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ON THE ULTRA-HYPERBOLIC WAVE OPERATOR

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Abstract: In this paper, we study the generalized wave equation of the form

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2(\Box)^k u(x,t) = 0$$

with the initial conditions

$$u(x,0) = f(x), \quad \frac{\partial}{\partial t}u(x,0) = g(x)\,,$$

where $u(x,t) \in \mathbb{R}^n \times [0,\infty)$, $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\Box^k$ is the ultra-hyperbolic operator iterated $k-$times defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k,$$

$p+q=n$, c is a positive constant, k is a nonnegative integer, f and g are continuous and absolutely integrable functions. We obtain $u(x,t)$ as a solution for such equation. Moreover, by ϵ-approximation we also obtain the asymptotic solution $u(x,t) = O(\epsilon^{-n/k})$. In particularly, if we put $n=1$, $k=2$ and $q=0$, the $u(x,t)$ reduces to the solution of the beam equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2\frac{\partial^4}{\partial x^4}u(x,t) = 0.$$

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1. Introduction

It is well known that for the 1-dimensional wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) = c^2\frac{\partial^2}{\partial x^2}u(x,t), \tag{1}$$

we obtain $u(x,t) = f(x+ct) + g(x-ct)$ as a solution of the equation, where f and g are continuous. Also for the n-dimensional wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2\triangle u(x,t) = 0, \tag{2}$$

with the initial condition

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t}u(x,0) = g(x)\,,$$

where f and g are given continuous functions. By solving the Cauchy problem for such equation, the Fourier transform has been applied and the solution is given by

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\cos\left(2\pi|\xi|\right)t + \widehat{g}(\xi)\frac{\sin\left(2\pi|\xi|\right)t}{2\pi|\xi|}$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2$ (see [2, p. 177]). By using the inverse Fourier transform, we obtain $u(x,t)$ in the convolution form, that is

$$u(x,t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x) \tag{3}$$

where Φ_t is an inverse Fourier transform of $\widehat{\Phi}_t(\xi) = \frac{\sin(2\pi|\xi|)t}{2\pi|\xi|}$ and Ψ_t is an inverse Fourier transform of $\widehat{\Psi}_t(\xi) = \cos\left(2\pi|\xi|\right)t = \frac{\partial}{\partial t}\widehat{\Phi}(\xi)$.

In this paper, we study the equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2\left(\Box\right)^k u(x,t) = 0 \tag{4}$$

with $u(x,0) = f(x)$ and $\frac{\partial}{\partial t}u(x,0) = g(x)$, where c is a positive constant, k is a nonnegative integer, f and g are continuous functions and absolutely integrable. The equation (4) is motivated by the heat equation of the form

$$\frac{\partial}{\partial t}u(x,t) = -c^2\left(\Box\right)^k u(x,t)$$

(see [3], more general: [1]-[4]). We obtain

$$u(x,t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x) \tag{5}$$

as a solution of (4) where Φ_t is an inverse Fourier transform of

$$\widehat{\Phi}_t(\xi) = \frac{\sin c\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k}$$

and Ψ_t is an inverse Fourier transform of $\widehat{\Psi}_t(\xi) = \cos c\left(\sqrt{s^2 - r^2}\right)^k t = \frac{\partial}{\partial t}\widehat{\Phi}_t(\xi)$ where $r^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2$ and $s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2$. Moreover, if we put $k = 1$ and $q = 0$ in (4) then (5) reduces to the solution of the $n-$ dimensional wave equation and also if $k = 2, n = 1$ and $q = 0$ in (4) then (5) reduces to the solution of beam equation.

We also study the asymptotic form of $u(x,t)$ in (5) by using ϵ approximation and obtain $u(x,t) = O(\epsilon^{-n/k})$.

2. Preliminaries

We shall need the following definitions.

Definition 1. Let $f \in L_1(\mathbb{R}^n)$-the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)dx\,, \tag{6}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the inner product in $\mathbb{R}^n$ and $dx = dx_1 dx_2 \ldots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f}(x)dx. \tag{7}$$

Lemma 2. *Given the function*

$$f(x) = \exp\left[-\sqrt{-\sum_{i=1}^{p} x_i^2 + \sum_{j=p+1}^{p+q} x_j^2}\,\right],$$

where $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $p + q = n$, $\sum_{i=1}^{p} x_i^2 < \sum_{j=p+1}^{p+q} x_j^2$. *Then*

$$\left|\int_{\mathbb{R}^n} f(x)dx\right| \leq \frac{\Omega_p \Omega_q}{2} \cdot \frac{\Gamma(n)\Gamma(\frac{p}{2})\Gamma(\frac{2-n}{2})}{\Gamma(\frac{2-q}{2})},$$

where Γ *denotes the Gamma function. That is* $\int_{\mathbb{R}^n} f(x)dx$ *is bounded.*

Proof.

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} \exp\left[-\sqrt{-\sum_{i=1}^{p} x_i^2 + \sum_{j=p+1}^{p+q} x_j^2}\,\right] dx\,.$$

Let us transform to bipolar coordinates defined by

$$x_1 = r\omega_1, \quad x_2 = r\omega_2, \ldots, \quad x_p = r\omega_p\,,$$

$$dx_1 = rd\omega_1, \quad dx_2 = rd\omega_2, \ldots, \quad dx_p = rd\omega_p\,,$$

and

$$x_{p+1} = s\omega_{p+1}, \quad x_{p+2} = s\omega_{p+2}, \ldots, \quad x_{p+q} = s\omega_{p+q}\,,$$

$$dx_{p+1} = sd\omega_{p+1}, \quad dx_{p+2} = sd\omega_{p+2}, \ldots, \quad dx_{p+q} = sd\omega_{p+q}\,,$$

where $\omega_1^2 + \omega_2^2 + \ldots + \omega_p^2 = 1$ and $\omega_{p+1}^2 + \omega_{p+2}^2 + \ldots + \omega_{p+q}^2 = 1$. Thus

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} \exp\left[-\sqrt{s^2 - r^2}\right] r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q\,,$$

where $dx = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area on the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively,

$$\left|\int_{\mathbb{R}^n} f(x)dx\right| \leq \int_{\mathbb{R}^n} \exp\left[-\sqrt{s^2 - r^2}\right] r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q\,.$$

By computing directly, we obtain

$$\int_{\mathbb{R}^n} f(x)dx = \Omega_p\Omega_q \int_0^\infty \int_0^s \exp\left[-\sqrt{s^2 - r^2}\right] r^{p-1}s^{q-1}drds\,,$$

where $\Omega_p = \frac{2\pi^{p/2}}{\Gamma(p/2)}$ and $\Omega_q = \frac{2\pi^{q/2}}{\Gamma(q/2)}$. Thus

$$\left|\int_{\mathbb{R}^n} f(x)dx\right| \leq \Omega_p\Omega_q \int_0^\infty \int_0^s \exp\left[-\sqrt{s^2 - r^2}\right] r^{p-1}s^{q-1}drds.$$

Put $r = s\sin\theta$, $dr = s\cos\theta d\theta$ and $0 \leq \theta \leq \frac{\pi}{2}$,

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} f(x)dx\right| &\leq \Omega_p\Omega_q \int_0^\infty \int_0^s e^{-\sqrt{s^2 - s^2\sin^2\theta}}(s\sin\theta)^{p-1}s^{q-1}s\cos\theta d\theta ds \\
&= \Omega_p\Omega_q \int_0^\infty \int_0^s e^{-s\cos\theta}s^{p+q-1}(\sin\theta)^{p-1}\cos\theta d\theta ds.
\end{aligned}$$

Put $y = s\cos\theta$, $ds = \frac{dy}{\cos\theta}$,

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} f(x)dx\right| &\leq \Omega_p\Omega_q \int_0^{\pi/2} \int_0^\infty e^{-y}\left(\frac{y}{\cos\theta}\right)^{n-1}(\sin\theta)^{p-1}\cos\theta d\theta \frac{dy}{\cos\theta} \\
&= \Omega_p\Omega_q \int_0^{\pi/2} \int_0^\infty e^{-y}y^{n-1}(\cos\theta)^{1-n}(\sin\theta)^{p-1}dyd\theta \\
&= \Omega_p\Omega_q\Gamma(n) \int_0^{\pi/2} (\cos\theta)^{1-n}(\sin\theta)^{p-1}d\theta \\
&= \frac{\Omega_p\Omega_q}{2}\Gamma(n)\beta\left(\frac{p}{2}, \frac{2-n}{2}\right)\,, \\
\left|\int_{\mathbb{R}^n} f(x)dx\right| &\leq \frac{\Omega_p\Omega_q}{2}\frac{\Gamma(n)\Gamma(\frac{p}{2})\Gamma(\frac{2-n}{2})}{\Gamma(\frac{2-n}{2})}.
\end{aligned}$$

That is $\int_{\mathbb{R}^n} f(x)dx$ is bounded. □

3. Main Results

Theorem 3. *Given the equation*

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2 (\Box)^k u(x,t) = 0 \tag{8}$$

with initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t}u(x,0) = g(x)\,, \tag{9}$$

where $u(x,t) \in \mathbb{R}^n \times [0,\infty)$, $\Box^k$ *is the ultra-hyperbolic operator iterated* $k-$*times,* c *is a positive constant,* k *is a nonnegative integer,* f *and* g *are continuous functions and absolutely integrable for* $x \in \mathbb{R}^n$. *Then (8) has a unique solution*

$$u(x,t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x) \tag{10}$$

and satisfy the condition (9), where Φ_t *is an inverse Fourier transform of*

$$\widehat{\Phi}_t(\xi) = \frac{\sin c\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k}$$

and Ψ_t *is an inverse Fourier transform of*

$$\widehat{\Psi}_t(\xi) = \cos c\left(\sqrt{s^2-r^2}\right)^k t = \frac{\partial}{\partial t}\widehat{\Phi}(\xi)\,,$$

where $r^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2$ *and* $s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2$.

Proof. By applying the Fourier transform defined by (6) to (8) and obtain

$$\frac{\partial^2}{\partial t^2}\widehat{u}(\xi,t) + c^2\left(-\xi_1^2 - \xi_2^2 - \cdots - \xi_p^2 + \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2\right)^k \widehat{u}(\xi,t) = 0\,,$$

$$\frac{\partial^2}{\partial t^2}\widehat{u}(\xi,t) + c^2\left(-\sum_{i=1}^{p}\xi_i^2 + \sum_{j=p+1}^{p+q}\xi_j^2\right)^k \widehat{u}(\xi,t) = 0$$

and let $s > r$. Thus we have

$$\frac{\partial^2}{\partial t^2}\widehat{u}(\xi,t) + c^2\left(s^2 - r^2\right)^k \widehat{u}(\xi,t) = 0$$

$$\widehat{u}(\xi,t) = A(\xi)\cos c\left(\sqrt{s^2-r^2}\right)^k t + B(\xi)\sin c\left(\sqrt{s^2-r^2}\right)^k t.$$

By (9), $\widehat{u}(\xi,0) = A(\xi) = \widehat{f}(\xi)$

$$\begin{aligned}\frac{\partial\widehat{u}(\xi,t)}{\partial t} &= -c\left(\sqrt{s^2-r^2}\right)^k A(\xi)\sin c\left(\sqrt{s^2-r^2}\right)^k t \\ &\quad +c\left(\sqrt{s^2-r^2}\right)^k B(\xi)\cos c\left(\sqrt{s^2-r^2}\right)^k t,\end{aligned}$$

$$\begin{aligned}\frac{\partial\widehat{u}(\xi,0)}{\partial t} &= 0 + c\left(\sqrt{s^2-r^2}\right)^k B(\xi) = \widehat{g}(\xi)\,,\\ B(\xi) &= \frac{\widehat{g}(\xi)}{c\left(\sqrt{s^2-r^2}\right)^k}\,,\end{aligned}$$

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\cos c\left(\sqrt{s^2-r^2}\right)^k t + \frac{\widehat{g}(\xi)}{c\left(\sqrt{s^2-r^2}\right)^k}\sin c\left(\sqrt{s^2-r^2}\right)^k t\,. \quad (11)$$

By applying the inverse Fourier transform (11), we obtain the solution $u(x,t)$ in the convolution form of (8). Now we need to show the existence of $\Phi_t(x)$ and $\Psi_t(x)$.

Let us consider the Fourier transform

$$\widehat{\Phi_t}(x) = \frac{\sin c\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k} \quad \text{and} \quad \Psi_t(x) = \cos c\left(\sqrt{s^2-r^2}\right)^k t.$$

They are all tempered distributions but they are not $L_1(\mathbb{R}^n)$ the space of integrable function. So we cannot compute the inverse Fourier transform $\Phi_t(x)$ and $\Psi_t(x)$ directly. Thus we compute the inverse $\Phi_t(x)$ and $\Psi_t(x)$ by using the method of $\epsilon-$approximation.

Let us define

$$\widehat{\phi_t^{\epsilon}}(\xi) = e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}\widehat{\phi_t}(\xi) = e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}\frac{\sin c\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k}$$

$$\text{for} \quad \epsilon > 0. \quad (12)$$

We see that $\phi_t^{\epsilon}(x) \in L_1(\mathbb{R}^n)$ and $\widehat{\phi_t^{\epsilon}}(x) \to \widehat{\phi_t}(x)$ uniformly as $\epsilon \to 0$. So that $\phi_t(x)$ will be limit in the topology of tempered distribution of $\phi_t^{\epsilon}(x)$. Now

$$\begin{aligned}\Phi_t^{\epsilon}(x) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{\Phi_t^{\epsilon}}(\xi)d\xi\\ &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}\frac{\sin c\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k}d\xi\\ |\Phi_t^{\epsilon}(x)| &\le \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\frac{e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}}{c\left(\sqrt{s^2-r^2}\right)^k}d\xi\,. \quad (13)\end{aligned}$$

By changing to bipolar coordinates. Now, put

$$\xi_1 = rw_1, \xi_2 = rw_2, \ldots, \xi_p = rw_p$$

and $$\xi_{p+1} = sw_{p+1}, \xi_{p+2} = sw_{p+2}, \ldots, \xi_p = sw_{p+q},\ p+q=n\,,$$

where $w_1^2 + w_2^2 + \cdots + w_p^2 = 1$ and $w_{p+1}^2 + w_{p+2}^2 + \cdots + w_{p+q}^2 = 1$.

$$|\Phi_t^\epsilon(x)| \le \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}}{c\left(\sqrt{s^2-r^2}\right)^k} r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q,$$

where $d\xi = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, where $\Omega_p = \frac{2\pi^{p/2}}{\Gamma(p/2)}$, $\Omega_q = \frac{2\pi^{q/2}}{\Gamma(q/2)}$,

$$|\Phi_t^\epsilon(x)| \le \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}} \int_0^\infty \int_0^s \frac{e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}}{c\left(\sqrt{s^2-r^2}\right)^k} r^{p-1}s^{q-1}drds\,.$$

Put $r = s\sin\theta, dr = s\cos\theta d\theta$ and $0 \le \theta \le \frac{\pi}{2}$.

$$\begin{aligned}
|\Phi_t^\epsilon(x)| &\le \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}} \int_0^\infty \int_0^{\pi/2} \frac{e^{-\epsilon c\left(\sqrt{s^2-s^2\sin^2\theta}\right)^k}}{c\left(\sqrt{s^2-s^2\sin^2\theta}\right)^k}(s\sin\theta)^{p-1}s^{q-1}s\cos\theta d\theta ds,\\
&= \frac{\Omega_p\Omega_q}{c(2\pi)^{n/2}} \int_0^\infty \int_0^{\pi/2} \frac{e^{-\epsilon c(s\cos\theta)^k}}{(s\cos\theta)^k}(s)^{p-1}s^{q-1}s(\sin\theta)^{p-1}\cos\theta d\theta ds.
\end{aligned}$$

Put $y = \epsilon c\,(s\cos\theta)^k = \epsilon c s^k\cos^k\theta$, $s^k = \frac{y}{c\epsilon\cos^k\theta}$, $ds = \frac{dy}{cks^{k-1}\epsilon\cos^k\theta} = \frac{sdy}{ky}$, thus

$$\begin{aligned}
|\Phi_t^\epsilon(x)| &\le \frac{\Omega_p\Omega_q}{c(2\pi)^{n/2}} \int_0^{\pi/2} \int_0^\infty \frac{e^{-y}s^{n-1}}{y/(\epsilon c)}(\sin\theta)^{p-1}\cos\theta\frac{s}{ky}dyd\theta\\
&= \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}} \int_0^{\pi/2} \int_0^\infty \frac{e^{-y}\epsilon}{ky^2}\left(\frac{y}{c\epsilon\cos^k\theta}\right)^{n/k}(\sin\theta)^{p-1}\cos\theta dyd\theta\\
&= \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}} \int_0^{\pi/2} \int_0^\infty \frac{e^{-y}y^{n/k-2}}{c^{n/k}k\epsilon^{n/k-1}}(\sin\theta)^{p-1}(\cos\theta)^{1-n}dyd\theta\\
&= \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}}\frac{\Gamma\left(\frac{n}{k}-1\right)}{k\epsilon^{\frac{n}{k}-1}c^{n/k}} \int_0^{\pi/2}(\sin\theta)^{p-1}(\cos\theta)^{1-n}d\theta\\
&= \frac{\Omega_p\Omega_q}{2c^{n/k}(2\pi)^{n/2}k\epsilon^{n/k-1}}\Gamma\left(\frac{n}{k}-1\right)\beta\left(\frac{p}{2},\frac{2-n}{2}\right),\\
|\Phi_t^\epsilon(x)| &\le \frac{\Omega_p\Omega_q}{2c^{n/k}(2\pi)^{n/2}k\epsilon^{n/k-1}}\frac{\Gamma\left(\frac{n}{k}-1\right)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)}.
\end{aligned}$$

Similarly, we defined $\widehat{\Psi^{\epsilon}_t}(\xi) = e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}\cos c\left(\sqrt{s^2-r^2}\right)^k t$ and

$$\begin{aligned}
\Psi^{\epsilon}_t(x) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{\Psi^{\epsilon}_t}(\xi)d\xi \\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}\cos c\left(\sqrt{s^2-r^2}\right)^k td\xi\,, \\
|\Psi^{\epsilon}_t(x)| &\leq \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}d\xi \\
&= \frac{1}{(2\pi)^{n/2}}\int_0^{\infty}\int_0^{s} e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}r^{p-1}s^{q-1}drds,
\end{aligned}$$

Put $r = s\sin\theta$, $dr = s\cos\theta d\theta$ and $0 \leq \theta \leq \frac{\pi}{2}$

$$\begin{aligned}
|\Psi^{\epsilon}_t(x)| &\leq \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}}\int_0^{\infty}\int_0^{\pi/2} e^{-\epsilon c(s\cos\theta)^k}(s\sin\theta)^{p-1}s^{q-1}s\cos\theta d\theta ds \\
&= \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}}\int_0^{\infty}\int_0^{\pi/2} e^{-\epsilon c(s\cos\theta)^k}s^{p+q-1}(\sin\theta)^{p-1}\cos\theta d\theta ds\,.
\end{aligned}$$

Put $y = \epsilon c(s\cos\theta)^k$, $ds = s\frac{dy}{ky}$,

$$\begin{aligned}
|\Psi^{\epsilon}_t(x)| &\leq \frac{\Omega_p\Omega_q}{k(2\pi)^{n/2}}\int_0^{\pi/2}\int_0^{\infty}\frac{e^{-y}}{y}\left(\frac{y}{c\epsilon\cos^k\theta}\right)^{n/k}(\sin\theta)^{p-1}\cos\theta dyd\theta \\
&= \frac{\Omega_p\Omega_q}{k(2\pi)^{n/2}}\int_0^{\pi/2}\int_0^{\infty}\frac{e^{-y}y^{n/k-1}}{c^{n/k}\epsilon^{n/k}}(\sin\theta)^{p-1}(\cos\theta)^{1-n}dyd\theta \\
&= \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}kc^{n/k}\epsilon^{n/k}}\Gamma\left(\frac{n}{k}\right)\int_0^{\pi/2}(\sin\theta)^{p-1}(\cos\theta)^{1-n}d\theta\,, \\
|\Psi^{\epsilon}_t(x)| &\leq \frac{\Omega_p\Omega_q}{2(2\pi)^{n/2}kc^{n/k}\epsilon^{n/k}}\frac{\Gamma\left(\frac{n}{k}\right)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)}.
\end{aligned}$$

Set

$$u^{\epsilon}(x,t) = f(x)*\Psi^{\epsilon}_t(x) + g(x)*\Phi^{\epsilon}_t(x) \tag{14}$$

which is $\epsilon-$approximation of $u(x,t)$ in (14) for $\epsilon \to 0$, $u^{\epsilon}(x,t) \to u(x,t)$ uniformly. Now

$$u^{\epsilon}(x,t) = \int_{\mathbb{R}^n} f(r)\Psi^{\epsilon}_t(x-r)dr + \int_{\mathbb{R}^n} g(r)\Phi^{\epsilon}_t(x-r)dr\,.$$

Thus

$$|u^{\epsilon}(x,t)| \leq |\Psi^{\epsilon}_t(x-r)|\int_{\mathbb{R}^n}|f(r)|dr + |\Phi^{\epsilon}_t(x-r)|\int_{\mathbb{R}^n}|g(r)|dr$$

$$
\begin{aligned}
&\leq \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}kc^{n/k}\epsilon^{n/k}}\frac{\Gamma\left(\frac{n}{k}\right)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)}M \\
&\quad+\frac{\Omega_p\Omega_q}{(2\pi)^{n/2}kc^{n/k}\epsilon^{n/k-1}}\frac{\Gamma\left(\frac{n}{k}-1\right)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)}N\,,
\end{aligned}
$$

$$
\begin{aligned}
\epsilon^{n/k}|u^{\epsilon}(x,t)| &\leq \frac{\Omega_p\Omega_q}{2(2\pi)^{n/2}kc^{n/k}}\frac{\Gamma\left(\frac{n}{k}\right)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)}M \\
&\quad+\frac{\Omega_p\Omega_q\epsilon}{2(2\pi)^{n/2}kc^{n/k}}\frac{\Gamma\left(\frac{n}{k}-1\right)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)}N\,,
\end{aligned}
$$

where $M = \int_{\mathbb{R}^n}|f(r)|dr$ and $N = \int_{\mathbb{R}^n}|g(r)|dr$, since f and g are absolutely integrable.

$$
\lim_{\epsilon\to 0}\epsilon^{n/k}|u^{\epsilon}(x,t)| \leq \frac{\Omega_p\Omega_q}{2(2\pi)^{n/2}kc^{n/k}}\frac{\Gamma\left(\frac{n}{k}\right)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)} = K.
$$

It follows that $u(x,t) = O\left(\epsilon^{-n/k}\right)$ for $n \neq k$ as $\epsilon \to 0$.

In particular, if we put $k = 2, n = 1$ and $q = 0$ then (8) reduces to the solution of the beam equation, see [1, p. 47]

$$
\frac{\partial^2}{\partial t^2}u(x,t) + c^2\frac{\partial^4}{\partial x^4}u(x,t) = 0,
$$

with the initial conditions

$$
u(x,0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t}u(x,0) = g(x),
$$

where f and g are continuous and absolutely integrable for $x \in \mathbb{R}^n$. Thus we obtain $u(x,t) = O(\epsilon^{-1/2})$ which is a solution of such beam equation.

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References

[1] J. David Logan, *An Introduction to Nonlinear Partial Differential Equations*, A Wiley-Interscience Publication, John Wiley and Sons (1997).

[2] G.B. Folland, *Introduction to Partial Differential Equation*, Princeton University Press, Princeton, New Jersey (1995).

[3] A. Kanathai, K. Nonlaopon, On the generalized heat kernel, *Computational Technologies*, **9**, No. 1 (2004), 3-10.