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LIMIT OF k-GONAL CURVES AND TOPOLOGICAL TYPES OF STABLE CURVES

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Abstract: Here we give examples of topological types for genus g stable curves such that no (or some) stable curve with that topological type is in the closure of the set of all smooth k-gonal curves.

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1. Introduction

For all integers $g \geq 2k-1 \geq 3$ let $\overline{\mathcal{M}}_g[k]$ denote the closure of the set of all smooth k-gonal curve of genus g. Thus $\overline{\mathcal{M}}_g[k]$ is an irreducible projective variety of dimension $2g+2k-5$. If $k \geq \lceil (g+1)/2 \rceil$, set $\overline{\mathcal{M}}_g[k] := \overline{\mathcal{M}}_g$. For any reduced projective curve Y let $\mathcal{B}(Y)$ denote the set of its irreducible components. Let X be a nodal and connected curve of arithmetic genus $g \geq 0$. Let $\mathrm{Sing}(X)''$ (resp. $\mathrm{Sing}'(X)$) denote the set of all singular points of X lying on exactly two (resp. one) irreducible components of X. Consider the following non-oriented marked graph $\|X\|$. The vertices of $\|X\|$ are the irreducible components of X. For any $T \in \mathcal{B}(X)$ let $[T]$ denote the associated vertex of $\|X\|$. For each $T \in \mathcal{B}(X)$ we give as a marking the non-negative integer q_T, where q_T is the geometric genus of T. $\|X\|$ contains $\sharp(\mathrm{Sing}(X)' \cap T)$ loops with $[T]$ as their vertex. For all $T, J \in \mathcal{B}(X)$, such that $T \neq J$ the vertices $[T]$ and $[J]$ of $\|X\|$ are joined by $\sharp(T \cap J)$ edges. Call τ the abstract marked graph $\|X\|$. If we forget the marking, i.e. if we forget the integers q_T, $T \in \mathcal{B}(X)$, then $\|X\|$ becomes the classical dual graph $|X|$ of the nodal curve X. The set of all nodal

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projective curves Y such that $\|Y\| \cong \tau$ (as marked graphs) is parametrized by an irreducible algebraic variety $\mathcal{M}[\tau]$. If $\mathbb{K} = \mathbb{C}$, then the topological type of the complex analytic space $X(\mathbb{C})$ is uniquely determined by the marked graph τ and two non-isomorphic marked graphs give topologically different complex analytic spaces. We may extend the notion of marked graph to pointed curves (see [1]). Fix a topological type τ for nodal connected curves, say $\tau = \|X\|$. For all $T, J \in \mathcal{B}(X)$, $T \neq J$, let $q_T \geq 0$ be the associated marking of $[T]$, a_T the number of loops based at $[T]$ and $a_{TJ} := \sharp(T \cap J)$ the number of edges with $[T]$ and $[J]$ as their vertices. To any nodal curve X we may also associate another graph $|X|$, obtained from $\|X\|$ deleting the loops and the markings, i.e. $\mathcal{B}(X)$ is the set of all vertices of $|X|$ and two vertices $[T]$, $J]$ of $|X|$, $T \neq j$, are joined by $\sharp(T \cap J)$ edges. We may also associate to X the following "partially marked" graph $\lfloor X \rfloor$, which is intermediate between $|X|$ and $\|X\|$: $\lfloor X \rfloor$ is like $|X|$, but at each vertex $[T]$ such that $p_a(T) > 0$ we add a symbol, say the symbol $>$. For any marked graph τ we obtain a partially marked graph $\lfloor \tau \rfloor$ and an unmarked graph $|\tau|$. Now assume $g \geq 2$ and call $\mathbb{G}(g)$ (resp. $\mathbb{G}(g)_1$, resp. $\mathbb{G}(g)_2$) the set of all marked (resp. partially marked, resp. unmarked) graphs assoociated to a stable curve of genus g. For any $\tau \in \mathbb{G}(g)_i$, $i = 1, 2$, we get a locally closed algebraic subset of $\overline{\mathcal{M}}_g$ corresponding to the curves with τ as partially marked (resp. unmarked) graph. For any partially marked grap $\lfloor X \rfloor$, let $a_0(\lfloor X \rfloor)$, (resp. $a_>(\lfloor X \rfloor)$) be the number of the unmarked (resp. marked) vertices of $\lfloor X \rfloor$. Set $w(\lfloor X \rfloor) = a_0(\lfloor X \rfloor) + 2a_>(\lfloor X \rfloor)$ (the weight of $\lfloor X \rfloor$).

Theorem 1. *Let η be a partially marked connected graph such that every unmarked graph of η has valence at least 3. Set $w := w(\eta)$ and fix an integer $k \geq w$. For each unmarked vertex v of η set $T_v := \mathbb{P}^1$ and $q_v = 0$. For every marked vertex v fix an integer $q_v > 0$ and an integral nodal curve Y_v such that $p_a(Y_v) = q_v$; if $q_v \geq 2$ assume that Y_v is hyperelliptic. Let g be the genus of an arbitrary nodal curve Y such that $\eta = \lfloor Y \rfloor$ and for every $v \in \eta$ the irreducible component of Y asssociated to v has arithmetic genus q_v. Then there exists $X \in \mathcal{M}[\eta]$ such that for every vertex v of η the irreducible component X_v of X associated to v is isomorphic to T_v.*

As a particular case of Theorem 1 we get the following result.

Corollary 1. *Fix integers $g \geq 2k - 1 \geq 3$ and $\eta \in (G)(g)_2$. Assume $w := w(\eta) \leq k$. Fix any $\tau \in \mathbb{G}(g)$ such that $\lfloor \tau \rfloor = \eta$. Then $\mathcal{M}[\tau] \cap \overline{\mathcal{M}}_g[k] \neq \emptyset$.*

There are many cases in which $\mathcal{M}[\eta] \cap \overline{\mathcal{M}}_g[k] = \emptyset$ for some integer $k \geq 2$ and some partially marked graph η. Of course, some examples are known in the literature (see the appendix by Sung Wong Park to [2] for the case of graph curves).

Here we prove the following result.

Theorem 2. *Fix $\eta = \lfloor Y \rfloor$ with the following properties. Set $t := \sharp(\mathcal{B}(Y))$ and $w := w(\eta)$. Fix an integer k such that $2 \le k \le w-1$ and any integer g such that there is a genus g example in $\mathcal{M}[\eta]$. Assume the existence of an ordering $Y_1, \dots, Y_t$ of the irreducible components of Y such that $\sharp((Y_1 \cup \cdots \cup Y_i) \cap Y_{i+1}) \ge k+1$ for all $i \in \{1, \dots, t-1\}$. Then $\mathcal{M}[\eta] \cap \overline{\mathcal{M}}_g[k] = \emptyset$.*

We also have example in which for each $X \in \mathcal{M}[\eta] \cap \overline{\mathcal{M}}_g[k]$ there are subsets of X with a special property with respect to any degree k admissible covering associated to a curve X' with X as stable reduction (see Example 1).

2. The Proofs and an Example

Example 1. Fix an integer $g \ge 2$ and any $X \in \overline{\mathcal{M}}_g$ with two irreducible and smooth components X_1 and X_2 and $\sharp(X_1 \cap X_2) = 2$. Since X is stable, $p_a(X_1) > 0$ and $p_a(X_2) > 0$. Assume the existence of an admissible covering $f : X' \to D$ with X as stable reduction of X'. Let $Y_i \cong X_i$, $i = 1, 2$, denote the irreducible component of X' corresponding to Y_i and take any $\{Q_1, Q_2\} \in X'$ with image $X_1 \cap X_2$ in X. Since D is a tree, it has no loop. Hence either $f(Q_1) = f(Q_2)$ or $f(Y_1) = f(Y_2)$. Since each X_i has positive degree, the latter case cannot occur if f has degree 2 or degree 3. Roughly speaking, if $2 \le k \le 3$ and $X \in \overline{\mathcal{M}}_g[k]$, then degree k admissible covering associated to X maps the two points of $X_1 \cap X_2$ into the same point of a $\mathbb{P}^1$.

Proof of Theorem 1. The condition on the unmarked vertices of η gives the stability of any nodal curve Y such that $\lfloor Y \rfloor = \eta$. Since $\overline{\mathcal{M}}_g[w] \subseteq \overline{\mathcal{M}}_g[k]$, it is sufficient to do the case $k = w$.

(a) Here we construct a pair (X, f), where $X \in \overline{\mathcal{M}}_g$, $f : X \to \mathbb{P}^1$ is a degree w finite morphism, $f|\mathrm{Sing}(X)$ is injective, $\lfloor X \rfloor = \eta$, and for every vertex v of η the irreducible component X_v of X corresponding to v is isomorphic to T_v. Set $W := \sqcup_v T_v$ (disjoint union). For each T_v we fix a morphisn $f_v : T_v \to \mathbb{P}^1$ such that $\deg(f_v) = 2$ if $q_v > 0$ and $\deg(f_v) = 1$ if $q_v = 0$. The map $\phi := \sqcup_v f_v : W \to \mathbb{P}^1$ is a degree w morphism. Composing some of the f_v with an automorphism of $\mathbb{P}^1$ we may produce ϕ such that $\phi|\mathrm{Sing}(W)$ is injective. We may get the pair (X, f) just gluing some pair of points of W according with the rules encoded by the unmarked graph associated to η and gluing only pairs of points with the same image in $\mathbb{P}^1$. It is possible to satisfy the latter condition, because each f_v is surjective. We may do the gluing in such a way that over each point of $\mathbb{P}^1$ at most one pair is glued and over each point of $\phi(\mathrm{Sing}(W))$ no pair is glued. Thus $f|\mathrm{Sing}(X)$ is injective.

(b) Take (X, f) as in step (a). Here we construct a degree w admissible covering (X', D, h) where X is obtained from X' contracting to a point each rational tail of X' and then taking the stable reduction. This would be sufficient to prove that $X' \in \overline{\mathcal{M}}_g$ by [3], p. 61 at the quotation of [4]. Remember that for any admissible covering (A, B, ϕ) we have $\mathrm{Sing}(A) = \phi^{-1}(\mathrm{Sing}(B))$. Let $\{P, \dots, P_a\}$, $a \geq 0$ (resp. $\{A_1, \dots, A_b\}$) be the set of all singular points of X lying on one (resp. two) irreducible components. Hence $\{P_1, \dots, P_a\}$ are exactly the singular points of X coming from one of the singular points of T_v for some vertex v, while b only depends from the unmarked graph associated to η. Set $D_0 := \mathbb{P}^1$ seen as the target of the morphism $f : X \to \mathbb{P}^1$. Let $u : X'' \to X$ be the partial normalization of X in which we only normalize the points $\{P_1, \dots, P_a\}$. Let X_1 be the quasistable curve obtained from X'' obtained inserting an exceptional curve $E_i \cong \mathbb{P}^1$, $1 \leq i \leq a$, for each P_i with $E_i \cap X'' = u^{-1}(P_i)$ for all i. We take as D the following tree with $a + b + 1$ irreducible components $D_0, D_1, \dots, D_{a+b}$, $D_i \cong \mathbb{P}^1$ for all i. We prescribe $D_i \cap D_j$ for all $1 \leq i < j \leq a+b$, $D_0 \cap D_i = \{f(P_i)\}$ for $1 \leq i \leq a+1$ and $D_0 \cap D_i = \{f(A_{i-a})\}$ for all $a + 1 \leq i \leq a + b$. For any integer i such that $1 \leq i \leq a$ fix a degree 2 morphism $u_i : E_i \to D_i$ such that $u_i^{-1}(u^{-1}(P_i)) = f(P_i)$. We obtain $f' : X_1 \to D$ just taking $f'|E_i = u_i$ for $1 \leq i \leq a$ and $f'|X'' = f \circ u$. Thus f' is finite, but not surjective (f has degree w over D_0, degree 2 over D_i for $1 \leq i \leq a$, and degree 0 over D_i for $a + 1 \leq i \leq a + b$). We will take as X' the union of X_1 and some rational tail to obtain from (X_1, D, f') a degree w admissible covering (X', D, h). Each additional tail has length 1 and there are $w - 2$ rational tails over each D_i, $1 \leq i \leq a$, and w rational tails over each D_i, $a + 1 \leq i \leq a + b$, the latter being needed only because for any admissible covering (A, B, ϕ) we have $\mathrm{Sing}(A) = \phi^{-1}(\mathrm{Sing}(B))$. $\square$

Proof of Theorem 2. Assume the existence of $X \in \mathcal{M}[\eta] \cap \overline{\mathcal{M}}_g[k]$ and choose an ordering $X_1, \dots, X_t$ of the irreducible components of X such that $\sharp((X_1 \cup \dots \cup X_i) \cap X_{i+1}) \geq k + 1$ for all $i \in \{1, \dots, t - 1\}$. Hence there is an admissible degree k covering (X', D, f) with X the stable reduction of X'. Let X'_i, $1 \leq i \leq t$, be the irreducible components of X' mapped birationally onto $X_1, \dots, X_s$ by the stable reduction map $X' \to X$. Since $k < w$, to get a contradiction it is sufficient to prove that all $f(X'_i)$ are mapped onto the same irreducible component of D. Set $A := f(X'_1)$. Since $\sharp(X_1 \cap X_2) \geq k + 1$, there is a union $B \subseteq X'$ of $X' \cup X'_2$ and some smooth rational curve with $X'_1 \cup X'_2$ as stable reduction and with cohomology group at least $\mathbb{Z}/\ell\mathbb{Z}^{\oplus k+1}$, ℓ a large prime, in the sense of étale topology. Since D is a tree, this implies $f(B) = f(X_1)$. Hence $f(X_2) = f(X_1)$. And so on, adding $X_3', \dots, X_t'$. □

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