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LINEAR SERIES ON RIGID STABLE CURVES

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Abstract: There is a stratification of $\overline{\mathcal{M}}_g$ by the topological type of the stable curves. The zero-dimensional strata are essentially given by the union of a graph curve with tails which are rational curves with one node. Here we study their gonality (admissible coverings and (but only in degree 1) Caporaso's semibalanced line bundles).

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1. Introduction

For all integers $g \geq 2k-1 \geq 3$ let $\overline{\mathcal{M}}_g[k]$ denote the closure of the set of all smooth k-gonal curve of genus g. Thus $\overline{\mathcal{M}}_g[k]$ is an irreducible projective variety of dimension $2g+2k-5$. For all integers $k \geq g/2-1$ set $\overline{\mathcal{M}}_g[k] := \overline{\mathcal{M}}_g$. Let N be an integral curve such that $p_a(N) = 1$ and N has an ordinary node. N is unique, up to isomorphisms and for all $P, Q \in N \backslash \mathrm{Sing}(N)$ there is an automorphism $f : N \to N$ such that $f(P) = Q$. This curve is our main building block. For any reduced projective curve Y let $\mathcal{B}(Y)$ denote the set of its irreducible components. Let X be a nodal and connected curve of arithmetic genus $g \geq 0$. Let $\mathrm{Sing}(X)''$ (resp. $\mathrm{Sing}'(X)$) denote the set of all singular points of X lying on exactly two (resp. one) irreducible components of X. Consider the following non-oriented marked graph $\|X\|$. The vertices of $\|X\|$ are the irreducible components of X. For any $T \in \mathcal{B}(X)$ let $[T]$ denote the associated

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vertex of $\|X\|$. For each $T \in \mathcal{B}(X)$ we give as a marking the non-negative integer q_T, where q_T is the geometric genus of T. $\|X\|$ contains $\sharp(\mathrm{Sing}(X)' \cap T)$ loops with $[T]$ as their vertex. For all $T, J \in \mathcal{B}(X)$, such that $T \neq J$ the vertices $[T]$ and $[J]$ of $\|X\|$ are joined by $\sharp(T \cap J)$ edges. Call τ the abstract marked graph $\|X\|$. If we forget the marking, i.e. if we forget the integers q_T, $T \in \mathcal{B}(X)$, then $\|X\|$ becomes the classical dual graph $|X|$ of the nodal curve X. The set of all nodal projective curves Y such that $\|Y\| \cong \tau$ (as marked graphs) is parametrized by an irreducible algebraic variety $\mathcal{M}[\tau]$. If $\mathbb{K} = \mathbb{C}$, then the topological type of the complex analytic space $X(\mathbb{C})$ is uniquely determined by the marked graph τ and two non-isomorphic marked graphs give topologically different complex analytic spaces. We may extend the notion of marked graph to pointed curves (see [1]). Fix a topological type τ for nodal connected curves, say $\tau = \|X\|$. For all $T, J \in \mathcal{B}(X)$, $T \neq J$, let $q_T \geq 0$ be the associated marking of $[T]$, a_T the number of loops based at $[T]$ and $a_{TJ} := \sharp(T \cap J)$ the number of edges with $[T]$ and $[J]$ as their vertices. Now assume $g \geq 2$ and call $\mathbb{G}(g))$ the set of all marked graphs assoociated to a stable curve of genus g. The algebraic sets $\{\mathcal{M}[\tau]\}_{\tau \in \mathbb{G}(g)\}}$ give an algebraic stratification of $\overline{\mathcal{M}}_g$. The zero-dimensional strata of this stratification are described in the following way. The zero-dimensional strata corresponding to curves with smooth components are exactly the set of all genus g graph curves (see [2]). Let $X \in \overline{\mathcal{M}}_g$ be a curve representing an isolated point of $\mathcal{M}([\|X\|])$. Let T be an integral and nodal projective curve. All deformations of T preserving the marked graph $\|T\|$ are trivial if and only if either $T \cong \mathbb{P}^1$ or $T \cong \mathbb{P}^1$. Hence each irreducible component of X is isomorphic either to $\mathbb{P}^1$ or to N. Any b-pointed smooth rational curve, $b \geq 4$, depends on $b-3$ parameters and in particular it may be deformed non-trivially preserving its marked graph. Any b-points curve $(N, P; P_1, \dots, P_b)$, $b \geq 2$, depends on $b-1$ parameters and in particular it may be deformed non-trivially preserving its marked graph. Hence there is an integer $a(X) \geq 0$ with the following properties:

(a) X contains $a(X)$ irreducible components N_i, $1 \leq i \leq a(X)$, isomorphic to N; let R_X denote the closure of $X \setminus \cup_{i=1}^{a(X)} N_i$ in X; $R_X = \emptyset$ if and only if $g = 2$, $a(X) = 2$ and $X = N_1 \cup N_2$; from now on assume $X \neq N_1 \cup N_2$; we have $\sharp(R_X \cap N_i) = 1$ and $N_i \cap \mathrm{Sing}(R_X) = \emptyset$ for all i; let $R[X]$ denote the $a(X)$-pointed curve $(R_X; R_X \cap N_1, \dots, R_X \cap N_{a(X)})$;

(b) $N_i \cap N_j = \emptyset$ for all $i \neq j$ (assuming $X \neq N_1 \cup N_2$);

(c) R_X is connected, $p_a(X) = g - a(X)$ and the pointed curve $R[X]$ is stable;

(d) every irreducible component of R_X contains exactly 3 marked points, i.e. point of $\mathrm{Sing}(R_X)$ or one of the $a(X)$ points $R_X \cap N_i$, $1 \le i \le a(X)$.

We will say that X is a *rigid* curve of genus g if these conditions are satisfied. If we allow that some N_i is a smooth elliptic curve, then we say that X is quasi-rigid. We write $a(X)$, N_i, $1 \le i \le a(X)$, R_X and $R[X]$ also for a quasi-rigid curve, without prescribing how may of the genus one tails $N_1, \dots, N_{a(X)}$ are smooth. A quasi-rigid curve X is called a *pseudotree* if $a(X) = p_a(X)$, i.e. if R_X is a tree.

The case $a(X) = 0$ is studied in detail in [2].

At the end of the paper we prove the following result.

Theorem 1. *Fix integers $g \ge 2k-1 \ge 3$. Let X be a genus g quasi-rigid curve such that $a(X) > 0$. $X \in \overline{\mathcal{M}}_g[k]$ if and only if either $p_a(R_X) \le 1$ (i.e. $a(X) \in \{g-1, g\}$) or $R_X \in \overline{\mathcal{M}}_{g-a(X)}[k]$.*

In Section 2 we consider some degree 1 examples of balanced line bundles in the sense of Caporaso with $h^0 = 2$ (see [3], [4], [5], [8]).

2. Semibalanced Line Bundles

Fix a quasi-rigid curve $X \in \overline{\mathcal{M}}_g$ eith $a(X) > 0$. Set $P_i := R_X \cap N_i$, $1 \le i \le a(X)$. Let $f_i : X_i \to X$ be the partial normalization of X in which we normalize only the point P_i. Set $F_i := f_{i*}(\mathcal{O}_{X_i})$. F_i is a depth 1 sheaf on X with pure rank 1 and $\mathrm{Sing}(F_i) = \{P_i\}$. Since f_i is finite, the Leray spectral sequence of f_i gives $h^0(X, F_i) = h^0(X_i, \mathcal{O}_{X_i})$ and $h^1(X, F_i) = h^1(X_i, \mathcal{O}_{X_i})$. Since $\chi(\mathcal{O}_{X_i}) = \chi(\mathcal{O}_X) + 1$, Riemann-Roch gives $\deg(F_i) = \deg(\mathcal{O}_{X_i}) + 1 = 1$. The reduced curve X_i has two connected components, one of them mapped isomorphically by f_i onto N_i, while the other component is mapped isomorphically onto $\overline{X \backslash N_i}$. Hence $h^0(X, F_i) = 2$. Since X is connected, no depth 1 sheaf on X with degree ≤ 0 has at least 2 linearly independent sections. Thus F_i is a spanned depth 1 sheaf on X with pure deptth 1 and degree 1. Thus there are at least $a(X)$ such sheaves. For some X there may be other sheaves with the same properties (even if X is a graph curve, as obvious from the enumeration of trivalent graphs). It is easy to check the existence of a bijection between the set of such sheaves and the set of all disconnecting nodes. Since $N_i \cong N$, there is a one-dimensional family $\overline{L}_i$ of degree 2 line bundles on N_i and each of them is spanned. Fix any $L \in \overline{L}_i$ and call $u_L : N_i \to \mathbb{P}^1$ the degree 2 morphism induced by the complete linear system $|L|$. Since $N_i \cap \overline{X \backslash N_i}$ is a unique point, we may extend u_L to a unique morphism $v_L : X \to \mathbb{P}^1$ such that $v_L|N_i = u_i$ and $v_L(X \backslash N_i) = u_L(X \cap N_i)$.

Set $L' : v_L^*(\mathcal{O}_{\mathbb{P}^1}(1))$. L' is a spanned line bundle on X, $\deg(L') = 2$ and $h^0(X, L') = 2$. Varying $L \in \overline{L}_i$ we get a one-dimensional family $\mathbb{L}_i$ of spanned line bundles on X with degree 2 and two linearly independent sections. For certain quasi-rigic curves X there may be other such line bundles (even if X is a graph curve); any such line bundle is spanned and it corresponds to an irreducible component T of X such that $T \cong \mathbb{P}^1$ and $X \backslash T$ is not connected.

Let Y be a semistable curve and $L \in \mathrm{Pic}(Y)$. As in [3], [4], [5] and [8] L is said to be *semibalanced* if

$$|\deg(L|Z) - \deg(L) \cdot \deg(\omega_Y|Z)/\deg(\omega_Y)| \leq \sharp(Z \cap \overline{Y\backslash Z})/2 \qquad (1)$$

for every proper subcurve $Z \subsetneq Y$. Now assume that Y is quasistable. L is said to be *balanced* if it is semibalanced and $\deg(L|E) = 1$ for every exceptional component E of Y.

Proposition 1. *Let X be a rigid curve of genus g. Assume the existence of a disconnecting node P of X. Let Z_1 and Z_2 be the closures in X of the two connected components of $X\backslash\{P\}$. Let Y_P be the quasistable curve with X as stable model, say $u_P : Y_P \to X$, with a unique exceptional curve E, and such that $u_P(E) = \{P\}$. Let L_P be the spanned line bundle on Y associated to P. The line bundle L_P is semibalanced if and only it $p_a(Z_1) = p_a(Z_2)$ and either each curve Z_i is 2-connected (i.e. either Z_i is irreducible or $\sharp(W \cap \overline{Z_i\backslash W}) \geq 2$ for every proper subcurve W of Z_i), or for any disconnecting node Q of Z_1 or Z_2, say Z_i, the connected component A of $Z_i\backslash\{Q\}$ not containing P satisfies $2p_a(A) \leq p_a(X) + 1$.*

Proof. By the definition of L_P we have $\deg(L_P) = \deg(L_P|E) = 1$ and $\deg(L_P|T) = 0$ for all other irreducible components of Y_P. Let $Y_P \to X$ be the stable reduction of Y_P. Let Y_i, $i = 0, 1$, denote the subcurves of Y_P mapped isomorphically onto Z_i. Hence $\sharp(Y_i \cap (Y_{2-i} \cup E)) = 1$, and $p_a(Y) = p_a(X) = p_a(Y_1) + p_a(Y_{2-i} \cup E) = p_a(Y_1) + p_a(Y_2)$.

(a) Assume that L_P is semibalanced. Just to fix the notation we assume $p_a(Y_1) \geq p_a(Y_2)$. Hence $\deg(\omega_{Y_1}) \geq \deg(\omega_{Y_P})/2$ and equality holds if and only $p_a(Y_1) = p_a(Y_2)$, i.e. if and only if $p_a(Z_1) = p_a(Z_2)$. We have $\deg(\omega_Y|Y_i) = \deg(\omega_{Y_i}) + 1$. Since $\deg(L_P|Y_1) = 0$ and $\sharp(Y_1 \cap (Y_2 \cup E)) = 1$, the inequality (1) is satisfied by the subcurve $Z := Y_1$ of Y if and only if $p_a(Y_1) = p_a(Y_2)$. Now assume that Z_i is not 2-connected, i.e. that Y_i is not 2-connected, and that the connected component A of $Z_i\backslash\{Q\}$ not containing P satisfies $2p_a(A) > p_a(X) + 1$, i.e. the connected component $B \cong A$ of $\overline{Y_i\backslash\{E \cap Y_i\}}$ not containing P satisfies $2p_a(B) > p_a(Y) + 1$. We have $B \cap \overline{Y_P\backslash B} = B \cap \overline{Y_i\backslash B}$. Hence $\sharp(B \cap \overline{Y_P\backslash B}) = 1$. Since $\deg(L_P|U) = 1$ and $2p_a(B) - 2 > (2p_a(Y) - 2)/2$, (1) is not satified.

(b) Now assume $p_a(Y_1) = p_a(Y_2)$ and either each curve Z_i is 2-connected or for any disconnecting node Q of Z_1 or Z_2, say Z_i, the connected component A of $Z_i\backslash\{Q\}$ not containing P satisfies $2p_a(A) \le p_a(X)+1$. If Y_1 or Y_2 are irreducible, then $p_a(X) = 2$, $Y_1 \cong Y_2 \cong N$ and $p_a(Y) = 2$. Since $\deg(L_P|Y_i) = 0$, $i = 0, 1$, and $\deg(L_P|E) = \deg(L_P|(Y_i \cup E)) = 1$, $i = 0, 1$,(1) is satisfied for every proper subcurve Z of Y. Now assume $p_a(X) \ge 3$. Let Z be a proper and connected subcurve of Y. We have $\deg(\omega_Y|Z) = \deg(\omega_Z) + \sharp(Z \cap \overline{Y\backslash Z})$. First assume $E \subseteq Z$. Hence $\deg(L|Z) = 1$. Since Y is quasistable, it is semistable. Hence either $\sharp(Z \cap \overline{Y\backslash Z}) \ge 2$ or $\sharp(Z \cap \overline{Y\backslash Z}) \ge 1$ and $\deg(\omega_Y) \ge \deg(\omega_Z) \ge 0$. Hence the inequality (1) is satisfied. Now assume that Z does not contains E. Since $Y_P\backslash E \cong X\backslash\{P\}$ is not connected, the connectedness of Z implies $Z \subseteq Y_i$ for some $i \in \{1, 2\}$. Hence $p_a(Z) \le p_a(Y_i) = p_a(Y)/2$. Since $p_a(Y) = p_a(X) \ge 3$, we get $|\deg(\omega_Z)| \le \deg(\omega_Y)/2$. If $\sharp(Z \cap \overline{Y_i\backslash Z}) \ge 2$, then $\sharp(Z \cap \overline{Y\backslash Z}) \ge \sharp(Z \cap \overline{Y_i\backslash Z}) \ge 2$. Hence Z satisfies the inequality (1) in this case. Now assume $\sharp(Z \cap \overline{Y_i\backslash Z}) \le 1$. Since Y_i is connected, we get that $Z \cap \overline{Y_i\backslash Z}$ is a disconnecting node Q' of Z_i. The point $u_P(Q')$ is a disconnecting node of Z_i. If Z intersects E, then $\sharp(Z \cap \overline{Y\backslash Z}) \ge 2$ and hence we just checked that (1) is satifsied. If $Z \cap E = \emptyset$, then $p_a(u_P(Z)) \le p_a(X) + 1$. Since $u_P(Z) \cong Z$ and $p_a(Y) = p_a(X)$, even in this case (1) holds. □

Take X and L_P as in the statement of Proposition 1. Since $\deg(L_P|E) = 1$, L_P is semibalanced if and only if it is balanced. Notice that if L_P is semibalanced, then the genus $p_A(X) = p_a(Z_1) + p_a(Z_2)$ is even.

Proof of Theorem 1. Set $Y := \overline{X\backslash N_1}$. Y is connected. Recall that $g \ge 3$. The pointed curve (Y, P_1) is stable, but Y may be semistable. Call Y' its stable reduction. Hence $Y' \in \overline{\mathcal{M}}_{g-1}$.

(a) Here we show that $X \in \overline{\mathcal{M}}_g[k]$ if and only if $Y' \in \overline{\mathcal{M}}_{g-1}[k]$. First assume $X \in \overline{\mathcal{M}}_g[k]$. Hence there are a nodal connected curve X', a morphism $u : X' \to X$ and an admissible degree k covering of X' such that $p_a(X') = g$, u contracts the rational tails of X' (if any) and then take the semistable reduction (use the construction of $\overline{\mathcal{M}}_g[k]$ made in [6], §4). Call N' the subcurve of X' mapped into N_1. N' is a connected nodal curve of arithmetic genus 1 union of a component $M \cong N$ and (perhaps) some rational tails contracted by u to points of N_1 (as in [7], Definition 1.3, and its iterations). It is easy to obtain from $u|\overline{X'\backslash N'}$ a degree k admissible covering of a curve Y'' with Y' as stable reduction (after deleting the rational tails) (as in [6] use that from an admissible $k-1$ covering of a curve D we get an admissible degree k covering of a curve D' such that D is obtained from D contracting some rational tail and making a partial semistable reduction). Thus as in [6], §4, we get $Y' \in \overline{\mathcal{M}}_{g-1}[k]$. For the converse we use that $p_a(N_1) = 1$. We may glue a degree k admissible covering of a nodal curve with Y' as its stable reduction and a degree k morphism $N_1 \to \mathbb{P}^1$ to get a degree k admissible covering $W \to D$ of a nodal curve W with X as its stable reduction (note that we need to insert a $\mathbb{P}^1$ at the singular point of N_1 because every singular point of W must be mapped into a singular point of the genus zero tree D).

(b) Use induction on the integer $a(X)$ and part (a) to conclude. □

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