International Journal of Pure and Applied Mathematics
Volume 52 No. 2 2009, 163-176

THE ALGEBRA OF SMOOTH FUNCTIONS OF RAPID DESCENT

Paul O. Oleche[1], N. Omolo-Ongati[2 §], John O. Agure[3]
[1,2,3]Department of Mathematics
Maseno University
P.O. Box 333, Maseno, KENYA
[1]e-mail: poleche@maseno.ac.ke
[2]e-mail: omolo_ongati@yahoo.com
[2]e-mail: johnagure@maseno.ac.ke

Abstract: A bounded operator with the spectrum lying in a compact set $V \subset \mathbb{R}$, has $C^\infty(V)$ functional calculus. On the other hand, an operator H acting on a Hilbert space $\mathcal{H}$, admits a $C(\mathbb{R})$ functional calculus if H is self-adjoint. So in a Banach space setting, we really desire a large enough intermediate topological algebra $\mathfrak{A}$, with $C_0^\infty(\mathbb{R}) \subset \mathfrak{A} \subseteq C(\mathbb{R})$ such that spectral operators or some sort of their restrictions, admit a $\mathfrak{A}$ functional calculus.

In this paper we construct such an algebra of smooth functions on $\mathbb{R}$ that decay like $(\sqrt{1+x^2})^\beta$ as $|x| \to \infty$, for some $\beta < 0$. Among other things, we prove that $C_c^\infty(\mathbb{R})$ is dense in this algebra. We demonstrate that important functions like $x \mapsto e^x$ are either in the algebra or can be extended to functions in the algebra. We characterize this algebra fully.

AMS Subject Classification: 46J15
Key Words: Banach algebra, smooth function, extension

1. Preliminaries

For $\beta \in \mathbb{R}$, we define $\mathfrak{S}^\beta$ to be the space of smooth functions $f : \mathbb{R} \to \mathbb{C}$ such that for each $r \geq 0$ there exists $c_r > 0$ so that

Received: February 21, 2009

§Correspondence author

$$\left|f^{(r)}(x)\right| := \left|\frac{d^r}{dx^r}f(x)\right| \leq c_r \langle x\rangle^{\beta-r}, \quad \text{all } x \in \mathbb{R}. \tag{1.1}$$

Remark 1.1. 1. Observe that $\mathfrak{S}^\beta\mathfrak{S}^\gamma \subseteq \mathfrak{S}^{\beta+\gamma}$ for all $\beta, \gamma \in \mathbb{R}$.

2. If $f \in \mathfrak{S}^\beta$ then so is $\bar{f}$ where $\bar{f}(z) := \overline{f(z)}$ for all $z \in \mathbb{C}$.

Define the *translation operator* τ_ϵ on the space of functions $f : \mathbb{R} \to \mathbb{C}$ by $\tau_\epsilon f(x) := f(x+\epsilon)$ for all $x \in \mathbb{R}$ and some $\epsilon \in \mathbb{R}$. Then we have the following lemma.

Lemma 1.2. *For $\beta < 0$, the space $\mathfrak{S}^\beta$ is invariant under translation τ_ϵ for $\epsilon > 0$.*

Proof. Let $f \in \mathfrak{S}^\beta$ then by (1.1) we can find $c_r > 0$ such that

$$\left|\frac{d^r}{dx^r}f(x)\right| \leq c_r \langle x\rangle^{\beta-r}, \quad \text{for all } x \in \mathbb{R}.$$

$$\text{But} \qquad \left|\frac{d^r}{dx^r}\tau_\epsilon f(x)\right| = \left|\frac{d^r}{dx^r}f(x+\epsilon)\right| \leq c_r \langle x+\epsilon\rangle^{\beta-r} \quad \text{by use of the chain rule.}$$

$$\text{Therefore} \qquad \langle x\rangle^{r-\beta}\left|\frac{d^r}{dx^r}\tau_\epsilon f(x)\right| \leq c_r \left(\frac{\langle x\rangle}{\langle x+\epsilon\rangle}\right)^{r-\beta}$$

with $\left(\frac{\langle x\rangle}{\langle x+\epsilon\rangle}\right)^{r-\beta}$ bounded on $\mathbb{R}$ and the bound goes to 1 as $\epsilon \to 0$, see Figure 1.

Now set $D_{r,\epsilon} := c_r \sup_{x\in\mathbb{R}} \left(\frac{\langle x\rangle}{\langle x+\epsilon\rangle}\right)^{r-\beta}$, then we have

$$\left|\frac{d^r}{dx^r}\tau_\epsilon f(x)\right| \leq D_{r,\epsilon} \langle x\rangle^{\beta-r}, x \in \mathbb{R}.$$

Thus $\tau_\epsilon f \in \mathfrak{S}^\beta$. □

Theorem 1.3. *The space*

$$\mathfrak{A} := \cup_{\beta<0}\mathfrak{S}^\beta \tag{1.2}$$

is an algebra under pointwise multiplication.

Proof. Let $f, g \in \mathfrak{A}$ and $\alpha, \lambda \in \mathbb{C}$. Then $f, g \in C^\infty(\mathbb{R})$ and we can find $c_{f,n}, c_{g,n} \in (0,\infty)$ such that

$$\left|\frac{d^n}{dx^n}f(x)\right| \leq \frac{c_{f,n}}{\langle x\rangle^{n-\beta_1}} \qquad \text{and} \qquad \left|\frac{d^n}{dx^n}g(x)\right| \leq \frac{c_{g,n}}{\langle x\rangle^{n-\beta_2}},$$

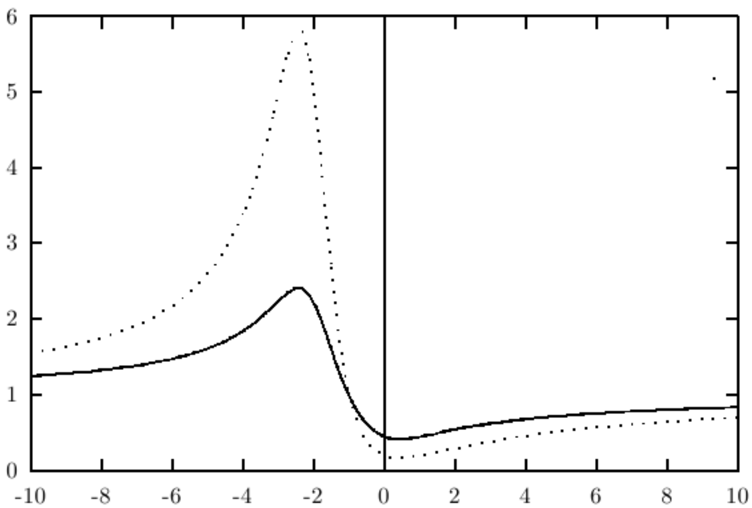

Figure 1: Graphs of $\left(\frac{\langle x\rangle}{\langle x+\epsilon\rangle}\right)^{r-\beta}$ for various ϵ's

for some $\beta_1, \beta_2 < 0$ and all $n \geq 0$. So we have

$$\frac{d^n}{dx^n}(\alpha f(x) + \lambda g(x)) = \alpha \frac{d^n}{dx^n} f(x) + \lambda \frac{d^n}{dx^n} g(x) \quad \text{(by linearity of } \frac{d^n}{dx^n}\text{)}.$$

Therefore

$$\begin{aligned}
\left|\frac{d^n}{dx^n}(\alpha f(x) + \lambda g(x))\right| &= \left|\alpha \frac{d^n}{dx^n} f(x) + \lambda \frac{d^n}{dx^n} g(x)\right| \\
&\leq |\alpha| \frac{c_{f,n}}{\langle x\rangle^{n-\beta_1}} + |\lambda| \frac{c_{g,n}}{\langle x\rangle^{n-\beta_2}} \\
&\leq \frac{|\alpha|\, c_{f,n} + |\lambda|\, c_{g,n}}{\langle x\rangle^{n-\beta}} \quad \text{(where } \beta := \max\{\beta_1, \beta_2\}\text{)} \\
&= \frac{c_{f+g,n}}{\langle x\rangle^{n-\beta}}, \qquad c_{f+g,n} > 0,\ \beta < 0 \quad \text{for all } \ n \geq 0.
\end{aligned}$$

Therefore $\alpha f + \lambda g \in \mathfrak{A}$, showing that $\mathfrak{A}$ is linear.

Next, by the Leibniz rule,

$$\begin{aligned}
\left|\frac{d^n}{dx^n}(f(x)g(x))\right| &= \left|\sum_{i=0}^{n} \frac{n!}{i!(n-i)!} \frac{d^i}{dx^i} f(x) \frac{d^{n-i}}{dx^{n-i}} g(x)\right| \\
&\leq \sum_{i=0}^{n} C_i \langle x\rangle^{\beta_1 - i} \langle x\rangle^{\beta_2 - (n-i)}
\end{aligned}$$

$$
\begin{aligned}
& \quad [\text{ where } C_i := \frac{n!}{i!(n-i)!} \max(c_{f,i}, c_{g,n-i}) \text{]} \\
&= \langle x\rangle^{\beta_1+\beta_2-n} \sum_{i=0}^{n} C_i \\
&= d_n \,\langle x\rangle^{\beta_1+\beta_2-n}, \quad d_n > 0.
\end{aligned}
$$

$$\text{Thus } \; fg \;\in\; \mathfrak{A}. \quad \square \tag{1.3}$$

Definition 1.4. The *support of* f is the set

$$\operatorname{supp}(f) := \overline{\{x \in \mathbb{R} \;:\; f(x) \neq 0\}}.$$

This notion of support of a function will feature prominently in the rest of our work.

The algebra $\mathfrak{A}$ contains the sub-algebra $C_c^\infty(\mathbb{R})$ of all smooth functions with compact support. The completions $\mathfrak{A}_n$ of $\mathfrak{A}$ or $C_c^\infty(\mathbb{R})$ with respect to the norms

$$\|f\|_n := \sum_{r=0}^{n} \int_{-\infty}^{\infty} \left|f^{(r)}(x)\right| \langle x\rangle^{r-1}\, dx \tag{1.4}$$

are also algebras under pointwise multiplication, and much of what we prove below could be extended to these spaces. In fact we have the following.

Lemma 1.5. *The space $C_c^\infty(\mathbb{R})$ is dense in $\mathfrak{A}$ for each norms $\|.\|_{n+1}$.*

Proof. Suppose that $f \in \mathfrak{S}^\beta$ for some $\beta < 0$. Let $\phi \in C_c^\infty$ such that

$$\phi(s) = \begin{cases} 1, & |s| < 1\,, \\ 0, & |s| > 2\,. \end{cases}$$

Set $\phi_m(s) := \phi(s/m)$ and $f_m := \phi_m f$. If $n \geq 1$ then

$$
\begin{aligned}
\|f - f_m\|_{n+1} &= \sum_{r=0}^{n+1} \int_{-\infty}^{\infty} \left|\frac{d^r}{dx^r}\{f(x)(1-\phi_m(x))\}\right| \langle x\rangle^{r-1}\, dx. \\
&\leq \sum_{r=0}^{n+1} \int_{-\infty}^{\infty} \sum_{k=0}^{r} \frac{r!}{k!(r-k)!} \left|\frac{d^k}{dx^k} f(x)\right| \left|\frac{d^{r-k}}{dx^{r-k}}(1-\phi_m(x))\right| \langle x\rangle^{r-1}\, dx,
\end{aligned}
$$

by the Leibniz formula.

We make the following observations:

1. For $k = r$,

$$
\begin{aligned}
\left|\frac{d^k}{dx^k} f(x)\right| & \left|\frac{d^{r-k}}{dx^{r-k}}(1-\phi_m(x))\right| \langle x\rangle^{r-1} = \left|\frac{d^r}{dx^r} f(x)\right| |1-\phi_m(x)|\, \langle x\rangle^{r-1} \\
&\leq c\, \langle x\rangle^{\beta-r}\, |1-\phi_m(x)|\, \langle x\rangle^{r-1}
\end{aligned}
$$

$$= \quad c\,|1-\phi_m(x)|\,\langle x\rangle^{\beta-1} \quad \text{for some } c \in (0,\infty).$$

2. $\operatorname{supp}\left(\frac{d^{r-k}}{dx^{r-k}}(1-\phi_m(x))\right) \subset \{x \;:\; m \leq |x| \leq 2m\}$ for $k < r$, while $\operatorname{supp}(1-\phi_m(x)) \subset \{x \;:\; |x| > m\}$.

3. For $s \geq 1$ we have the bound,

$$\left|\frac{d^s}{dx^s}(1-\phi_m(x))\right| \leq c_s m^{-s}\chi_m(x) \leq c'_s\,\langle x\rangle^{-s}\,\chi_m(x)$$

valid for $m \geq 2$, where χ_m is the characteristic function of $\{x \;:\; m \leq |x| \leq 2m\}$.

4. From 1, we conclude that $\left|\frac{d^k}{dx^k}f(x)\right|\left|\frac{d^{r-k}}{dx^{r-k}}(1-\phi_m(x))\right|\langle x\rangle^{r-1} \leq cc'_s\,\langle x\rangle^{\beta-1}\,\chi_m$ for $0 \leq k < r$.

These yield

$$\|f - f_m\|_{n+1} \leq \tilde{c}\sum_{r=0}^{n+1}\int_{|x|>m}\langle x\rangle^{\beta-1}\,dx \quad \text{for some } \tilde{c} > 0$$

which converges to 0 as $m \to \infty$. □

It is important for application that the functions in $\mathfrak{A}$ need not be $\mathbb{R}$-integrable.

2. Functions that Lie in $\mathfrak{A}$

Definition 2.1. Let $\mathbf{B}_b(\mathbb{R})$ denote the space of bounded complex valued functions on $\mathbb{R}$ with the uniform norm. A set $\mathfrak{F} \subset \mathbf{B}_b(\mathbb{R})$ is said to *distinguish between points of* $\mathbb{R}$ if for each pair $s, t \in \mathbb{R}$ with $s \neq t$, there is a function $f \in \mathfrak{F}$ such that $f(s) \neq f(t)$.

Lemma 2.2. (Stone-Weierstrass Theorem) *Let $\mathfrak{F}$ be a closed sub-algebra of $C_0(\mathbb{R})$, with the supremum norm $\|\cdot\|_\infty$, and closed with respect to complex conjugation. Then $\mathfrak{F} = C_0(\mathbb{R})$ if and only if $\mathfrak{F}$ distinguishes between points of $\mathbb{R}$ and for each finite point of $\mathbb{R}$, contains a function which does not vanish there.*

Proof. See for example Dunford and Schwartz [2, p. 274]. □

Example 2.3. Let $w \in \mathbb{C} \setminus \mathbb{R}$ and set $r_w := \frac{1}{w-x}$, $x \in \mathbb{R}$ then $r_w \in \mathfrak{A}$.

Indeed

$$\frac{d^n}{dx^n}r_w(x) = \frac{n!}{(w-x)^{n+1}} \qquad \text{for all} \quad n \geq 0$$

showing that r_w is smooth on $\mathbb{R}$.

Next,

$$\begin{aligned}\left|\frac{d^n}{dx^n} r_w(x)\right| &= \frac{n!}{|w-x|^{n+1}} \leq \frac{2^{(n+1)/2} n! \langle w \rangle^{n+1}}{\left(\sqrt{\beta_0}\,\langle x \rangle\right)^{n+1}} \\ &= \frac{n! \left(\sqrt{2}\,\langle w \rangle\right)^{n+1}}{(\beta_0)^{(n+1)/2}} \langle x \rangle^{-1-n} \quad \text{for all } x \in \mathbb{R}, \text{ and all } n \geq 0. \qquad (2.1)\end{aligned}$$

With $\beta_0 \in (0, 1 - \langle \Im w \rangle^{-1})$ in this case.

Thus $r_w \in \mathfrak{S}^{-1} \subset \mathfrak{A}$. □

Corollary 2.4. *$\mathfrak{A}$ is dense in $C_0(\mathbb{R})$ with respect to uniform norm.*

Proof. Note that $\mathfrak{A}$ is closed with respect to complex conjugation, see Remark 1.1.

For $x, y \in \mathbb{R}$,

$$x \neq y \iff r_w(x) \neq r_w(y) \quad \text{for some } w \notin \mathbb{R}.$$

But from Example 2.3, $r_w \in \mathfrak{A}$ for all $w \notin \mathbb{R}$. Thus $\mathfrak{A}$ distinguishes points of $\mathbb{R}$. Therefore by Stone-Weierstrass Theorem (Lemma 2.2), $\overline{\mathfrak{A}} = C_0(\mathbb{R})$ with respect to the uniform norm. □

We are now in a position to prove the following perturbation result:

Lemma 2.5. *If $f \in \mathfrak{A}$ and $c, w \in \mathbb{C}$ with $\Im w \neq 0$ then $(x+c)(w-x)^{-1} f$, $(f+c)(w-x)^{-1} \in \mathfrak{A}$.*

Proof.

$$(x+c)(w-x)^{-1} f = \{-1 + (c+w)(w-x)^{-1}\} f = -f + (c+w) r_w f$$

(where $r_w := (w-x)^{-1}$), and

$$(f+c)(w-x)^{-1} = f r_w + c r_w.$$

Hence the result follows from Example 2.3 and Theorem 1.3. □

Theorem 2.6. *For an arbitrary $t \in \mathbb{R}$ and $f \in \mathfrak{A}$, define $\acute{f}_t$ by*

$$\acute{f}_t(x) := \begin{cases} \frac{f(t)-f(x)}{t-x}, & x \neq t\,, \\ f'(t), & x = t\,. \end{cases}$$

Then $\acute{f}_t \in \mathfrak{A}$.

Proof. For $x \neq t$,

$$\begin{aligned}\acute{f}_t^{(m)}(x) &= \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} f^{(k)}(x)(-1)^{m-k}(m-k)!(t-x)^{k-m-1} \\ &\quad + (m!) f(t)(t-x)^{-m-1}(-1)^m.\end{aligned}$$

Thus

$$
\begin{aligned}
s\left|\acute{f}_t^{(m)}(x)\right| &\le \sum_{k=0}^{m}\frac{m!}{k!(m-k)!}\left|f^{(k)}(x)\right|\left|t-x\right|^{k-m-1}+m!\left|f(t)\right|\left|t-x\right|^{-m-1}\\
&\le \frac{m!}{\left|t-x\right|^{m+1}}\left(\sum_{k=0}^{m}\frac{c_k}{k!(m-k)!}\langle x\rangle^{\beta-k}\left|t-x\right|^{k}+c_m\langle t\rangle^{\beta}\right)\\
&\le \frac{m!}{\left|t-x\right|^{m+1}}\left(\sum_{k=0}^{m}\frac{c_k}{k!(m-k)!}\langle x\rangle^{\beta-k}2^k\langle t\rangle^{k}\langle x\rangle^{k}+c_m\langle t\rangle^{\beta}\right)\\
&\quad (\text{Using } \langle u+v\rangle\le 2\langle u\rangle\langle v\rangle)\\
&\le \frac{m!}{\left|t-x\right|^{m+1}}\left(\langle x\rangle^{\beta}\sum_{k=0}^{m}\frac{c_k 2^k}{k!(}\langle t\rangle^{k}+c_m\langle t\rangle^{\beta}\right);\ x\neq t\\
&\le d_m\langle x\rangle^{\beta-1-m}+d'_m\langle x\rangle^{-1-m}\\
&\le (d_m+d'_m)\langle x\rangle^{-1-m}\quad \text{since } \beta<0.
\end{aligned}
$$

Next, the fact that $f\in C^\infty(\mathbb{R})$ implies that there exists a function f_m, continuous on some neighbourhood $(t-\delta_m, t+\delta_m)$, $\delta_m>0$; of t such that

$$
f_m(x) := \begin{cases} \frac{f^{(m)}(t)-f^{(m)}(x)}{t-x}, & x\in(t-\delta_m,t+\delta_m)\setminus\{t\}\\ f^{(m+1)}(t), & x=t. \end{cases}
$$

From Taylor's expansion

$$
f(t)=f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2}f''(x)+\frac{1}{2}\int_x^t (t-y)^2 f'''(y)dy
$$

we have

$$
\frac{f(t)-f(x)}{t-x}=f'(x)+\frac{(t-x)}{2}f''(x)+\frac{1}{2(t-x)}\int_x^t (t-y)^2 f'''(y)dy.
$$

Therefore

$$
\begin{aligned}
\acute{f}_t^{(1)}(t) &:= \lim_{x\to t}\frac{\acute{f}_t(t)-\acute{f}_t(x)}{t-x}\\
&= \lim_{x\to t}\frac{f'(t)-\frac{f(t)-f(x)}{t-x}}{t-x}\\
&= \lim_{x\to t}\left(\frac{f'(t)-f'(x)}{t-x}-\frac{1}{2}f''(x)-\frac{1}{2(t-x)^2}\int_x^t (t-y)^2 f'''(y)dy\right)\\
&= \frac{1}{2}f''(t).
\end{aligned}
$$

Inductively, $\acute{f}_t^{(m)}(t)=\frac{1}{(m+1)!}f^{(m+1)}(t)$.

Consider $[t-\epsilon,t+\epsilon]\subset(t-\delta_m,t+\delta_m)$ for some $\epsilon\ :\ 0<\epsilon<\delta_m$. Then:

1. f_m and $\acute{f}_t^{(m)}$ are continuous and bounded on $[t-\epsilon, t+\epsilon]$.
2. $(m+1)!\acute{f}_t^{(m)}(t) = f_m(t) = f^{(m+1)}(t)$.

Because of continuity of $\acute{f}_t^{(m)}$ and $f^{(m+1)}$, we can find $\rho_m \in \mathbb{R}$ such that

$$\begin{aligned} \left|\acute{f}_t^{(m)}(x)\right| &\leq \left|f^{(m+1)}(x)\right| + \rho_m, \quad \text{on } [t-\epsilon, t+\epsilon] \\ &\leq c_{m+1} \langle x \rangle^{\beta-m-1} + |\rho_m| \\ \text{which implies } \langle x \rangle^{m+1-\beta} \left|\acute{f}_t^{(m)}(x)\right| &\leq c_{m+1} + |\rho_m| \langle x \rangle^{m+1-\beta} . \end{aligned}$$

Since $\langle x \rangle^{m+1-\beta}$ is continuous on $[t-\epsilon, t+\epsilon]$, it is bounded and attains its bounds there. Let $c'_{m+1} := c_{m+1} + |\rho_m| \max_{x\in[t-\epsilon,t+\epsilon]} \left\{ \langle x \rangle^{m+1-\beta} \right\}$. Then

$$\begin{aligned} \left|\acute{f}_t^{(m)}(x)\right| &\leq c'_{m+1} \langle x \rangle^{\beta-m-1} \\ &\leq c'_{m+1} \langle x \rangle^{-m-1} \text{ (since } \beta < 0 \text{ and } \langle x \rangle \geq 1) \\ & \quad x \in [t-\epsilon, t+\epsilon]. \end{aligned}$$

Thus $\acute{f}_t \in \mathfrak{S}^{-1}$. □

3. Extensions of $C^\infty([0,\infty))$ Functions to $\mathbb{R}$

We next present a series of results about smooth functions initially defined on the half real line but extendible to the whole real line. In particular we wish to obtain an extension preserving the decay condition (1.1).

Lemma 3.1. *There are sequences $\{a_k\}$, $\{b_k\}$ such that:*

1. $b_k < 0$ for all k.

2. $\sum_{k=0}^{\infty} |a_k| |b_k|^n < \infty$, $n = 0, 1, 2, \ldots$.

3. $\sum_{k=0}^{\infty} a_k (b_k)^n = 1$ for $n = 0, 1, 2, \ldots$.

4. $b_k \to -\infty$ as $k \to \infty$.

Proof. see Seeley, [5]. □

Theorem 3.2. (Seeley) *Let $\phi \in C^\infty(\mathbb{R})$ be such that ϕ is bounded on $\mathbb{R}$ and*

$$\phi(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & t \geq 2. \end{cases}$$

Define $E : C^\infty([0,\infty)) \to C^\infty(\mathbb{R})$ *by*

$$(Ef)(t) := \begin{cases} \sum_{k=0}^{\infty} a_k \phi(b_k t) f(b_k t), & t < 0\,, \\ f(t), & t \geq 0. \end{cases}$$

Here $\{a_k\}, \{b_k\}$ *are the sequences described in Lemma 3.1.*

Then E *is a continuous linear extension operator.*

Proof. Again, see Seeley, [5]. □

Lemma 3.3. *If* $f \in C^\infty(\mathbb{R}^+)$ *with*

$$\left| \frac{d^r}{dx^r} f(x) \right| \leq c_r \langle x \rangle^{\beta - r} \tag{3.1}$$

for some $\beta < 0$, *all* $r \geq 0$ *and for all* $x \geq 0$, *then* $Ef \in \mathfrak{S}^\beta \subset \mathfrak{A}$, *where* E *is Seeley's extension operator.*

Proof. Using notations of Theorem 3.2 and Lemma 3.1, first observe that $\phi^{(r-\nu)}(b_k x)$ vanishes everywhere except on the set $Q := \{x \ : \ 1 \leq b_k x \leq 2\}$. So for $x \in Q$, we have $1 \leq (b_k x)^2 \leq 4$ whence $2 \leq 1 + (b_k x)^2 \leq 5$ or equivalently $\frac{1}{\sqrt{5}} \leq \frac{1}{\langle b_k x \rangle} \leq \frac{1}{\sqrt{2}}$. So we can find a constant n_r such that $\langle b_k x \rangle^{\beta - \nu} \leq n_r \langle b_k x \rangle^{\beta - r}$, $\beta < 0$ and all $0 \leq \nu \leq r$. Thus

$$\begin{aligned} \left| \frac{d^r}{dx^r} (Ef)(x) \right| &\leq \sum_{k=0}^{\infty} |a_k|\, |b_k|^r \sum_{\nu=0}^{r} \frac{r!}{\nu!(r-\nu)!} \left| \phi^{(r-\nu)}(b_k x) \right| \left| f^{(\nu)}(b_k x) \right| \\ &\leq \sum_{k=0}^{\infty} |a_k|\, |b_k|^r \sum_{\nu=0}^{r} \frac{r!}{\nu!(r-\nu)!} \left| \phi^{(r-\nu)}(b_k x) \right| c_\nu \langle b_k x \rangle^{\beta - \nu} \\ &\leq \sum_{k=0}^{\infty} |a_k|\, |b_k|^r M_r \sum_{\nu}^{r} c_\nu n_r \langle b_k x \rangle^{\beta - \nu} \qquad \text{for all } \ x < 0\,, \end{aligned}$$

where

$$\begin{aligned} M_r &:= \max_{0 \leq \nu \leq r} \left\{ \frac{(r+1)!}{\nu!(r-\nu)!} \sup_{x<0} \left| \phi^{(r-\nu)}(b_k x) \right| \right\} \\ &< \infty, \quad \text{since } \phi^{(m)} \text{ is bounded on } \mathbb{R} \quad \text{for all } \ m. \end{aligned}$$

Next, since $b_k \to -\infty$ as $k \to \infty$ we can find $\tilde{c} \in \mathbb{R}$ such that $\left\langle \frac{1}{b_k} \right\rangle \leq \frac{\tilde{c}}{\sqrt{2}}$ for all k and hence $\langle x \rangle = \left\langle \frac{1}{b_k} b_k x \right\rangle \leq \sqrt{2} \left\langle \frac{1}{b_k} \right\rangle \langle b_k x \rangle \leq \tilde{c} \langle b_k x \rangle$. Thus

$$\begin{aligned} \frac{1}{\langle b_k x \rangle} &\leq \frac{\tilde{c}}{\langle x \rangle} \quad \text{for all } \ x \in \mathbb{R} \ \text{ and all } \ b_k \\ \text{implies } \langle b_k x \rangle^{\beta - \nu} &\leq \tilde{c}^{\nu - \beta} \langle x \rangle^{\beta - \nu} \ \text{ for all } \ x \in \mathbb{R}, \text{ and all } \ k, \nu \in \mathbb{N}. \end{aligned}$$

So we can choose c'_r so that

$$r \cdot \max_{0\le\nu\le r}(c_\nu) n_r \langle b_k x\rangle^{\beta-r} \le c'_r \tilde{c}^{r-\beta} \langle x\rangle^{\beta-r} \qquad \text{for all} \;\; x \in \mathbb{R}$$

and hence,

$$\begin{aligned} \left|\frac{d^r}{dx^r}(Ef)(x)\right| &\le \sum_{k=0}^{\infty} |a_k|\,|b_k|^r\, c'_r M_r \tilde{c}^{r-\beta}\langle x\rangle^{\beta-r} \\ &\le c'_r \tilde{c}^{r-\beta}\langle x\rangle^{\beta-r}\sum_{k=0}^{\infty}|a_k|\,|b_k|^r \\ &=: N_r \langle x\rangle^{\beta-r}, \quad x<0 \quad \text{and some } N_r>0 \end{aligned} \tag{3.2}$$

after summing up the series which converges by Lemma 3.1.

Now set $D_r := \max\{c_r, N_r\}$ then

$$\left|\frac{d^r}{dx^r}(Ef)(x)\right| \le D_r \langle x\rangle^{\beta-r}$$

for some $D_r > 0$, for all $r \ge 0$ and for all $x \in \mathbb{R}$. That is $Ef \in \mathfrak{S}^\beta \subset \mathfrak{A}$. □

Theorem 3.4. *Let $f \in C^\infty(\mathbb{R}^+)$ satisfying (3.1) and define $\|.\|_n^+$ by*

$$\|f\|_n^+ := \sum_{r=0}^{n}\int_0^\infty \left|f^{(r)}(x)\right| \langle x\rangle^{r-1}\,dx.$$

Then

$$\|Ef\|_n \le c_n \|f\|_n^+$$

for some $c_n > 0$ (where E is Seeley's extension operator defined in Theorem 3.2).

Proof.

$$\|Ef\|_n = \sum_{r=0}^{n}\left\{\int_0^\infty \left|f^{(r)}(x)\right|\langle x\rangle^{r-1}\,dx + \int_{-\infty}^0 \left|F^{(r)}(x)\right|\langle x\rangle^{r-1}\,dx\right\},$$

where

$$F(x) := \sum_{k=0}^{\infty} a_k \phi(b_k x) f(b_k x).$$

Therefore

$$\begin{aligned} F^{(r)}(x) &= \sum_{k=0}^{\infty} a_k \sum_{\nu=0}^{r} \frac{r!}{\nu!(r-\nu)!}\frac{d^{r-\nu}}{dx^{r-\nu}}\phi(b_k x)\frac{d^\nu}{dx^\nu}f(b_k x) \\ &= \sum_{k=0}^{\infty} a_k \sum_{\nu=0}^{r} \frac{r!}{\nu!(r-\nu)!} b_k^{r-\nu}\phi^{(r-\nu)}(b_k x) b_k^\nu f^{(\nu)}(b_k x) \end{aligned}$$

$$= \sum_{k=0}^{\infty} a_k b_k^r \sum_{\nu=0}^{r} \frac{r!}{\nu!(r-\nu)!} \phi^{(r-\nu)}(b_k x) f^{(\nu)}(b_k x).$$

Also,

$$\begin{aligned} \langle x \rangle^{r-1} dx &\leq -\frac{\langle b_k x \rangle^{r-1}}{|b_k|^{r-1}} \left\langle \frac{\langle b_k^2 \rangle}{|b_k|} \right\rangle^{r-1} \frac{1}{-b_k} d(b_k x) \\ &= \frac{\langle b_k x \rangle^{r-1}}{|b_k|^{r}} \left\langle \frac{\langle b_k^2 \rangle}{|b_k|} \right\rangle^{r-1} d(b_k x) \end{aligned}$$

using second part of Lemma 3.1. Thus

$$\begin{aligned} \|Ef\|_n &= \|f\|_n^+ + \sum_{r=0}^{n} \left\{ \int_{-\infty}^{0} \left| F^{(r)}(x) \right| \langle x \rangle^{r-1} dx \right\} \\ &\leq \|f\|_n^+ + \sum_{r=0}^{n} \sum_{k=0}^{\infty} |a_k| \, |b_k|^r \sum_{\nu=0}^{r} \frac{r!}{\nu!(r-\nu)!} \int_{-\infty}^{0} \left| \phi^{(r-\nu)}(b_k x) \right| \\ &\qquad \times \left| f^{(\nu)}(b_k x) \right| \left\langle \frac{\langle b_k^2 \rangle}{|b_k|} \right\rangle^{r} \frac{\langle b_k x \rangle^{r-1}}{|b_k|^r} d(b_k x) \\ &\leq \|f\|_n^+ + \sum_{r=0}^{n} M_r \sum_{k=0}^{\infty} |a_k| \left\langle \frac{\langle b_k^2 \rangle}{|b_k|} \right\rangle^{r-1} \sum_{\nu=0}^{r} \int_{0}^{\infty} \left| f^{(\nu)}(t) \right| \langle t \rangle^{\nu-1} dt \end{aligned}$$

$$\text{(where } M_r := \max_{0\leq \nu \leq r} \left\{ \frac{r!}{\nu!(r-\nu)!} \sup_{x<0} \left| \phi^{(r-\nu)}(b_k x) \langle b_k x \rangle^{r-\nu} \right| \right\} < \infty,$$

$$\text{since} \quad \phi^{(m)}(b_k x) \langle b_k x \rangle^m = 0 \quad \text{for all } m \text{ and all } x : b_k x > 2).$$

That is

$$\begin{aligned} \|Ef\|_n &\leq \|f\|_n^+ + \sum_{r=0}^{n} M_r \|f\|_r^+ \sum_{k=0}^{\infty} |a_k| \left\langle \frac{\langle b_k^2 \rangle}{|b_k|} \right\rangle^{r} \\ &\leq \|f\|_n^+ (1+n) L_n \|f\|_n^+ \\ \text{where} \quad L_n &:= \max_{0\leq r\leq n} \left(M_r \sum_{k=0}^{\infty} |a_k| \left\langle \frac{\langle b_k^2 \rangle}{|b_k|} \right\rangle^{r} . \right) \end{aligned}$$

But

$$\left\langle \frac{\langle b_k^2 \rangle}{|b_k|} \right\rangle^{r} = \left\langle |b_k| \left\langle \frac{1}{b_k^2} \right\rangle \right\rangle^{r} = |b_k|^r \left\langle \frac{1}{b_k^2} \right\rangle^{r} \left\langle \frac{1}{|b_k| \left\langle \frac{1}{b_k^2} \right\rangle} \right\rangle^{r}.$$

Since $b_k \to -\infty$ as $k \to \infty$, we have $\left\langle \frac{1}{b_k^2} \right\rangle \to 1$ as $k \to \infty$ and

$\frac{1}{\left\langle \frac{1}{b_k^2} \right\rangle} \leq 1$ for any k. Therefore we can find a constant $N_r > 0$ such that $\left\langle \frac{1}{b_k^2} \right\rangle^r \left\langle \frac{1}{|b_k| \left\langle \frac{1}{b_k^2} \right\rangle} \right\rangle^r \leq N_r$ for all k and hence

$$\begin{aligned} L_n &\leq \max_{0\leq r\leq n} \left(M_r N_r \sum_{k=0}^{\infty} |a_k| \, |b_k|^r \right) \\ (&< \infty \qquad \text{by Lemma 3.1.)} \end{aligned}$$

So,

$$\|Ef\|_n \leq c_n \, \|f\|_n^+ \tag{3.3}$$

with $c_n = 1 + (n+1)L_n$. □

Example 3.5. Let $f(x) := e^{-x^n t}$, $t > 0$, integer $n \geq 1$. Then $Ef \in \mathfrak{A}$, where E is Seeley's extension operator.

Indeed,

$$f^{(r)}(x) \ \to \ u_r < \infty \ as \ x \ \to \ 0 \quad \text{for all } \ r \geq 0. \tag{3.4}$$

Thus by Theorem 3.2 $Ef \in C^\infty(\mathbb{R})$.

Further,

$$f^{(r)}(x) = \sum_{k=1}^{r} e_{r,k}(n)(-1)^k t^k x^{nk-r} f(x), \quad r \geq 1\,,$$

where $e_{r,k}(n) \in \mathbb{Z}$ is defined by

$$e_{r,k}(n) = \begin{cases} \prod_{s=0}^{r-1}(n-s), & \text{if } k = 1\,, \\ n^r, & \text{if } \ k = r\,, \\ (nk - r + 1)e_{r-1,k}(n) + n e_{r-1,k-1}(n), & \text{if } 2 \ \leq k \leq r-1\,. \end{cases}$$

Therefore for $x > 1$, and $r \geq 1$

$$\begin{aligned} \left|f^{(r)}(x)\right| &= \sum_{k=1}^{r} e_{r,k}(n) t^k \left|x^{nk-r}\right| |f(x)| \\ &\leq c_r \, |x|^{nr-r} \, |f(x)| \sum_{k=1}^{r} t^k \\ &= c_r \, |x|^{nr-r} \, |f(x)| \, t^r \sum_{k=0}^{r-1} \frac{1}{t^k} \qquad (3.5) \end{aligned}$$

$$\text{with } c_r \ := \ \max_{1\leq k\leq r} \{e_{r,k}(n)\}$$

Also by means of Taylor series expansion,

$$|f(x)| = \left|e^{-x^n t}\right| \leq \frac{(r+1)!}{t^{r+1}\,|x|^{nr+n}} \quad x>0, \quad r \geq 0. \tag{3.6}$$

Substituting (3.6) into (3.5) and using $\frac{1}{|x|} = \frac{\langle 1/x\rangle}{\langle x\rangle} \leq \frac{\sqrt{2}}{\langle x\rangle}$ for $x>1$ we get,

$$\begin{aligned}
\left|f^{(r)}(x)\right| &\leq c_r\,|x|^{nr-r}\,\frac{(r+1)!t^r\sum_{k=0}^{r-1}t^{-k}}{t^{r+1}\,|x|^{nr+n}} \\
&= c_r\frac{(r+1)!\sum_{k=0}^{r-1}t^{-k}}{t}\,|x|^{-n-r}, \quad x>1 \\
&\leq c_r\frac{(r+1)!\sum_{k=0}^{r-1}t^{-k}\left(\sqrt{2}\right)^{n+r}}{t}\,\langle x\rangle^{-n-r} \\
&=: d_r\,\langle x\rangle^{-n-r}, \quad r \geq 1.
\end{aligned}$$

From (3.6) and comments following it we can set $d_0 := \frac{(\sqrt{2})^n}{t}$.

For the case $x \leq 1$, since $f^{(r)}(x)$ is bounded on $[0,1]$ for all $r \geq 0$,

$$\begin{aligned}
\left|f^{(r)}(x)\right| &\leq \sup_{x\in[0,1]}\left|f^{(r)}(x)\right| =: \left|f^{(r)}\right|_I < \infty \\
(\text{with } I &:= [0,1]).
\end{aligned}$$

But then we can find a constant $M_r' > 0$ such that

$$\left|f^{(r)}\right|_I \leq d_r M_r'\,\langle x\rangle^{-n-r}, \qquad x \in I,$$

since $1 \leq \langle x\rangle \leq \sqrt{2}$ for $x \in [0,1]$. Now set

$$p_r := d_r \max\left\{1, M_r'\right\}, \quad r \geq 0, \tag{3.7}$$

whence

$$\left|f^{(r)}(x)\right| \leq p_r\,\langle x\rangle^{-n-r} \qquad \text{for all } x \in [0,\infty); \quad r \geq 0. \tag{3.8}$$

Thus by Lemma 3.3, $Ef \in \mathfrak{S}^{-n} \subset \mathfrak{A}$. □

Remark 3.6. Note that if $t \geq 1$, then the constant d_r (and hence p_r), does not depend on t, since in this case

$$\begin{aligned}
d_r &= c_r\frac{(r+1)!\left(\sqrt{2}\right)^{n+r}\sum_{k=0}^{r-1}t^{-k}}{t} \\
&\leq c_r\frac{r(r+1)!\left(\sqrt{2}\right)^{n+r}}{t} \leq c_r r(r+1)!\left(\sqrt{2}\right)^{n+r} \\
&=: d_r'.
\end{aligned}$$

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