International Journal of Pure and Applied Mathematics
Volume 52 No. 2 2009, 275-277

SEPARATORS OF ZERO-DIMENSIONAL SCHEMES IN $\mathbb{P}^n$

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Abstract: Here we extend the definition and the main properties of *separators* of a connected component mP of a zero-dimensional scheme $Z := mP \cup Z'' \subset \mathbb{P}^n$ introduced by Guardo, Marino and Van Tuyl if Z'' is a disjoint union of fat points.

AMS Subject Classification: 13D40, 13D02, 14M05, 14N05
Key Words: zero-dimensional scheme, fat point, Hilbert function

1. Introduction

Let $\mathbb{K}$ be an algebraically closed base field. Fix $P \in \mathbb{P}^n$, an integer $m > 0$ and a zero-dimensional scheme $Z'' \subset \mathbb{P}^n \backslash \{P\}$. Assume either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > m$. Set $Z := mP \cup Z''$ and $Z' := (m-1)P \cup Z''$ (with the convention $0P := \emptyset$). A *separator* of mP in Z or of mP from $Z'' \cup (m-1)P$ is a homogeneous form on $\mathbb{P}^n$ vanishing on Z', but not on Z, i.e. vanishing on Z'' and vanishing at P with order exactly $m-1$. Fix homogeneous coordinates $x_0, \dots, x_n$ of $\mathbb{P}^n$ and set $R := \mathbb{K}[x_0, \dots, x_n]$. For any closed subscheme $A \subseteq \mathbb{P}^n$ let I_A denote the homogeneous ideal of A. Notice that $\mathcal{I}_Z/\mathcal{I}_{Z'}$ is a $\mathbb{K}$-vector space of dimension $\beta := \binom{n+m-2}{n-1}$. Since $\mathcal{I}_Z/\mathcal{I}_{Z'} \cong I_Z/I_{Z'}$, it is a homogeneous R-module and we may take a basis of it as a $\mathbb{K}$-vector space aformed by homogeneous forms. E. Guardo, L. Marino and A. Van Tuyl introduced this concept in the case in which Z'' is a finite union of fat points (see [2]). See the introduction of [2] for the history of this concept in the classical case $m = 1$ and Z'' reduced. Our starting point was that most of the proofs in [2] works verbatim in our set-up.

Received: March 15, 2009

Thus we get the following results.

Theorem 1. *There is a basis $F_1, \dots, F_\beta$ of $I_Z/I_{Z'}$ which induces a minimal set of generators of it as an R-module.*

We order the basis $F_1, \dots, F_\beta$ given by Theorem 1 so that $\deg(F_i) \le \deg(F_j)$ for all $i \le j$. The β-ple $\underline{deg}(mP, Z'') := (\deg(F_1), \dots, \deg(F_\beta))$ is called the *degrees of the minimal separators* of mP in Z or of mP from $Z'' \cup (m-1)P$. The next result shows that this β-ple depends only from n, Z'', P and m. For any scheme $A \subset \mathbb{P}^n$, $H_A : \mathbb{N} \to \mathbb{N}$ denote its Hilbert function and ΔH_A its first difference. For any graded R-module M and any integer t let M_t denote the degree t part of M.

Theorem 2. *For every integer $t \ge 0$ we have*

$$\dim_{\mathbb{K}}(I_{Z'}/I_Z)_t) = \sharp(\{d_j \in \underline{deg}(mP, Z'') : d_j =\le t\}),$$

$$\Delta H_{Z'}(t) = \Delta_H(Z) - \sharp(\{d_j \in \underline{deg}(mP, Z'') : d_j = t\}).$$

Theorem 3. *Set $(d_1, \dots, d_\beta) := \underline{deg}(mP, Z'')$. Let $\mathbb{F}_{n-1}$ (resp. $\mathbb{F}'_{n-1}$) be the last piece of the minimal free resolution of Z (resp. Z'). Then*

$$\mathbb{F}_{n-1} \cong \mathbb{F}'_{n-1} \oplus \bigoplus_{i=1}^{\beta} R(-d_i - n).$$

We tried to extend the definition of separators to connected components of zero-dimensional schemes, which are more general than the fat point mP. We won only in a very particular case, which we list below. As stressed in [1], Remark 2.4, it is very easy to give more general definition: the difficult part is to prove anything with them.

Remark 1. Fix $P \in \mathbb{P}^n$, an integer $m > 0$ and a zero-dimensional scheme $Z'' \subset \mathbb{P}^n \backslash \{P\}$. Let Z_1 be any subscheme such that $(m-1)P \subsetneq Z_1 \subseteq mP$ and set $Z := Z_1 \cup Z''$, $Z' := (m-1)P \cup Z''$ and $\beta := \dim_{\mathbb{K}}(\mathcal{I}_Z/\mathcal{I}_{Z'})$. Thus $1 \le \beta \le \binom{n+m-2}{n-1}$ and the last inequality is an equality if and only if $Z_1 = mP$. A separator of Z_1 inside Z or of Z_1 from $Z'' \cup (m-1)P$ is a homogeneous form in $I_Z \backslash I_{Z'}$. The proof of Theorem 1 works in this set-up, because it only use that $I_{Z'}/I_Z$ is killed by the multiplication with any homogeneous form vanishing at P.

Proof of Theorem 1. Let s be the minimal integer such that there homogeneous polynomials $F_1, \dots, F_s \in I_{Z'}$ whose images in $I_{Z'}/I_Z$ generates $I_{Z'}/I_Z$ as an R-module. Obviously, there images in $I_{Z'}/I_Z$ are linearly indepent. Hence

$s \le \beta$. The proof of the inequality $s \ge \beta$ given in [2], Proof of Theorem 3.3, works verbatim here. □

Proof of Theorem 2. A formal consequence of Theorem 1 is the statement of [2], Lemma 3.6, which is equivalent to both assertions of Theorem 2. □

Proof of Theorem 3. The key observation is that [2], Lemma 3.6, works verbatim, because it only use the point P. Indeed, it uses that the ideal $(I_P)^x$ is a primary ideal for all integers $x > 0$ (see [2], Lemma 2.2). This result is used in line 4 from the bottom of p. 11 of [2]. Everything else works verbatim. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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