International Journal of Pure and Applied Mathematics
Volume 52 No. 2 2009, 211-227

INFERENCE AND PREDICTION FOR A GENERALIZED EXPONENTIAL DISTRIBUTION BASED ON THE k-TH LOWER RECORDS

Iwona Malinowska[1], Dominik Szynal[2 §]

[1]Department of Mathematics
Lublin University of Technology
38a, Nadbystrzycka, Lublin, 20-618, POLAND
e-mail: i.malinowska@pollub.pl

[2]Institute of Mathematics
Maria Curie–Skłodowska University
1, Pl. M. Curie-Skłodowskiej, Lublin, 20–031, POLAND
e-mail: szynal@poczta.umcs.lublin.pl

Abstract: The minimum variance unbiased estimator (MVU estimator), the maximum likelihood estimator (ML estimator) and the Bayesian estimator for the parameter of the generalized exponential distribution are obtained based on k-th lower record values. The Bayes estimators are obtained using the symmetric loss functions: squared error, squared log error and Kullback-Leibler divergence type loss function (KLD) and the asymmetric loss functions: LINEX, General Entropy and Modified General Entropy (MGE) loss function. Interval prediction for future k-th lower record values is also presented from a Bayesian point of view. Numerical computations are given to illustrate these procedures.

AMS Subject Classification: 62F10, 62F15, 62G32
Key Words: generalized exponential distribution, minimum variance estimators, maximum likelihood estimators, Bayes estimators, k-th record values, LINEX loss function, general entropy loss function, modified general entropy, Kullback-Leibler divergence, prediction

Received: March 3, 2009

§Correspondence author

1. Introduction

Let $\{X_n,\, n \geq 1\}$ be a sequence of independent identically distributed (iid) random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. The j-th order statistic of a sample $(X_1, ..., X_n)$ is denoted by $X_{j:n}$. For a fixed $k \geq 1$ we define the sequence $L_k(n)$, $n \geq 1$, of k-th lower record times of $\{X_n,\, n \geq 1\}$ as follows:

$$L_k(1) = 1, \quad L_k(n+1) = \min\left\{j > L_k(n) : X_{k:L_k(n)+k-1} > X_{k:j+k-1}\right\},\ n \geq 1.$$

The sequence $\{Z_n^{(k)}, n \geq 1\}$ with $Z_n^{(k)} = X_{k:L_k(n)+k-1},\ n \geq 1$, is called the sequence of k-th lower record values of $\{X_n,\, n \geq 1\}$. Note that $Z_1^{(k)} = \max\{X_1, ...,\ X_k\}$ and $Z_n^{(1)} = X_{L(n)},\ n \geq 1$, are lower record values. It is known that

$$f_{Z_n^{(k)}}(z) = \frac{k^n}{(n-1)!}\left[-\ln F(z)\right]^{n-1}\left(F(z)\right)^{k-1} f(z), \quad z \in \mathbb{R}, \tag{1}$$

and

$$f_{Z_1^{(k)},...,Z_n^{(k)}}(z_1, ...z_n) = k^n \left(F(z_n)\right)^{k-1} f(z_n) \prod_{i=1}^{n-1} \frac{f(z_i)}{F(z_i)}, \quad z_1 > ... > z_n,$$

(cf. Pawlas and Szynal [16]). (2)

The probability density function (pdf) and the corresponding cumulative distribution function (cdf) of the generalized exponential distribution is

$$f(x|\theta) = \theta e^{-x}\left(1 - e^{-x}\right)^{\theta-1}, \quad 0 < x < +\infty;\ \theta > 0, \tag{3}$$

and

$$F(x|\theta) = \left(1 - e^{-x}\right)^{\theta}, \quad 0 < x < +\infty;\ \theta > 0. \tag{4}$$

Estimators of parameters for various classes of distributions for example Gumbel, power, Weibull, Rayleigh, logistic, Pareto, based on record values were given in Ahsanullah [2]. Moreover, two types of predictors of the sth record values based on the first m ($m < s$) record values were presented there. The Bayesian estimators of parameters for the Gumbel distribution and the Burr X distribution based on record values and Bayes prediction bounds for the s-th lower record were discussed in Ali Mousa [3], [4], Malinowska and Szynal [14], [15]. Bayes estimates under squared error and LINEX loss function for a generalized exponential distribution and exponential distribution via record values was obtained in Jaheen [10] and [11]. Prediction bounds for future lower record values by using Bayes and empirical Bayes techniques were given there. Inferences for generalized exponential distribution based on record values were considered in Raqab [17]. Bayesian prediction of rainfall records using the generalized exponential distribution was given in Madi and Raqab [13]. Moreover, in Ahmadi et al [1] was considered estimation and prediction in a two-parameter exponential distribution based on k-record values under LINEX loss function.

This contribution contains the MVU estimator, the ML estimator and the Bayesian estimator for the parameter of generalized exponential distribution expressed in terms of k-th lower record values. The Bayes estimators which assume that the prior distribution is a gamma distribution, are obtained using both symmetric and asymmetric error loss functions. The classical squared error, squared log error loss functions and Kullback-Leibler divergence type loss function are applied in the symmetric case. As asymmetric functions we use LINEX, General Entropy function and Modified General Entropy function. Interval prediction for future k-th lower record values is also discussed from a Bayesian point of view. A simulation study is included.

2. Non-Bayesian Inference

2.1. Point Estimation

Suppose we observe the first m k-th lower record values $Z_1^{(k)} = x_1^{(k)},\ Z_2^{(k)} = x_2^{(k)}, ..., Z_m^{(k)} = x_m^{(k)}$ from the generalized exponential distribution, with cdf and pdf given by (3) and (4), respectively. By (2) the likelihood function is as follows:

$$L(\theta|\underline{x}^{(k)}) = k^m \left[1 - e^{-x_m^{(k)}}\right]^{-k\theta} \prod_{i=1}^{m} \frac{\theta e^{-x_i^{(k)}}}{1 - e^{-x_i^{(k)}}} \propto k^m \theta^m e^{-k\theta T_m^{(k)}},$$

$$x_1^{(k)} > x_2^{(k)} > ... > x_m^{(k)}, \quad (5)$$

where $\underline{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, ..., x_m^{(k)})$, and

$$T_m^{(k)} = -\ln\left(1 - e^{-x_m^{(k)}}\right). \quad (6)$$

Then using (5) the ML estimator of θ is given by

$$\hat{\theta}_{ML}^{(k,m)} = \frac{m}{kT_m^{(k)}}, \quad (7)$$

where $T_m^{(k)}$ is given by (6).

From (5) one can see that the statistic $T_m^{(k)}$ is sufficient and complete for the parameter θ and from (1) that it is distributed as gamma $(m, k\theta)$ with pdf

$$f(T_m^{(k)}|\theta) = \frac{(k\theta)^m}{\Gamma(m)} \left(T_m^{(k)}\right)^{m-1} e^{-k\theta T_m^{(k)}}, \quad T_m^{(k)} > 0, \tag{8}$$

and the random variable $Z = \left(1/T_m^{(k)}\right)$ has the inverted gamma $(m, k\theta)$ distribution with

$$E(Z|\theta) = \frac{k\theta}{m-1}, \quad \mathrm{Var}(Z|\theta) = \frac{k^2\theta^2}{(m-1)^2(m-2)} \qquad \text{(cf. Lin et al [12]).}$$

Hence

$$E(\hat{\theta}_{ML}^{(k,m)}|\theta) = \frac{m\theta}{m-1}, \quad \mathrm{Var}(\hat{\theta}_{ML}^{(k,m)}|\theta) = \frac{m^2\theta^2}{(m-1)^2(m-2)}.$$

As mentioned earlier, the statistic $T_m^{(k)}$ is sufficient and complete for θ. It then follows from the Rao–Blackwell and Lehmann–Scheffe Theorems that the MVU estimator of θ is

$$\hat{\theta}_{MVU}^{(k,m)} = \frac{m-1}{kT_m^{(k)}}$$

with

$$E(\hat{\theta}_{MVU}^{(k,m)}|\theta) = \theta, \quad \mathrm{Var}(\hat{\theta}_{MVU}^{(k,m)}|\theta) = \frac{\theta^2}{(m-2)}.$$

2.2. Interval Estimation

Writing $Q = 2k\theta T_m^{(k)}$, then from (8), Q is distributed as a Chi-squared rv with 2m degrees of freedom. Q is therefore a pivotal quantity for the parameter θ, and a $(1-\tau)100\%$ classical confidence interval for θ is given by

$$\left(\frac{q_1}{2kT_m^{(k)}}, \frac{q_2}{2kT_m^{(k)}}\right),$$

where $q_1 = \chi^2\left((\tau/2), 2m\right)$ and $q_2 = \chi^2\left(1-(\tau/2), 2m\right)$.

3. Bayesian Inference

Assume that the parameter θ is a realization of a rv Θ which has the gamma conjugate prior distribution of the form

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}, \quad \theta > 0;\ \alpha,\ \beta > 0. \tag{9}$$

Combining (5) and (9), the posterior density of θ is a gamma distribution with parameters $(m+\alpha, kT_m^{(k)}+\beta)$, given by

$$\pi^*(\theta|\underline{x}^{(k)}) = \frac{(\beta+kT_m^{(k)})^{m+\alpha}}{\Gamma(m+\alpha)}\theta^{m+\alpha-1}e^{-(\beta+kT_m^{(k)})\theta}, \quad \theta > 0. \tag{10}$$

3.1. Symmetric Bayes Estimation

Assuming a squared error loss function

$$L(\phi(\theta), \widehat{\phi}(\theta)) = (\phi(\theta) - \widehat{\phi}(\theta))^2$$

-the Bayes estimate of θ is its posterior mean, or

$$\hat{\theta}_B^{(k,m)} = \frac{m+\alpha}{\beta+kT_m^{(k)}}. \tag{11}$$

Note that

$$\hat{\theta}_B^{(1,m)} = \frac{m+\alpha}{\beta+T_m^{(1)}}.$$

For $m=1$, $\hat{\theta}_B^{(1,1)} = \frac{1+\alpha}{\beta+T_1^{(1)}}$ is the estimator based on a sample of size 1. Our approach allows us to give the Bayesian estimator of θ using a sample of size k. Namely, for $m=1$ we have $\hat{\theta}_B^{(k,1)} = \frac{1+\alpha}{\beta+kT_1^{(k)}}$. Note that when α and β tend to zero then the estimator (11) tends to the estimator

$$\hat{\theta}_B^{(k,m)} = \frac{m}{kT_m^{(k)}}$$

which is the ML estimator $\hat{\theta}_{ML}^{(k,m)}$ in (7). This was given by Jaheen [10] for the case $k=1$. Combining (8) and (9) the marginal density of $T_m^{(k)}$ is

$$f(T_m^{(k)}) = \frac{\beta^\alpha k^m}{B(m,\alpha)} \frac{\left(T_m^{(k)}\right)^{m-1}}{(\beta+kT_m^{(k)})^{m+\alpha}}, \quad T_m^{(k)} > 0, \tag{12}$$

where $B(a,b)$ is the beta function, i.e. $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt, \quad a > 0,\ b > 0$. It follows from (12) that for $\hat{\theta}_B^{(k,m)}$ in (11) we have

$$E(\hat{\theta}_B^{(k,m)}) = \frac{\alpha}{\beta}, \quad \mathrm{Var}(\hat{\theta}_B^{(k,m)}) = \frac{\alpha m}{(m+\alpha+1)\beta^2}.$$

Now we study the estimation of θ under the squared log error loss (cf. Ferguson [8]). The squared log error has the form

$$L(\widehat{\phi}(\theta), \phi(\theta)) = (\ln\widehat{\phi}(\theta) - \ln\phi(\theta))^2, \tag{13}$$

whose minimum occurs at $\widehat{\phi}(\theta) = \phi(\theta)$. The Bayes estimator of $\phi(\theta)$ denoted by $\widehat{\phi}_{BLG}$ under the squared log error loss (13) has the form:

$$\widehat{\phi}_{BLG} = \exp\left[\mathrm{E}_\phi\left(\ln\phi(\theta)\right)\right],$$

provided that $\mathrm{E}_\phi\left(\ln\phi(\theta)\right)$ – the expectation of $(\ln\phi(\theta))$ under the posterior distribution – exists and is finite.

From (10) we have

$$\widehat{\theta}_{BLG}^{(k,m)} = \frac{\exp(\psi(m+\alpha))}{\beta + kT_m^{(k)}}, \tag{14}$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$. It follows from (12) that for $\hat{\theta}_{BLG}^{(k,m)}$ in (14) we have

$$E(\hat{\theta}_{BLG}^{(k,m)}) = \frac{\alpha\exp(\psi(m+\alpha))}{\beta(m+\alpha)}, \quad \mathrm{Var}(\hat{\theta}_{BLG}^{(k,m)}) = \frac{\alpha m\left(\exp(\psi(m+\alpha))\right)^2}{(m+\alpha)^2(m+\alpha+1)\beta^2}.$$

We consider a new symmetric error loss function, called the Kullback-Leibler divergence type loss function for $\phi(\theta)$, given by:

$$L\left(\widehat{\phi}(\theta), \phi(\theta)\right) = (\phi(\theta))^r \ln\frac{\phi(\theta)}{\widehat{\phi}(\theta)} + \left(\widehat{\phi}(\theta)\right)^r \ln\frac{\widehat{\phi}(\theta)}{\phi(\theta)}, \tag{15}$$

where $\widehat{\phi}$ is an estimate of ϕ and r is a positive number. The posteriori expectation of this loss function in (15) is

$$\begin{aligned}\mathrm{E}_\phi\left(L\left(\widehat{\phi}(\theta), \phi(\theta)\right)\right) &= \mathrm{E}_\phi\left((\phi(\theta))^r\ln\phi(\theta)\right) - \ln\widehat{\phi}(\theta)\,\mathrm{E}_\phi(\phi(\theta))^r \\ &\quad + \left(\widehat{\phi}(\theta)\right)^r\ln\widehat{\phi}(\theta) - \left(\widehat{\phi}(\theta)\right)^r\mathrm{E}_\phi\ln\phi(\theta),\end{aligned}$$

where $\mathrm{E}_\phi(.)$ denotes posteriori expectation with respect to the posterior density of ϕ. The Bayes estimator of $\phi(\theta)$, denoted by $\widehat{\phi} := \widehat{\phi}_{BKL}$ under the error loss function (15) is the solution of the equation

$$r\left(\widehat{\phi}_{BKL}\right)^r\ln\widehat{\phi}_{BKL} + \left(\widehat{\phi}_{BKL}\right)^r - r\left(\widehat{\phi}_{BKL}\right)^r\mathrm{E}_\phi\ln\phi(\theta) = \mathrm{E}_\phi\left(\phi(\theta)\right)^r,$$

provided that $\mathrm{E}_\phi(\cdot)$ exists and is finite.

From (10) we obtain $\mathrm{E}_\theta(\theta^r)$, $\mathrm{E}_\theta(\ln\theta)$ and $\widehat{\theta}_{BKL}^{(k,m)}$ is the solution of the equation

$$\begin{aligned}r\left(\widehat{\theta}_{BKL}^{(k,m)}\right)^r\ln\widehat{\theta}_{BKL}^{(k,m)} + \left(\widehat{\theta}_{BKL}^{(k,m)}\right)^r - r\left(\widehat{\theta}_{BKL}^{(k,m)}\right)^r\left(\psi(m+\alpha) - \ln\left(\beta + kT_m^{(k)}\right)\right) \\ = \frac{\Gamma(m+\alpha+r)}{\Gamma(m+\alpha)}\left(\beta + kT_m^{(k)}\right)^{-r}, \quad a.s. \text{ (almost surely)}.\end{aligned}$$

3.2. Bayes Estimation Based on the LINEX Loss Function

An useful asymmetric loss function, known as the LINEX loss function (linear-exponential), was introduced in Varian [19] and was widely used by several authors. This function rises approximately exponentially on one side of zero and approximately linearly on other side. Under the assumption that the minimal loss occurs at $\widehat{\phi} = \phi$, the LINEX loss function for $\phi := \phi(\theta)$ can be expressed as

$$L(\triangle) = e^{a\triangle} - a\triangle - 1, \quad a \neq 0,$$

where $\triangle = \widehat{\phi}(\theta) - \phi(\theta)$, and $\widehat{\phi}$ is an estimate of ϕ (cf. Varian [19]). The sign of a represents the direction and its magnitude represents the degree of symmetry. First, for $a = 1$ the LINEX loss function is quite asymmetric about zero with overestimation being more costly than underestimation. Second, if $a < 0$, $L(\triangle)$ rise exponentially when $\triangle < 0$ (underestimation) and almost linearly when $\triangle > 0$ (overestimation). For a near zero $L(\triangle)$ approximates the squared error loss function, and so is almost symmetric. The Bayes estimator of $\phi(\theta)$, denoted by $\widehat{\phi}_{BL}$ under the LINEX loss function is the value $\widehat{\phi}(\theta)$ given by

$$\widehat{\phi}_{BL} = -\frac{1}{a} \ln \left(\mathrm{E}_\phi \, e^{-a\phi(\theta)} \right)$$

provided that $\mathrm{E}_\phi \left(e^{-a\phi(\theta)} \right)$ exists and is finite (see Calambria and Plucini [5]).

From (10) we have

$$\widehat{\theta}_{BL}^{(k,m)} = -\frac{1}{a} \ln \left[\left(\frac{\beta + kT_m^{(k)}}{\beta + kT_m^{(k)} + a} \right)^{\alpha+m} \right]. \tag{16}$$

It follows from (12) that for $\hat{\theta}_{BL}^{(k,m)}$ in (16) we have

$$\mathrm{E}(\hat{\theta}_{BL}^{(k,m)}) = -\frac{1}{a} \frac{\beta^\alpha k^m}{B(m,\alpha)} \int_0^\infty \ln \left[\left(\frac{\beta + kT_m^{(k)}}{\beta + kT_m^{(k)} + a} \right)^{\alpha+m} \right] \frac{\left(T_m^{(k)}\right)^{m-1}}{(\beta + kT_m^{(k)})^{m+\alpha}} dT_m^{(k)}$$

$$= \frac{(\alpha)_{m+1}}{a} \sum_{j=1}^{\infty} \frac{(-1)^{j+1} \left(\frac{a}{\beta}\right)^j}{j(\alpha+j)_m}, \text{ when } \left| \frac{a}{\beta + kT_m^{(k)}} \right| < 1, \; a.s.$$

From the fact (Choi et al [6])

$$\frac{1}{(k)_m} = \sum_{i=0}^{m-1} \frac{A_i}{k+i}, \quad \text{where} \quad A_i = \frac{(-1)^i}{(m-1)!} \binom{m-1}{i}$$

we have

$$\mathrm{E}(\hat{\theta}_{BL}^{(k,m)}) = \frac{(\alpha)_{m+1}}{\beta} \sum_{i=0}^{m-1} A_i \sum_{j=0}^{\infty} \frac{\left(-\frac{a}{\beta}\right)^j}{(j+\alpha+i+1)(j+1)}.$$

Using the following formula (cf. Hansen [9])

$$\sum_{k=0}^{\infty} \frac{z^k}{(kx+y)^n (ku+v)^m}$$
$$= \frac{1}{(m-1)!} x^{m-n}(xv-yu)^{-m} \sum_{k=0}^{n-1} \frac{(m+k-1)!}{k!} \left(\frac{y}{x}-\frac{v}{u}\right)^{-k} \zeta(n-k, \frac{y}{x}, z)$$
$$+ \frac{1}{(n-1)!} u^{n-m}(yu-xv)^{-n} \sum_{k=0}^{m-1} \frac{(n+k-1)!}{k!} \left(\frac{v}{u}-\frac{y}{x}\right)^{-k} \zeta(m-k, \frac{v}{u}, z).$$

we obtain

$$\mathrm{E}(\hat{\theta}_{BL}^{(k,m)}) = \frac{(\alpha)_{m+1}}{\beta} \sum_{i=0}^{m-1} \frac{(-1)^i}{(m-1)!} \binom{m-1}{i} \frac{\left[\zeta(1,1,-\frac{a}{\beta}) - \zeta(1,\alpha+i+1,-\frac{a}{\beta})\right]}{\alpha+i}.$$

Next from (cf. Hansen [9])

$$(\log(1-x))^2 = 2\sum_{i=2}^{\infty} \frac{x^i}{i!}[\psi(i)-\psi(1)], \ -1 \le x < 1,$$

we have

$$\mathrm{E}(\hat{\theta}_{BL}^{(k,m)})^2 = \frac{2(m+\alpha)^2}{a^2 B(m,\alpha)} \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \left(\frac{a}{\beta}\right)^n [\psi(n)-\psi(1)] \int_0^1 z^{n+\alpha-1}(1-z)^{m-1} dz$$
$$= \frac{2(\beta+\alpha)(\alpha)_{m+1}}{a^2} \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \left(\frac{a}{\beta}\right)^n \frac{[\psi(n)-\psi(1)]}{(n+\alpha)_m}.$$

Hence

$$\mathrm{Var}(\hat{\theta}_{BL}^{(k,m)}) = \frac{2(\beta+\alpha)(\alpha)_{m+1}}{a^2} \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \left(\frac{a}{\beta}\right)^n \frac{[\psi(n)-\psi(1)]}{(n+\alpha)_m}$$
$$- \left(\frac{(\alpha)_{m+1}}{\beta} \sum_{i=0}^{m-1} \frac{(-1)^i}{(m-1)!} \binom{m-1}{i} \frac{\left[\zeta(1,1,-\frac{a}{\beta}) - \zeta(1,\alpha+i+1,-\frac{a}{\beta})\right]}{\alpha+i} \right)^2,$$
$$\text{when } \left| \frac{a}{\beta + kT_m^{(k)}} \right| < 1, \quad \text{a.s.},$$

where $(z)_n$ denotes the Pochhammer symbol defined by $(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)} = (z)(z+$

1)...($z + n - 1$), $n \in \mathbb{N}$. Note that when parameter a tends to zero then the estimator (16) tends to the estimator

$$\hat{\theta}_B^{(k,m)} = \frac{m+\alpha}{\beta + kT_m^{(k)}}$$

which is the Bayes estimator under squared error loss function $\hat{\theta}_B^{(k,m)}$.

3.3. Bayes Estimators Using General Entropy Loss

Another useful asymmetric loss function is the General Entropy loss, for which

$$L(\widehat{\phi}(\theta), \phi(\theta)) \propto (\widehat{\phi}(\theta)/\phi(\theta))^q - q\ln(\widehat{\phi}(\theta)/\phi(\theta)) - 1 \qquad (17)$$

and its minimum occurs at $\widehat{\phi}(\theta) = \phi(\theta)$. This function is a generalization of the entropy loss used by several authors, where the shape parameter q is equal to 1 (see for example, Dey et al [7], Soliman [18]). The Bayes estimator denoted by $\widehat{\phi}(\theta)_{BGE}$ under the General Entropy loss is:

$$\widehat{\phi}_{BGE} = \left[\mathrm{E}_\phi\left(\phi(\theta)^{-q}\right)\right]^{-\frac{1}{q}}$$

provided that $\mathrm{E}_\phi\left(\phi(\theta)^{-q}\right)$ exists and is finite.

From (10) we have

$$\widehat{\theta}_{BGE}^{(k,m)} = \left[\frac{\Gamma(\alpha+m)}{\Gamma(\alpha+m-q)}\right]^{\frac{1}{q}} \frac{1}{\left(\beta + kT_m^{(k)}\right)}. \qquad (18)$$

It follows from (12) that for $\hat{\theta}_{BGE}^{(k,m)}$ in (18) we have

$$E(\hat{\theta}_{BGE}^{(k,m)}) = \left(\frac{\Gamma(m+\alpha)}{\Gamma(m+\alpha-q)}\right)^{1/q} \frac{\alpha}{\beta(m+\alpha)},$$

$$\mathrm{Var}(\hat{\theta}_{BGE}^{(k,m)}) = \left(\frac{\Gamma(m+\alpha)}{\Gamma(m+\alpha-q)}\right)^{2/q} \frac{\alpha m}{(m+\alpha)^2(m+\alpha+1)\beta^2}.$$

Now we consider a negative loss function for which the condition $L\left(\widehat{\phi}(\theta), \phi(\theta)\right) \geq -c > -\infty$ (cf. Ferguson [8]) is satisfied, and we modify (17) as follows

$$L(\widehat{\phi}(\theta), \phi(\theta)) \propto (\widehat{\phi}(\theta)/\phi(\theta))^q - q\phi(\theta)\ln(\widehat{\phi}(\theta)/\phi(\theta)) - 1.$$

We call it the Modified General Entropy loss function. The Bayes estimator denoted by $\widehat{\phi}(\theta)_{BMGE}$ under the Modified General Entropy loss is:

$$\widehat{\phi}_{BMGE} = \left[\frac{\mathrm{E}_\phi\left(\phi(\theta)\right)}{\mathrm{E}_\phi\left(\phi(\theta)^{-q}\right)}\right]^{\frac{1}{q}},$$

provided that $\mathrm{E}_\phi(\cdot)$ exists and is finite.

From (10) we have

$$\widehat{\theta}_{BMGE}^{(k,m)} = \left[\frac{\Gamma(\alpha+m+1)}{\Gamma(\alpha+m-q)} \frac{1}{\left(\beta+kT_m^{(k)}\right)^{q+1}}\right]^{\frac{1}{q}}. \quad (19)$$

It follows from (12) that for $\hat{\theta}_{MBGE}^{(k,m)}$ in (19) we have

$$E(\hat{\theta}_{MBGE}^{(k,m)}) = \left(\frac{\Gamma(m+\alpha+1)}{\beta\Gamma(m+\alpha-q)}\right)^{1/q} \frac{(\alpha)_m}{\beta(\alpha+1+1/q)_m},$$

$$\mathrm{Var}(\hat{\theta}_{MBGE}^{(k,m)}) = \left(\frac{\Gamma(m+\alpha+1)}{\Gamma(m+\alpha-q)}\right)^{2/q} \frac{(\alpha)_m}{\beta^{2+2/q}} \left[\frac{1}{(\alpha+2+2/q)_m} - \frac{(\alpha)_m}{(\alpha+1+1/q)_m^2}\right].$$

3.4. Interval Estimation

Writing $R = 2(\beta + kT_m^{(k)})\theta$ then from (10), R is distributed as a Chi-squared rv with $2(m+\alpha)$ degrees of freedom. R is therefore a pivotal quantity for the parameter θ and a $(1-\tau)100\%$ Bayesian confidence interval for θ is given by

$$\left(\frac{r_1}{2(\beta+kT_m^{(k)})}, \frac{r_2}{2(\beta+kT_m^{(k)})}\right),$$

where $r_1 = \chi^2\left((\tau/2), 2(m+\alpha)\right)$ and $r_2 = \chi^2\left(1-(\tau/2), 2(m+\alpha)\right)$.

4. Bayesian Prediction

Assume that we have the first m k-th lower records $Z_1^{(k)} = x_1^{(k)}$, $Z_2^{(k)} = x_2^{(k)}, ..., Z_m^{(k)} = x_m^{(k)}$ from the generalized exponential distribution. Based on that sample, we would like to predict the s-th one of the k-th lower record values, $1 < m < s$. Now let $Y^{(k)} = Z_s^{(k)}$ be the s-th lower record value, $1 < m < s$. The conditional pdf of $Y_s^{(k)}$ given the parameter θ and the observed value $x_m^{(k)}$ of $Z_m^{(k)}$ is

$$f^*(y_s^{(k)}|\theta) = \frac{k^{s-m}}{\Gamma(s-m)}[H(y_s^{(k)}) - H(x_m^{(k)})]^{s-m-1} \left(\frac{F(y^{(k)})}{F(x_m^{(k)})}\right)^{k-1} \frac{f(y_s^{(k)})}{F(x_m^{(k)})}, \quad (20)$$

where $f(\cdot)$, $F(\cdot)$ are the pdf and cdf, respectively, and $H(y_s^{(k)}) = -\ln F(y_s^{(k)}) = \theta T_s^{(k)}$, $H(x_m^{(k)}) = -\ln F(x_m^{(k)}) = \theta T_m^{(k)}$, where $T_s^{(k)}$ is obtained from (6). Applying (3) and (4) in (20) we get

$$f^*(y_s^{(k)}|\theta) = \frac{k^{s-m}}{\Gamma(s-m)}[T_s^{(k)} - T_m^{(k)}]^{s-m-1}\theta^{s-m}\frac{e^{-y_s^{(k)}}}{1-e^{-y_s^{(k)}}}e^{-\left(k\theta(T_s^{(k)}-T_m^{(k)})\right)},$$
$$0 < y_s^{(k)} < x_m^{(k)} < \infty.$$

Combining the posterior density, given by (10), and integrating out the parameters θ, we obtain the Bayesian predictive density function of $Y_s^{(k)} = Z_s^{(k)}$ given the past m k-th lower record values, in the form

$$f(y_s^{(k)}|\underline{x}^{(k)}) = \int_0^\infty f^*(y_s^{(k)}|\theta)\pi^*(\theta|\underline{x}^{(k)})d\theta$$
$$= Ak^{s-m}\frac{e^{-y_s^{(k)}}}{1-e^{-y_s^{(k)}}}\left[\frac{T_s^{(k)} - T_m^{(k)}}{\beta + kT_s^{(k)}}\right]^{s-m-1}\left[\frac{\beta + kT_m^{(k)}}{\beta + kT_s^{(k)}}\right]^{\alpha+m+1},$$
$$0 < y_s^{(k)} < x_m^{(k)} < \infty,$$

where

$$A^{-1} = (\beta + kT_m^{(k)})B(s-m, \alpha+m).$$

The Bayesian prediction bounds for $Y_s^{(k)} = Z_s^{(k)}$ are obtained by evaluating

$$Pr(Y_s^{(k)} \geq \gamma|\underline{x}^{(k)}) = \int_\gamma^{x_m^{(k)}} f(y_s^{(k)}|\underline{x}^{(k)})dy_s^{(k)}$$
$$= Ak\int_\gamma^{x_m^{(k)}}\frac{e^{-y_s^{(k)}}}{1-e^{-y_s^{(k)}}}\left[\frac{kT_s^{(k)} - kT_m^{(k)}}{\beta + kT_s^{(k)}}\right]^{s-m-1}\left[\frac{\beta + kT_m^{(k)}}{\beta + kT_s^{(k)}}\right]^{\alpha+m+1} dy_s^{(k)}.$$

Using the transformation $w = \frac{kT_s^{(k)} - kT_m^{(k)}}{\beta + kT_s^{(k)}}$, the above integral simplifies to

$$Pr(Y_s^{(k)} \geq \gamma|\underline{x}^{(k)}) = \frac{1}{B(s-m,\alpha+m)}\int_0^{w_\gamma} w^{s-m-1}(1-w)^{m+\alpha-1}dw$$
$$= F_B(w_\gamma), \tag{21}$$

where

$$\underline{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, ..., x_m^{(k)}),\ w_\gamma = \frac{kT_\gamma^{(k)} - kT_m^{(k)}}{\beta + kT_\gamma^{(k)}},\ T_\gamma^{(k)} = -\ln\left(1 - e^{-\gamma}\right). \tag{22}$$

and $F_B(\cdot)$ is the incomplete Beta cdf with parameters $(s-m, \alpha+m)$. The $(1-\tau)100\%$ predictive interval for $Y_s^{(k)} = Z_s^{(k)}$ is obtained by evaluating both the lower, $L(\underline{x}^{(k)})$, and upper, $U(\underline{x}^{(k)})$, limits which satisfy

$$Pr(Y_s^{(k)} > L(\underline{x}^{(k)})|\underline{x}^{(k)}) = 1 - \frac{\tau}{2}, \quad Pr(Y_s^{(k)} > U(\underline{x}^{(k)})|\underline{x}^{(k)}) = \frac{\tau}{2}.$$

m	k	$\widehat{\theta}_{ML}$	$\widehat{\theta}_{MVU}$	$\widehat{\theta}_{B}$	$\widehat{\theta}_{BLG}$	$\widehat{\theta}_{2,BKL}$	$\widehat{\theta}_{3,BKL}$	$\widehat{\theta}_{4,BKL}$	$\widehat{\theta}_{2,BL}$	$\widehat{\theta}_{-2,BL}$	$\widehat{\theta}_{BGE}$	$\widehat{\theta}_{BMGE}$
3	1	4.173	2.782	1.839	1.658	1.836	1.929	2.030	1.378	3.325	**2.520**	**2.231**
	2	2.665	1.776	1.599	1.442	1.597	1.678	1.766	1.236	**2.553**	**2.192**	1.996
	3	1.789	1.193	1.360	1.226	1.358	1.427	1.501	1.085	**1.963**	**1.864**	1.753
	4	1.342	0.894	1.180	1.064	1.178	1.238	1.303	0.966	**1.597**	**1.618**	1.565
	5	1.418	0.945	1.215	1.095	1.213	1.275	1.341	0.990	**1.664**	**1.665**	1.602
	6	1.182	0.788	1.102	0.993	1.100	1.156	1.216	0.912	1.453	**1.510**	**1.481**
4	1	2.505	1.879	1.668	1.531	1.666	1.737	1.812	1.326	**2.436**	**2.189**	1.976
	2	3.160	**2.370**	1.837	1.686	1.835	1.913	1.996	1.433	2.843	**2.410**	2.134
	3	**2.368**	1.776	1.626	1.493	1.624	1.693	1.767	1.299	**2.344**	2.134	1.936
	4	1.776	1.332	1.411	1.295	1.409	1.469	1.533	1.156	**1.907**	**1.851**	1.728
	5	1.431	1.073	1.251	1.148	1.250	1.303	1.359	1.046	**1.619**	**1.642**	1.570
	6	1.488	1.116	1.279	1.174	1.278	1.332	1.390	1.065	**1.668**	**1.679**	1.598
5	1	**2.384**	1.907	1.708	1.588	1.707	1.769	1.835	1.391	**2.344**	2.169	1.948
	2	3.584	2.867	2.062	1.916	2.060	2.135	**2.214**	1.612	3.113	2.617	**2.265**
	3	2.633	2.106	1.795	1.669	1.794	1.859	1.928	1.449	**2.518**	**2.279**	2.027
	4	**2.056**	1.644	1.579	1.468	1.578	1.635	1.696	1.303	**2.100**	2.005	1.830
	5	1.776	1.421	1.454	1.351	1.453	1.505	1.561	1.216	**1.879**	**1.846**	1.712
	6	1.491	1.193	1.307	1.215	1.306	1.354	1.404	1.111	**1.637**	**1.660**	1.573
6	1	1.835	1.529	1.518	1.424	1.517	1.565	1.616	1.287	**1.909**	**1.878**	1.728
	2	4.173	3.477	**2.327**	2.183	**2.325**	1.399	2.477	1.834	3.487	2.879	2.432
	3	2.867	**2.390**	1.955	1.834	1.953	2.015	2.081	1.591	2.683	**2.419**	2.115
	4	**2.370**	1.975	1.765	1.656	1.764	1.820	1.879	1.462	**2.329**	2.184	1.949
	5	**1.973**	1.645	1.587	1.489	1.586	1.636	1.689	1.336	**2.022**	1.964	1.791
	6	1.776	1.480	1.487	1.395	1.487	1.534	1.583	1.264	**1.860**	**1.841**	1.700

$\widehat{\theta}_{r,BKL}$ estimators $\widehat{\theta}^{(k,m)}_{BKL}$ with $r = 2, 3, 4$ respectively.
$\widehat{\theta}_{a,BL}$ estimators $\widehat{\theta}^{(k,m)}_{BL}$ with $a = 2, -2$ respectively.

Table 1: The estimators $\widehat{\theta}^{(k,m)}_{ML}$, $\widehat{\theta}^{(k,m)}_{MVU}$, $\widehat{\theta}^{(k,m)}_{B}$, $\widehat{\theta}^{(k,m)}_{BLG}$, $\widehat{\theta}^{(k,m)}_{BKL}$, $\widehat{\theta}^{(k,m)}_{BL}$, $\widehat{\theta}^{(k,m)}_{BGE}$ and $\widehat{\theta}^{(k,m)}_{BMGE}$ of $\theta = 2.336$

Thus, applying (21), one can obtain the lower and upper prediction bounds of $Y_s^{(k)} = Z_s^{(k)}$, analytically, in the forms

$$L(\underline{x}^{(k)}) = \ln\left\{\frac{1}{1 - \exp\left(-\frac{kT_m^{(k)} + \beta\delta_1(s)}{k(1-\delta_1(s))}\right)}\right\},$$

$$U(\underline{x}^{(k)}) = \ln\left\{\frac{1}{1 - \exp\left(-\frac{kT_m^{(k)} + \beta\delta_2(s)}{k(1-\delta_2(s))}\right)}\right\},$$

where

$$\delta_1(s) = F_B^{-1}\left(1 - \frac{\tau}{2}\right), \quad \delta_2(s) = F_B^{-1}\left(\frac{\tau}{2}\right),$$

$F_B^{-1}(\cdot)$ is the inverse incomplete Beta cdf with parameters $(s - m, \alpha + m)$.

For the special case, when predicting the next k-th lower record $Y_{m+1}^{(k)} =$

m	k	$\widehat{\theta}_{ML}$	$\widehat{\theta}_{MVU}$	$\widehat{\theta}_{B}$	$\widehat{\theta}_{BLG}$	$\widehat{\theta}_{2,BKL}$	$\widehat{\theta}_{3,BKL}$	$\widehat{\theta}_{4,BKL}$	$\widehat{\theta}_{2,BL}$	$\widehat{\theta}^{(k,m)}_{-2,BL}$	$\widehat{\theta}_{BGE}$	$\widehat{\theta}_{BMGE}$
3	1	11.163	3.854	0.476	0.703	0.479	0.384	0.297	1.116	1.434	**0.132**	**0.129**
	2	11.208	4.052	0.556	0.787	0.559	0.461	0.371	1.118	1.436	**0.177**	**0.175**
	3	9.495	3.476	0.589	0.823	0.592	0.492	0.399	1.215	1.310	**0.185**	**0.192**
	4	11.686	4.403	0.600	0.834	0.603	0.503	0.410	1.225	1.384	**0.193**	**0.199**
	5	9.503	3.584	0.623	0.859	0.626	0.523	0.429	1.246	1.278	**0.198**	**0.209**
	6	8.924	3.329	0.617	0.853	0.620	0.518	0.424	1.241	1.265	**0.195**	**0.206**
4	1	5.671	2.461	0.353	0.515	0.355	0.289	0.232	0.876	1.117	**0.156**	**0.125**
	2	5.077	2.294	0.439	0.606	0.441	0.371	0.309	0.956	1.085	**0.199**	**0.176**
	3	4.606	2.099	0.458	0.629	0.460	0.389	0.325	0.976	1.032	**0.203**	**0.186**
	4	4.823	2.279	0.483	0.656	0.485	0.412	0.346	1.001	1.009	**0.210**	**0.199**
	5	4.602	2.173	0.496	0.669	0.498	0.424	0.358	1.012	1.018	**0.218**	**0.207**
	6	4.230	2.000	0.503	0.677	0.505	0.431	0.364	1.019	0.980	**0.219**	**0.211**

m	k	$\widehat{\theta}_{ML}$	$\widehat{\theta}_{MVU}$	$\widehat{\theta}_{B}$	$\widehat{\theta}_{BLG}$	$\widehat{\theta}_{2,BKL}$	$\widehat{\theta}_{3,BKL}$	$\widehat{\theta}_{4,BKL}$	$\widehat{\theta}_{2,BL}$	$\widehat{\theta}^{(k,m)}_{-2,BL}$	$\widehat{\theta}_{BGE}$	$\widehat{\theta}_{BMGE}$
5	1	3.255	1.593	0.283	0.403	0.284	0.237	0.198	0.711	0.868	**0.169**	**0.124**
	2	3.385	1.735	0.356	0.479	0.357	0.309	0.267	0.776	0.953	**0.222**	**0.172**
	3	3.293	1.741	0.375	0.501	0.376	0.324	0.280	0.800	0.889	**0.215**	**0.180**
	4	3.066	1.639	0.403	0.532	0.404	0.351	0.305	0.828	0.880	**0.229**	**0.198**
	5	2.824	1.533	0.425	0.556	0.426	0.372	0.323	0.851	0.843	**0.235**	**0.211**
	6	2.789	1.503	0.417	0.548	0.418	0.364	0.316	0.844	0.837	**0.231**	**0.206**
6	1	2.483	1.363	0.256	0.346	0.257	0.223	**0.195**	0.604	0.770	**0.195**	**0.135**
	2	2.327	1.311	0.309	0.402	0.310	0.273	0.243	0.655	0.786	**0.225**	**0.170**
	3	2.344	1.347	0.327	0.423	0.328	0.290	0.259	0.675	0.786	**0.231**	**0.181**
	4	2.240	1.313	0.351	0.449	0.352	0.312	0.279	0.701	0.766	**0.239**	**0.196**
	5	1.992	1.183	0.370	0.471	0.371	0.329	0.294	0.724	0.718	**0.240**	**0.208**
	6	1.919	1.145	0.367	0.470	0.368	0.326	0.289	0.725	0.687	**0.230**	**0.205**

$\widehat{\theta}_{r,BKL}$ estimators $\widehat{\theta}^{(k,m)}_{BKL}$ with $r = 2, 3, 4$ respectively.
$\widehat{\theta}_{a,BL}$ estimators $\widehat{\theta}^{(k,m)}_{BL}$ with $a = 2, -2$ respectively.

Table 2: MSE of the estimators $\widehat{\theta}^{(k,m)}_{ML}$, $\widehat{\theta}^{(k,m)}_{MVU}$, $\widehat{\theta}^{(k,m)}_{B}$, $\widehat{\theta}^{(k,m)}_{BLG}$, $\widehat{\theta}^{(k,m)}_{BKL}$, $\widehat{\theta}^{(k,m)}_{2,BL}$, $\widehat{\theta}^{(k,m)}_{BGE}$ and $\widehat{\theta}^{(k,m)}_{BMGE}$ of $\theta = 2.336$.

$Z^{(k)}_{m+1}$ $(s = m + 1)$, (21) reduces to

$$Pr(Y^{(k)}_{m+1} \geq \gamma | \underline{x}^{(k)}) = 1 - (1 - w_\gamma)^{m+\alpha}, \tag{23}$$

where w_γ is given by (22). By using (23), the lower and upper prediction bounds for $Y^{(k)}_{m+1}$ are given by

$$L_1(\underline{x}^{(k)}) = \ln \left\{ \frac{1}{1 - \exp\left(-\frac{kT^{(k)}_m + \beta(1-\zeta_1)}{k\zeta_1} \right)} \right\},$$

$$U_1(\underline{x}^{(k)}) = \ln \left\{ \frac{1}{1 - \exp\left(-\frac{kT^{(k)}_m + \beta(1-\zeta_2)}{k\zeta_2} \right)} \right\},$$

where

$$\zeta_1 = \left(\frac{\tau}{2}\right)^{1/m+\alpha}, \quad \zeta_2 = \left(1 - \frac{\tau}{2}\right)^{1/m+\alpha}.$$

m	k	$\widehat{\theta}_B$	$\widehat{\theta}_{BLG}$	$\widehat{\theta}_{2,BKL}$	$\widehat{\theta}_{3,BKL}$	$\widehat{\theta}_{4,BKL}$	$\widehat{\theta}_{2,BL}$	$\widehat{\theta}^{(k,m)}_{-2,BL}$	$\widehat{\theta}_{BGE}$	$\widehat{\theta}_{BMGE}$
3	1	0.489	0.703	2.688	4.973	9.080	1.236	1.142	0.421	1.770
	2	0.576	0.787	2.877	5.384	10.028	1.293	1.372	0.864	1.576
	3	0.613	0.823	2.963	5.572	10.461	1.317	1.413	1.273	1.515
	4	0.632	0.834	3.016	5.689	10.736	1.323	1.443	1.385	1.464
	5	0.643	0.859	3.031	5.721	10.814	1.325	1.466	1.229	1.429
	6	0.661	0.853	3.088	5.844	11.096	1.327	1.459	1.435	1.379
4	1	0.373	0.155	2.240	4.085	7.187	1.021	0.935	0.360	2.181
	2	0.457	0.192	2.416	4.450	7.984	1.081	1.161	0.763	1.982
	3	0.485	0.206	2.476	4.577	8.265	1.117	1.209	1.011	1.927
	4	0.508	0.217	2.546	4.733	8.633	1.126	1.257	1.230	1.866
	5	0.515	0.219	2.583	4.817	8.851	1.131	1.261	1.126	1.817
	6	0.544	0.233	2.650	4.962	9.180	1.133	1.233	1.314	1.749
5	1	0.295	0.114	1.840	3.255	5.319	0.853	0.800	0.336	2.571
	2	0.374	0.147	2.049	3.693	6.284	0.928	0.995	0.679	2.335
	3	0.414	0.164	2.111	3.802	6.453	0.952	1.073	0.902	2.255
	4	0.426	0.171	2.159	3.924	6.794	0.974	1.107	1.076	2.219
	5	0.432	0.173	2.206	4.035	7.084	0.981	1.113	1.109	2.177
	6	0.466	0.187	2.302	4.240	7.546	0.988	1.145	1.219	2.067
6	1	0.272	0.098	1.595	2.726	4.045	0.741	0.735	0.407	2.809
	2	0.317	0.116	1.738	3.033	4.734	0.805	0.851	0.595	2.651
	3	0.363	0.133	1.822	3.179	4.953	0.830	0.985	0.793	2.537
	4	0.382	0.142	1.870	3.290	5.239	0.847	0.972	0.963	2.491
	5	0.379	0.143	1.888	3.350	5.452	0.855	0.987	1.081	2.491
	6	0.402	0.151	1.957	3.490	5.738	0.866	0.998	1.114	2.408

Table 3: ER of the estimators $\widehat{\theta}^{(k,m)}_B$, $\widehat{\theta}^{(k,m)}_{BLG}$, $\widehat{\theta}^{(k,m)}_{BKL}$, $\widehat{\theta}^{(k,m)}_{BL}$, $\widehat{\theta}^{(k,m)}_{BGE}$ and $\widehat{\theta}^{(k,m)}_{BMGE}$ of $\theta = 2.336$

5. Numerical Illustration

Here we discuss some numerical results of a simulation study testing the performance of non-Bayesian and Bayesian methods in estimating the parameter of the generalized exponential distribution. We generate the first m ($m = 6$) lower k-th record values ($k = 1, 2, 3, 4, 5, 6$) from the generalized exponential distribution with the parameter $\theta = 2.336$ which was generated via the gamma conjugate priori distribution with the parameters $\alpha = 2.0$ and $\beta = 2.0$. In the prediction we take $m = 7$. The values of the shape parameter a in the LINEX loss function, and value of q in GE and MGE loss functions, respectively, are $a = 2$, $a = -2$ and $q = -5$. In the Kullback-Leibler divergence type loss function we take with $r = 2$, 3 and 4. The point estimators $\widehat{\theta}^{(k,m)}_{ML}$, $\widehat{\theta}^{(k,m)}_{MVU}$, $\widehat{\theta}^{(k,m)}_{B}$, $\widehat{\theta}^{(k,m)}_{BKL}$, $\widehat{\theta}^{(k,m)}_{BL}$, $\widehat{\theta}^{(k,m)}_{BGE}$, $\widehat{\theta}^{(k,m)}_{BMGE}$ and $\widehat{\theta}^{(k,m)}_{BLG}$ of θ are given in Table 1, where we omit (k,m) and bold numbers in rows are for the two closest estimates to the estimated values. Table 2 contains the MSE of estimators. Table 3 contains the Bayesian risk (ER) under symmetric and asymmetric error loss functions which are considered in this paper. The 95% classical and Bayesian confidence intervals for θ and also the 95% Bayesian prediction bounds for the next k-th record $x^{(k)}_{m+1}$ are presented in the Table 4. The MSE and ER of θ, based on 1000 replications.

m	k	θ	Conf.int.	Bayes pred. int.	m	Conf.int.	Bayes pred. int.
3	1	2.336	$(0.362, 4.230)^a$	0.149^c	4	$(0.551, 4.434)^a$	0.115^c
			$(0.438, 2.762)^b$	$(0.003, 0.195)^d$		$(0.553, 2.934)^b$	$(0.005, 0.148)^d$
	2		$(1.001, 11.694)^a$	1.254^c		$(1.623, 13.057)^a$	0.764^c
			$(0.620, 3.912)^b$	$(0.193, 1.307)^d$		$(0.824, 4.368)^b$	$(0.261, 1.240)^d$
	3		$(0.667, 7.796)^a$	1.254^c		$(1.082, 8.705)^a$	1.227^c
			$(0.554, 3.499)^b$	$(0.292, 1.311)^d$		$(0.732, 3.880)^b$	$(0.364, 1.244)^d$
	4		$(1.329, 15.529)^a$	1.647^c		$(1.274, 10.251)^a$	1.325^c
			$(0.658, 4.154)^b$	$(0.605, 2.184)^d$		$(0.771, 4.086)^b$	$(0.589, 1.635)^d$
	5		$(1.063, 12.423)^a$	1.647^c		$(1.019, 8.201)^a$	1.325^c
			$(0.629, 3.967)^b$	$(0.706, 2.188)^d$		$(0.717, 3.802)^b$	$(0.653, 1.637)^d$
	6		$(0.886, 10.353)^a$	1.647^c		$(0.849, 6.834)^a$	1.325^c
			$(0.602, 3.796)^b$	$(0.787, 2.190)^d$		$(0.671, 3.554)^b$	$(0.708, 1.638)^d$
5	1		$(0.731, 4.641)^a$	0.073^c	6	$(0.829, 4.395)^a$	0.055^c
			$(0.667, 3.094)^b$	$(0.006, 0.113)^d$		$(0.742, 3.098)^b$	$(0.004, 0.072)^d$
	2		$(1.294, 8.166)^a$	0.645^c		$(1.481, 7.851)^a$	0.350^c
			$(0.865, 4.013)^b$	$(0.189, 0.757)^d$		$(0.991, 4.137)^b$	$(0.187, 0.640)^d$
	3		$(1.560, 9.841)^a$	0.764^c		$(1.170, 6.203)^a$	0.645^c
			$(0.925, 4.295)^b$	$(0.430, 1.218)^d$		$(0.889, 3.716)^b$	$(0.288, 0.759)^d$
	4		$(1.314, 8.288)^a$	1.254^c		$(1.639, 8.689)^a$	1.227^c
			$(0.869, 4.036)^b$	$(0.543, 1.317)^d$		$(1.083, 4.314)^b$	$(0.576, 1.248)^d$
	5		$(1.051, 6.631)^a$	1.254^c		$(1.312, 6.951)^a$	1.227^c
			$(0.794, 3.684)^b$	$(0.596, 1.318)^d$		$(0.939, 3.920)^b$	$(0.624, 1.249)^d$
	6		$(0.876, 5.526)^a$	1.254^c		$(1.093, 5.793)^a$	1.227^c
			$(0.730, 3.389)^b$	$(0.635, 1.319)^d$		$(0.860, 3.592)^b$	$(0.659, 1.249)^d$

^{a} The 95% classical confidence interval for θ.
^{b}-the 95% Bayesian confidence interval for θ.
^{c} The generated lower k-th record $x_{m+1}^{(k)}$.
^{d} The 95% Bayesian prediction interval for $x_{m+1}^{(k)}$.

Table 4: The 95% classical and Bayesian confidence intervals for θ, the 95% Bayesian prediction bounds for the next k-th record $x_{m+1}^{(k)}$

Comments. As can be seen from Table 1, Table 2 and Table 4:

1. Simulations show the performance of estimators under many classical and modified loss funcions.

2. The Bayes estimations $\widehat{\theta}_{2,BKL}^{(k,m)}$ under Kullback-Leibler divergence type loss function with $r = 2$ are almost equal to the Bayes estimators $\widehat{\theta}_{B}^{(k,m)}$.

3. The MSE estimators $\widehat{\theta}_{ML}^{(k,m)}$, $\widehat{\theta}_{MVU}^{(k,m)}$ and Bayes estimators under squared error, squared log error, Kullback - Leibler divergence type and LINEX, General Entropy, Modified General Entropy loss functions decrease when m increases.

4. The generated k-th lower record $x_{m+1}^{(k)}$ is in the 95% Bayesian prediction interval for $x_{m+1}^{(k)}$.

References

[1] J. Ahmadi, M. Doostrasat, A. Parsian, Estimation and prediction in a two-parameter exponential distribution based on k-th-record values under LINEX loss function, *Commun. Statist. Theor. Meth.*, **34** (2005), 795-805.

[2] M. Ahsanullah, *Record Values – Theory and Applications*, University Press of America, Lanham, Maryland, USA (2004).

[3] M.A.M. Ali Mousa, Inference and prediction for the Burr type X model based on records, *Statistics,* **35** No. 4 (2001), 415-425.

[4] M.A.M. Ali Mousa, Z.F. Jaheen, A.A. Ahmed, Bayesian estimation, prediction and characterization for the Gumbel model based on records, *Statistics*, **36**, No. 1 (2002), 65-74.

[5] G. Calambria, G. Plucini, Point estimation under asymmetric loss functions for left truncated exponential sample, *Commun. Statist. Theor. Meth.*, **25**, No. 3 (1996), 285-600.

[6] J. Choi, Y. J. Cho, H.M. Srivastava, Series involving the Zeta function and multiple Gamma functions, *Appl. Math. Comput.,* **159** (2004), 509-537.

[7] D. K. Dey, M. Ghosh, C. Srinivasan, Simultaneous estimation of parameters under entropy loss, *J. Statist. Plann. Infer.*, **25** (1987), 347-363.

[8] T.S. Ferguson, *Mathematical Statistics*, Academic Press, New York and London, England (1967).

[9] E.R. Hansen, *A Table of Series and Products*, Prentice-Hall, Inc., Englewood Cliffs, NJ (1975).

[10] Z.F. Jaheen, Empirical Bayes inference for generalized exponential distribution based on records, *Commun. Statist. Theor. Meth.*, **33**, No. 8 (2004), 1851-1861.

[11] Z.F. Jaheen, Empirical Bayes analisis of record statistics based on LINEX and quadratic loss functions, *Computers and Mathematics with Applications*, **47** (2004), 947-954.

[12] C.T. Lin, B.S. Duran, T.O. Lewis, Inverted gamma as a life distribution, *Microelectron. Reliab.,* **29**, No. 4 (1989), 619-626.

[13] M. Madi, M.Z. Raqab, Bayesin prediction of rainfall records using the generalized exponential distribution, *Environmetrics*, **18** (2007), 541-549.

[14] I. Malinowska, D. Szynal, On a family of Bayesian estimators and predictors for Gumbel model based on the k-th lower records, *Appl. Math.,* **31**, No. 1 (2004), 107-115.

[15] I. Malinowska, D. Szynal, Inference and prediction for the Burr type X model based on the k-th lower record, *J. Statist. Theor. Appl.*, **4**, No. 3 (2005), 282-291.

[16] P. Pawlas, D. Szynal, Relations for single and product moments of k-th record values from exponential and Gumbel distributions, *J. Appl. Statist. Sci.,* **7**, No. 1 (1998), 53-62.

[17] M. Z. Raqab, Inference for generalized exponential distribution based on record statistics, *J. Statist. Plann. Infer.,* **104** (2002), 339-350.

[18] A. A. Soliman, Estimators for finite mixture of Rayleight model based on progressively censored date, *Commun. Statist. Theor. Meth.*, **32** (2006), 803-820.

[19] H.R. Varian, A Bayesian approch to real estate assessment, In: *Studies in Bayesian Econometrics and Statistics in Honer of Leonard J. Savage*, North Holland, Amsterdam (1975) 195-208.