $1 + 1/2^n < 5/4$ for $n \geq 2$, so

$$|f'(16x)| < \frac{1}{4} + \frac{1}{2}4^{n^2} + \frac{5}{4} \frac{4^{1/3}}{4} \frac{1}{4}4^{n^2+2n/3}$$

and therefore

$$\begin{aligned} \frac{1}{4}4^{n^2+2n/3} - |f'(16x)| &> \left(1 - \frac{5 \cdot 4^{1/3}}{16}\right) \frac{1}{4}4^{n^2+2n/3} - \frac{1}{2}4^{n^2} - \frac{1}{4} \\ &> \left(1 - \frac{5 \cdot 4^{1/3}}{16}\right) \frac{1}{4}4^{2^2+2 \cdot 2/3} - \frac{1}{2}4^{2^2} - \frac{1}{4} > 76 > 0. \end{aligned}$$

Therefore, $|f'(16x)| < 4^{n^2+2n/3-1}$ for $x \in I_n$. Also

$$\begin{aligned} 16x &\geq 16(4 \cdot 16^n - 8^n) = 4 \cdot 16^{n+1} - 2 \cdot 8^{n+1}, \\ 16x &\leq 16(4 \cdot 16^n + 8^n) = 4 \cdot 16^{n+1} + 2 \cdot 8^{n+1}. \end{aligned}$$

Then

$$|f'(x)| < 4^{n^2+2n/3}$$

for $x \in I_{n+1} \subset [4 \cdot 16^{n+1} - 2 \cdot 8^{n+1}, 4 \cdot 16^{n+1} + 2 \cdot 8^{n+1}]$. □

Claim 4.15. $|f(x)| < \frac{1}{2}2^n 4^{n^2}$ for $x \in I_n$ and $n \geq 2$.

Proof. We have $\prod_{n=1}^{\infty} (1 - 1/16^n) < 1$, $\prod_{n=1}^{\infty} (1 - 4/(x \cdot 16^{n-1})) < 1$ and $|1 - x/(4 \cdot 16^{k-1})| < 1$ for $k \geq n + 2$. Now we estimate:

$$\begin{aligned} \prod_{k=1}^{\infty} \left|1 - \frac{x}{4 \cdot 16^{k-1}}\right| &= \prod_{k=1}^{n+1} \left|1 - \frac{x}{4 \cdot 16^{k-1}}\right| \prod_{k=n+2}^{\infty} \left|1 - \frac{x}{4 \cdot 16^{k-1}}\right| \\ &< \left(\prod_{k=1}^n \left|1 - \frac{x}{4 \cdot 16^{k-1}}\right|\right) \left|1 - \frac{x}{4 \cdot 16^n}\right|. \end{aligned}$$

We have $x < 4 \cdot 16^n + 8^n$, so

$$\begin{aligned} \prod_{k=1}^{\infty} \left| 1 - \frac{x}{4 \cdot 16^{k-1}} \right| &< \left(\prod_{k=0}^{n-1} \left| 1 - \frac{4 \cdot 16^n + 8^n}{4 \cdot 16^k} \right| \right) \left| 1 - \frac{4 \cdot 16^n + 8^n}{4 \cdot 16^n} \right| \\ &< \frac{1}{4 \cdot 2^n} \prod_{k=0}^{n-1} \left| 1 - 16^{n-k} + \frac{8^n}{4 \cdot 16^k} \right| \\ &= \frac{1}{4 \cdot 2^n} \prod_{k=0}^{n-1} 16^{n-k} \left(1 - \frac{1}{16^{n-k}} - \frac{1}{4 \cdot 2^n} \right) \\ &< \frac{1}{4 \cdot 2^n} \prod_{k=0}^{n-1} 16^{n-k} = 16^{n(n+1)/2} 2^{-n-2} = \frac{1}{4} 2^n 4^{n^2}. \end{aligned}$$

Then

$$\begin{aligned} |f(x)| &< \prod_{n=1}^{\infty} \left| 1 - \frac{1}{16^n} \right| \prod_{n=1}^{\infty} \left| 1 - \frac{4}{x \cdot 16^{n-1}} \right| \prod_{n=1}^{\infty} \left| 1 - \frac{x}{4 \cdot 16^{n-1}} \right| + 2 \\ &< \frac{1}{4} 2^n 4^{n^2} + 2 < \frac{1}{2} 2^n 4^{n^2} \end{aligned}$$

for $n \geq 2$. □

Claim 4.16. $|f'(x)| > \frac{1}{5} 4^{(n-1)^2 + 2(n-1)/3}$ for $x \in I_n$ and n sufficiently large.

Proof. We use induction on n . When $n = 2$ we must show that $|f'(x)| > 4^{5/3}/5$ for $x \in [960, 1088]$. Compute:

$$f'(x) = \frac{1}{4} - \frac{x}{128} + \frac{3x^2}{4^9} - \frac{4x^3}{4^{16}} + \sum_{n=2}^{\infty} \left[\frac{(2n+1)x^{2n}}{4^{(2n+1)^2}} - \frac{(2n+2)x^{2n+1}}{4^{(2n+2)^2}} \right].$$

We have

$$\frac{(2n+1)x^{2n}}{4^{(2n+1)^2}} - \frac{(2n+2)x^{2n+1}}{4^{(2n+2)^2}} > 0$$

for $x \in [960, 1088]$ and the function

$$h(x) = \frac{1}{4} - \frac{x}{128} + \frac{3x^2}{4^9} - \frac{4x^3}{4^{16}}$$

has minimum $h(960) = 10129/4096$. Therefore

$$f'(x) > h(x) \geq h(960) > \frac{4^{5/3}}{5}.$$

Now let $n \geq 2$ and suppose that $|f'(x)| > 4^{(n-1)^2 + 2(n-1)/3}$ for $x \in I_n$. Then

$$|f'(16x)| > \frac{1}{4} |x| |f'(x)| - \frac{1}{4} |f(x)| - \frac{1}{4}.$$

By Claim 4.15, $|f(x)| < (1/2)2^n 4^{n^2}$, so

$$\begin{aligned} |f'(16x)| &> \frac{1}{4}(4 \cdot 16^n - 8^n) \frac{1}{5} 4^{(n-1)^2+2(n-1)/3} - \frac{1}{8} 2^n 4^{n^2} - \frac{1}{4} \\ &> \frac{1}{4} \cdot 16^n \left(1 - \frac{1}{4 \cdot 2^n}\right) \frac{1}{5} 4^{n^2-2n+1+2n/3-2/3} - \frac{1}{8} 4^{n^2} 2^n - \frac{1}{4}. \end{aligned}$$

Now $1 - 1/(4 \cdot 2^n) > 15/16$ because $n \geq 2$. Then

$$\begin{aligned} |f'(16x)| &> \frac{3 \cdot 4^{1/3}}{16} 4^{n^2+2n/3} - \frac{1}{8} 4^{n^2+n/2} - \frac{1}{4}, \\ |f'(16x)| - \frac{1}{5} 4^{n^2+2n/3} &> \frac{4^{1/3} \cdot 15 - 16}{80} \cdot 4^{n^2+2n/3} - \frac{1}{8} 4^{n^2+n/2} - \frac{1}{4}. \end{aligned}$$

The right-hand side is increasing and has minimum for $n = 2$, so

$$|f'(16x)| - \frac{1}{5} 4^{n^2+2n/3} > \frac{4^{1/3} \cdot 15 - 16}{80} \cdot 4^{16/3} - \frac{1}{8} 4^5 - \frac{1}{4} > 30 > 0.$$

Therefore, $|f'(x)| > (1/5)4^{n^2+2n/3}$ for $x \in I_{n+1}$, since $x/16 \in I_n$. \square

Lemma 4.17. $|g'(x)| > \frac{1}{40} 4^{(n-1)^2+2(n-1)/3}$ for $x \in I_n$ and n sufficiently large.

Proof. We have $(4 \cdot 16^n + 8^n)/k^2 < 16^{n-m+1} + 8^{n-m+1}$ for $2 \cdot 4^{m-1} \leq k \leq 2 \cdot 4^m - 1$. By Lemma 4.14, $|f(x/k^2)| < 4^{(n-m)^2+2(n-m)/3-1}$. Then

$$\begin{aligned} \sum_{k=8}^{2 \cdot 4^n - 1} \frac{1}{k^2} \left| f' \left(\frac{x}{k^2} \right) \right| &= \sum_{m=2}^n \sum_{k=2 \cdot 4^{m-1}}^{2 \cdot 4^m - 1} \frac{1}{k^2} \left| f' \left(\frac{x}{k^2} \right) \right| \\ &< \sum_{m=2}^n \sum_{k=2 \cdot 4^{m-1}}^{2 \cdot 4^m - 1} \frac{1}{4} 4^{(n-m)^2+2(n-m)/3} \\ &< \sum_{m=2}^n 6 \cdot 4^{m-1} \frac{1}{4} 4^{(n-m)^2+2(n-m)/3} \\ &< \frac{3 \cdot 4^{1/3}}{2} (n-1) 4^{(n-2)^2+2(n-1)/3} \end{aligned}$$

because the maximal value of $h(m) = m - 1 + (n - m)^2 + 2(n - m)/3$ on the interval $[2, n]$ is $h(2) = (n - 2)^2 + 2(n - 1)/3 + 1/3$. So

$$\sum_{k=8}^{2 \cdot 4^n - 1} \frac{1}{k^2} \left| f' \left(\frac{x}{k^2} \right) \right| < \frac{6 \cdot 4^{1/3} (n-1)}{16^{n-1}} 4^{(n-1)^2+2(n-1)/3} < \frac{1}{50} 4^{(n-1)^2+2(n-1)/3}$$

because $6 \cdot 4^{1/3} (n - 1)/16^{n-1} < 1/50$ for $n \geq 4$. Recall $\widehat{f}_2 = \max\{|f'(x)| : 0 \leq$

$x \leq 2\}$. We have

$$\sum_{k=2 \cdot 4^n}^{\infty} \frac{1}{k^2} \left| f' \left(\frac{x}{k^2} \right) \right| < \sum_{k=2 \cdot 4^n}^{\infty} \frac{\widehat{f}_2}{k^2} < \sum_{k=1}^{\infty} \frac{\widehat{f}_2}{k^2} = \frac{\pi^2}{6} \widehat{f}_2 < 2\widehat{f}_2$$

because $x/k^2 < 2$ and $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$. Also

$$\sum_{k=2}^7 \frac{1}{k^2} \left| f' \left(\frac{x}{k^2} \right) \right| < \frac{1}{4} 4^{(n-1)^2+2(n-1)/3} \sum_{k=2}^7 \frac{1}{k^2} < \frac{3}{20} 4^{(n-1)^2+2(n-1)/3}$$

using $x/k^2 \leq x/4 < (4 \cdot 16^n + 8^4)/4 < 16^n + 8^n$ and Lemma 4.14. Then

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{1}{k^2} \left| f' \left(\frac{x}{k^2} \right) \right| \\ &= \sum_{k=2}^7 \frac{1}{k^2} \left| f' \left(\frac{x}{k^2} \right) \right| + \sum_{k=8}^{2 \cdot 4^n - 1} \frac{1}{k^2} \left| f' \left(\frac{x}{k^2} \right) \right| + \sum_{k=2 \cdot 4^n}^{\infty} \frac{1}{k^2} \left| f' \left(\frac{x}{k^2} \right) \right| \\ &< \left(\frac{3}{40} + \frac{1}{50} \right) 4^{(n-1)^2+2(n-1)/3} + 2\widehat{f}_2 < \frac{7}{40} 4^{(n-1)^2+2(n-1)/3}. \end{aligned}$$

By Claim 4.16, $|f'(x)| > (1/5)4^{(n-1)^2+2(n-1)/3}$ for $x \in I_n$. Therefore

$$\begin{aligned} |g'(x)| &> |f'(x)| - \sum_{k=2}^{\infty} \frac{1}{k^2} \left| f' \left(\frac{x}{k^2} \right) \right| \\ &> \frac{1}{5} 4^{(n-1)^2+2(n-1)/3} - \frac{7}{40} 4^{(n-1)^2+2(n-1)/3} \\ &= \frac{1}{40} 4^{(n-1)^2+2(n-1)/3} \end{aligned}$$

as claimed. \square

Corollary 4.18. $|G'_i(x)| > \frac{1}{100} 4^{(n-1)^2+2(n-1)/3}$ for $x \in I_n$ and n sufficiently large.

Proof.

$$\begin{aligned} |G'_i(x)| &> 2|g'(x)| - \left| \frac{\pi}{2\sqrt{2x}} \right| > \frac{1}{40} 4^{(n-1)^2+2(n-1)/3} - \frac{\pi}{2\sqrt{2(4 \cdot 16 - 8)}} \\ &> \frac{1}{40} 4^{(n-1)^2+2(n-1)/3} - \frac{\pi}{64\sqrt{2}} > \frac{1}{100} 4^{(n-1)^2+2(n-1)/3}. \end{aligned} \quad \square$$

4.3. Conclusion

We have shown that the functions G_1 and G_2 have zeros on the intervals I_n while they have no zeros on the remaining intervals J_n when n is large enough. From Corollary 4.18 the derivatives $G'_i(x)$ have no zeros on the intervals I_n and the functions $G_i(x)$ are either increasing or decreasing on I_n . Therefore the functions G_1 and G_2 have unique zeros on the intervals I_n for sufficiently large values of n .

5. The Eigenvalues of T

In this section we prove (as announced in Theorem 2.1) that the eigenvalues of T satisfy:

$$(i) \left| A_n - \frac{\sqrt{2}}{\pi 4^n} \right| < \frac{1}{8^{n+1}}, \quad \text{for } n \text{ sufficiently large positive integer.}$$

$$(ii) \left| A_{-n} + \frac{\sqrt{2}}{\pi 4^{-n}} \right| < \frac{1}{8^{1-n}}, \quad \text{for } n < 0 \text{ and } |n| \text{ sufficiently large.}$$

For (i), let $A_n = 1/a_n$ be an eigenvalue of T , where $n > 0$ is large enough that the results in Section 4 hold. The function G_1 has unique zeros on the intervals I_n and $G_1(a_n^2/(2\pi^2)) = 0$, where $a_n^2/(2\pi^2) = 1/(2\pi^2 A_n^2) \in I_{n-1}$. So

$$4 \cdot 16^{n-1} - 8^{n-1} < \frac{1}{2\pi^2 A_n^2} < 4 \cdot 16^{n-1} + 8^{n-1},$$

$$\frac{1}{\sqrt{2\pi\sqrt{4 \cdot 16^{n-1} + 8^{n-1}}}} < A_n < \frac{1}{\sqrt{2\pi\sqrt{4 \cdot 16^{n-1} - 8^{n-1}}}}.$$

The Mean Value Theorem states $h(x) = h(x_0) + h'(\theta)(x - x_0)$, where θ is a number between x and x_0 . Applying this for $h(x) = 1/\sqrt{x}$, we get

$$\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{x_0}} - \frac{1}{2\theta^{3/2}}(x - x_0).$$

Using $x = 4 \cdot 16^{n-1} - 8^{n-1}$ and $x_0 = 4 \cdot 16^{n-1}$, we get

$$\frac{1}{\sqrt{4 \cdot 16^{n-1} - 8^{n-1}}} = \frac{1}{\sqrt{4 \cdot 16^{n-1}}} + \frac{8^{n-1}}{2} \frac{1}{\theta^{3/2}}$$

for some θ with $4 \cdot 16^{n-1} - 8^{n-1} < \theta < 4 \cdot 16^{n-1}$, so $\theta > 4 \cdot 16^{n-1} - 8^{n-1} > 3.9 \cdot 16^{n-1}$ for $n \geq 5$. Thus

$$\frac{1}{\sqrt{4 \cdot 16^{n-1} - 8^{n-1}}} < \frac{1}{2 \cdot 4^{n-1}} + \frac{8^{n-1}}{2} \frac{1}{(3.9 \cdot 16^{n-1})^{3/2}} < \frac{2}{4^n} + \frac{13}{25 \cdot 8^n}. \quad (5.1)$$

Using $x = 4 \cdot 16^{n-1} + 8^{n-1}$ and $x_0 = 4 \cdot 16^{n-1}$, we get

$$\frac{1}{\sqrt{4 \cdot 16^{n-1} + 8^{n-1}}} = \frac{1}{\sqrt{4 \cdot 16^{n-1}}} - \frac{8^{n-1}}{2} \frac{1}{\theta^{3/2}}$$

for some θ with $4 \cdot 16^{n-1} < \theta < 4 \cdot 16^{n-1} + 8^{n-1}$. Thus

$$\frac{1}{\sqrt{4 \cdot 16^{n-1} + 8^{n-1}}} > \frac{1}{2 \cdot 4^{n-1}} - \frac{8^{n-1}}{2} \frac{1}{(4 \cdot 16^{n-1})^{3/2}} > \frac{2}{4^n} - \frac{13}{25 \cdot 8^n} \quad (5.2)$$

From (5.1) and (5.2),

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \left(\frac{2}{4^n} - \frac{13}{25 \cdot 8^n} \right) &< A_n < \frac{1}{\sqrt{2\pi}} \left(\frac{2}{4^n} + \frac{13}{25 \cdot 8^n} \right) \\ -\frac{13}{\sqrt{2\pi} 25 \cdot 8^n} &< A_n - \frac{2}{\sqrt{2\pi} 4^n} < \frac{13}{\sqrt{2\pi} 25 \cdot 8^n} \\ \left| A_n - \frac{2}{\sqrt{2\pi} 4^n} \right| &< \frac{13}{\sqrt{2\pi} 25 \cdot 8^n} < \frac{1}{8^{n+1}}. \end{aligned}$$

This proves (i).

(ii) If $A_n = 1/a_n$ is an eigenvalue of T with $n < -1$, then $G_2(a_n^2/(2\pi^2)) = 0$. The proof of (ii) is similar to the proof of (i).

Remarks. Theorem 2.1 establishes upper and lower bounds for the asymptotic behavior of the eigenvalues of T . These bounds allow us to prove properties (2) for these eigenvalues.

We showed that Theorem 2.1 holds for values of n large enough so that all statements in Section 4 hold. Numerical calculations suggest that in fact Theorem 2.1 holds for all integer values of n except $n = -1$ and $n = -2$.

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