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# ON THE IMAGE OF THE CANONICAL MAP OF A STABLE POINTED CURVE

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**Abstract:** Here we describe (in some particular cases) the canonical model of a 2-connected, but not 3-connected, pointed stable curve.

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## 1. Introduction

Fix any pointed nodal curve $Y = (X; P_1, \dots, P_n) \in \overline{\mathcal{M}}_{g,n}$ with $g + n \geq 2$ and $n \geq 3$ if $g = 0$. $X$ is a nodal and connected projective curve, $p_a(X) = g$, $P_1, \dots, P_n$ are distinct points of $X_{reg}$ and each smooth and rational irreducible component of $X$ contains at least 3 among the points in $\mathrm{Sing}(X) \cup \{P_1, \dots, P_n\}$. The canonical line bundle $\omega_Y$ of the pointed curve $Y$ is the line bundle $\omega_X(P_1 + \cdots + P_n)$. There is a non-empty maximal open subset $U_Y$ of $X$ such that the complete linear system $|\omega_X(P_1 + \cdots + P_n)|$ induces a morphism $\phi_Y : U_Y \to \mathbb{P}^x$, where $x = g - 1$ if $n = 0$ and $x = g + n - 2$ if $n > 0$. In many cases (even with large $g$ and $n = 0$ or with any $g$ and large $n$) $U_Y$ is not Zariski dense in $X$. Let $Y_\omega$ be the closure of $\phi_Y(U_Y)$ in $\mathbb{P}^x$. The curve $Y_\omega$ may be called the *canonical image* of the pointed curve $Y$. The quasi-projective algebraic set $\phi_Y(U_Y)$ is called the *effective canonical image* of $Y$. A. Artamkin gave a necessary and sufficient condition for the equality $U_Y = X$ and a necessary and sufficient condition of the very ampleness (up to a few classified "honestly hyperelliptic pointed curves") of $\omega_Y$ ([1], Theorem 1.2; see [3], Theorems 3.3 and 3.6, for

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the case $n = 0$). From [1] it only remains to describe $Y_\omega$ when $Y$ is not 3-connected. First assume $Y$ not 2-connected, i.e. $\omega_Y$ not spanned. In [2] we described $U_Y$ and proved that $Y_\omega$ is the image of a disconnected pointed curve $Y' = \sqcup_i Y_i$ in which each $Y_i$ is a semistable pointed curve with $\omega_{Y_i}$ is spanned. Taking the stable reduction of each $Y_i$ we see that to describe (in principle) the canonical image of any stable pointed curve it would be sufficient to describe the canonical images of all 2-connected pointed curves. By [1], Theorem 1.2, we know the canonical image of any 3-connected pointed curve. Hence from now on we fix a stable pointed curve $Y = (X; P_1, \dots, P_n)$ which is 2-connected, but not 3 connected. Let $\mathbb{B}(Y)$ denote the set of all sets $\{P, Q\}$ formed by 2 distinct points of $X$ such that $X \backslash \{P, Q\}$ is not connected and (if $n > 0$) one of the connected components of $X \backslash \{P, Q\}$ contains all marked points $P_1, \dots, P_n$. The 2-connectedness of $X$ implies $n \neq 1$.

For any subscheme $A$ of a projective space, let $\langle A \rangle$ denote the linear span of $A$.

**Theorem 1.** *Assume $n > 0$, $Y$ 2-connected, $\sharp(\mathbb{B}(Y)) = 1$, say $\mathbb{B}(Y) = \{P, Q\}$, and that $\{P, Q\} \cap \{P_1, \dots, P_n\} = \emptyset$. Let $B_1$ and $B_2$ be the closures in $X$ of the 2 connected components of $X \backslash \{P, Q\}$. One of these components, say $B_2$, contains all points $P_1, \dots, P_n$. The line bundle $\omega_Y$ is spanned. Let $h_{\omega_Y} : X \to \mathbb{P}^x$ be the morphism induced by the spanned line bundle $\omega_Y$. Set $f_i := h_{\omega_Y}(B_i)$, $i = 1, 2$. Then:*

*(i) $h_{\omega_Y}(P) = h_{\omega_Y}(Q)$; and $\langle f_1(B_1) \rangle \cap \langle f_2(B_2) \rangle = \{h_{\omega_Y}(P)\}$.*

*(ii) $f_1$ is induced by the complete linear system $|\omega_{B_1}(P + Q)|$.*

*(iii) $f_2$ is induced by the codimension 1 linear subsystem of $|\omega_{B_2}(P + Q + P_1 + \cdots + P_n)|$ which identifies $P$ and $Q$.*

*(iv) If $B_1$ is 3-connected and $p_a(B_1) \geq 2$, then either $f_1$ is an embedding or it is a degree 2 finite morphism with as image a rational normal curve of $\langle f_1(B_1) \rangle$. The latter case occurs if and only if there is a degree 2 morphism $\beta : B_1 \to \mathbb{P}^1$ and $\{P, Q\}$ is a fiber of $\beta$.*

*(v) Assume $h^0(B_2, \mathcal{O}_{B_2}(P + Q)) = 1$. $f_2^{-1}(f_2(P)) = \{P, Q\}$ (scheme-theoretically) if and only if there is no length 3 zero-dimensional scheme $E \subset B_2$ such that $\{P, Q\} \subset E$ and $h^0(B_2, \mathcal{O}_{B_2}(E)) \geq 2$ (or, equivalently, $= 2$).*

*(vi) Assume $n \geq 3$ and that $B_2$ is 3-connected. Then $f_2$ is unramified and $\{P, Q\}$ are the only distinct points of $B_2$ in one of its fibers.*

The second part of (i) in the statement of Theorem 1 means that $Y_\omega$ is obtained by gluing together the curves $h_{\omega_Y}(B_1)$ and $h_{\omega_Y}(B_2)$ at a unique point, $h_{\omega_Y}(P)$.

**Theorem 2.** *Assume $n = 0$, $Y$ 2-connected, and $\sharp(\mathbb{B}(Y)) = 1$, say $\mathbb{B}(Y) =$*

*$\{P, Q\}$. Let $B_1$ and $B_2$ be the closures in $X$ of the 2 connected components of $X \backslash \{P, Q\}$. Assume that for each index $i \in \{1, 2\}$ $B_i$ is 3-connected and that there is no length 4 zero-dimensional scheme $W_i \subset B_i$ such that $\{P, Q\} \subset W_i$ and $h^0(B_i, \mathcal{O}_{B_i}(W_i) \geq 2$. Then $h_{\omega_Y}$ is unramified outside $P$ and $Q$, $h_{\omega_Y}|B_1$ and $h_{\omega_Y}|B_2$ are unramified, $P$ and $Q$ are the only distinct points of $X$ in the same fiber of $h_{\omega_Y}$, $\langle h_{\omega_Y}(B_1)\rangle \cap \langle h_{\omega_Y}(B_2)\rangle = \{h_{\omega_Y}(P)\}$ and $h_{\omega_Y}(P)$ is a seminormal point with multiplicity 4 and hence embedding dimension 4.*

**Theorem 3.** *Assume $n > 0$, that $Y$ is 2-connected, and that there is $\{P, Q\} \in \mathbb{B}(Y)$ such that $P \in \{P_1, \dots, P_n\}$, say $P = P_1$, and $Q \notin \{P_1, \dots, P_n\}$. Let $B_1$ and $B_2$ be the closure in $X$ of the 2 connected components of $X \backslash \{P, Q\}$ with, say, $P_1 \in B_1$. Then $P_i \in B_2$ for all $i \geq 2$. We have $h_{\omega_Y}(P) = h_{\omega_Y}(Q)$, $h^0(X, \omega_Y) = h^0(B_1, \omega_Y|B_1) + h^0(B_2, \omega_Y|B_2) - 1$, $\langle h_{\omega_Y}(B_1)\rangle \cap \langle h_{\omega_Y}(B_2)\rangle = \{h_{\omega_Y}(P)\}$, $h_{\omega_Y}|B_1$ is induced by the complete linear system $|\omega_{B_1}(P_1 + Q)|$, while $h_{\omega_Y}|B_2$ is induced by the complete linear system $|\omega_{B_2}(P_1 + \cdots + P_n)|$. Assume that $B_2$ is 3-connected. The morphism $h_{\omega_Y}|B_2$ is either an embedding or a degree 2 finite morphism onto a rational normal curve, the latter being the case if and only if $n = 2$, $B_2$ has a degree 2 finite morphism $\beta : B_2 \to \mathbb{P}^1$ and $\{P_1, P_2\}$ is a fiber of $\beta$.*

If in the statement of Theorem 3 $h_{\omega_Y}(B_1)$ has an ordinary node at $h_{\omega_Y}(Q)$ (see Lemmas 1 and 3 for conditions assuring it) and $h_{\omega_Y}|B_2$ is an embedding (see Lemmas 2, 3 or 4 for conditions assuring it) $h_{\omega_Y}(Q)$ is a seminormal point of $Y_\omega$ with multiplicity 3 and hence embedding dimension 3.

**Theorem 4.** *Assume $g \geq 3$, $X$ 3-connected, but $Y$ not 3-connected. Then $n = 2$ and $\mathbb{B}(Y) = \{P_1, P_2\}$. Either $h_{\omega_Y}$ is unramified and $\{P, Q\}$ are the only distinct points of $X$ in one of its fibers or it is a degree 2 finite morphism with as image a rational normal curve. The latter case occurs if and only if there is a degree 2 morphism $\beta : B_1 \to \mathbb{P}^1$ and $\{P, Q\}$ is a fiber of $\beta$.*

## 2. The Proofs

Fix a nodal curve $X$, an integer $s \geq 2$, and $A_i \in X_{reg}$, $1 \leq i \leq s$. Set $A := \{A_1, \dots, A_s\}$. For any zero-dimensional scheme $W \subset X$ let $\mathrm{Res}_A(W)$ denote the residual of $W$ with respect to the effective Cartier divisor $A$ of $X$, i.e. the closed subscheme of $X$ with $\mathcal{I}_W : \mathcal{I}_A$ as its ideal sheaf. Since $A$ is a Cartier divisor, we have $\mathrm{length}(W) = \mathrm{length}(W \cap A) + \mathrm{length}(\mathrm{Res}_A(W))$. We will often use without further mention that the 2-connectedness of $Y$ is equivalent to the spannedness of $\omega_Y$ ([1], part I of Theorem 1.2). Any of these two conditions immediately gives $n \neq 1$, i.e. either $n = 0$ or $n \geq 2$.

The proofs of the following two lemmas are elementary and omitted (the assumption "$\omega_X$ very ample " implies that $X$ 3-connected; see step (i) of the proof of Lemma 3 for the use of the 3-connectedness).

**Lemma 1.** *Let $X$ be a stable curve such that $\omega_X$ is very ample. Fix $P, Q \in X_{reg}$ such that $P \neq Q$. Set $L := \omega_X(P+Q)$. Then $L$ is spanned, the morphism $h_L : X \to \mathbb{P}^g$, $g := p_a(X)$, induced by $|L|$ is unramified and $P, Q$ are the only different points of $X$ in the same fiber of $h_L$.*

**Lemma 2.** *Let $X$ be a stable curve such that $\omega_X$ is very ample. Fix an integer $m \geq 3$ and $P_i \in X_{reg}$, $1 \leq i \leq m$. Then $\omega_X(P_1 + \cdots + P_m)$ is very ample.*

**Lemma 3.** *Let $X$ be a 3-connected stable curve such that $\omega_X$ is spanned and the canonical map $h_{\omega_X} : X \to \mathbb{P}^{g-1}$, $g := p_a(X)$, has degree 2 onto its image, which is a rational normal curve. Fix $P, Q \in X_{reg}$ such that $P \neq Q$ and set $L := \omega_X(P+Q)$. Then $L$ is spanned and the morphism $h_L : X \to \mathbb{P}^g$ induced by $|L|$ is described in the following way:*

*(a) If $h_{\omega_X}(P) \neq h_{\omega_X}(Q)$, then $h_L$ is unramified and $P, Q$ are the only different points of $X$ in the same fiber of $h_L$.*

*(b) Assume $h_{\omega_X}(P) = h_{\omega_X}(Q)$. Then $h_L$ has degree 2 onto its image, which is a rational normal curve. The fibers of $h_L$ are the same as the fibers of $h_{\omega_X}$.*

*Proof.* The value of $h^0(X, L)$ and the fact that $h_L(P) = h_L(Q)$ is obvious, by Riemann-Roch and the 3-connectedness of $X$. Fix $O, A \in X$, such that $O \neq A$ and $\{O, A\} \cap \{P, Q\} = \emptyset$. Since $|\omega_X| + P + Q \subset |L|$, if $h_{\omega_X}(O) \neq h_{\omega_X}(A)$, then $h_L(O) \neq h_L(A)$ and if $h_{\omega_X}$ is unramified at $O$, then $h_L$ is unramified at $A$.

(i) Assume $h_{\omega_X}(P) \neq h_{\omega_X}(Q)$. By Rieman-Roch to check part (a) it is sufficient to prove that $h^1(X, \mathcal{I}_Z \otimes \omega_X(P+Q)) = 0$ for every zero-dimensional scheme $Z \subset X$ such that length$(Z) = 2$ and $Z \neq \{P, Q\}$. Fix any such $Z$. By Serre duality it is sufficient to prove $h^0(X, \mathcal{I}_Z(P+Q)) = 0$. Assume that this in not true and take any nonzero $\sigma \in H^0(X, \mathcal{I}_Z(P+Q))$. Since $p_a(X) > 0$, $\sigma$ vanishes identically on a proper subcurve of $X$. Call $D$ the maximal such subcurve and set $B := \overline{X \backslash D}$. Since $X$ is 3-connected, $\sharp(D \cap B) \geq 3$. Thus $\sigma|B$ is a section of the line bundle $\mathcal{O}_X(P+Q)|B$ with at least 3 zeroes, but only finitely many zeroes, contradiction.

(ii) Assume $h_{\omega_X}(P) = h_{\omega_X}(Q)$. Notice that $L \cong f^*(L_g)$, where $L_g$ is the degree $g$ line bundle on the rational normal curve of $\mathbb{P}^{g-1}$. Then use that $h^0(X, L) = g$ and the universal property of a projective space. □

**Lemma 4.** *Let $X$ be a 3-connected stable curve. Fix an integer $n \geq 3$ and*

*$P_i \in X_{reg}$, $1 \le i \le n$, such that $P_i \neq P_j$ for all $i \neq j$. Then $\omega_X(P_1 + \cdots + P_n)$ is very ample.*

*Proof.* Since $X$ is 3-connected, $\omega_X$ is spanned and the canonical morphism $h_{\omega_X}$ is either an embedding or a degree 2 finite morphism onto a smooth rational curve ([1], Theorem 1.2, or [3], Theorem 3.6). In the first case use Lemma 2. In the second case use Lemma 3 and that at most 2 among the points $P_1, \dots, P_n$ are in the same fiber of $h_{\omega_X}$. □

**Lemma 5.** *Let $X$ be a 3-connected stable curve such that its canonical morphism $h : X \to \mathbb{P}^{g-1}$, $g := p_a(X) = g$, is an embedding. Fix $P, Q \in X_{reg}$ such that $P \neq Q$. Set $D := \langle\{h(P), h(Q)\}\rangle \subset \mathbb{P}^{g-1}$. For any $O \in D \backslash D \cap h(X)$ let $h_O : X \to \mathbb{P}^{g-2}$ be the composition of $h$ with the linear projection from $O$.*

*(a) Assume the non-existence of a zero-dimensional scheme $W$ such that $\{P, Q\} \subset W$, length$(W) = 4$ and $h^0(X, \mathcal{O}_X(W)) \ge 2$. Then $D \cap h(X) = \{h(P), h(Q)\}$. For every $O \in D\backslash\{P, Q\}$ the morphism $h_O$ is unramified and $P, Q$ is the only pair of distinct points in a fiber of $h_O$.*

*(b) Assume $h^0(X, \mathcal{O}_X(P + Q)) = 1$. There is a zero-dimensional scheme $E \subset X$ such that $\{P, Q\} \subset E$, length$(E) = 3$ and $h^0(X, \mathcal{O}_X(E)) \ge 2$ if and only if $h^{-1}(D) \neq \{P, Q\}$ (scheme-theoretically). In this case for each $O \in D \backslash D \cap h(X)$ the scheme $E$ is contained in the fiber $h_O^{-1}(h_O(P))$.*

*(c) Assume the existence of a zero-dimensional scheme $W \subset X$ such that $\{P, Q\} \subset W$, length$(W) = 4$ and $h^0(X, \mathcal{O}_X(W)) \ge 2$, but the non-existence of a zero-dimensional scheme $E \subset X$ such that $\{P, Q\} \subset E$, length$(E) = 3$ and $h^0(X, \mathcal{O}_X(E)) \ge 2$. Then $h^{-1}(D) = \{P, Q\}$ (scheme-theoretically) and there is $O \in D\backslash\{h(P), h(Q)\}$ such that the morphism $h_O$ is either ramified or contains a pair distinct points in a fiber of $h_O$ different from the pair $(P, Q)$.*

*Proof.* Since $P, Q$ are smooth points of $X$, taking the residual with respect to $\{P, Q\}$ induces a bijection between the set of all length 4 (resp. 3) zero-dimensional schemes containing $\{P, Q\}$ and the length 2 zero-dimensional schemes (resp. the points of $X$). Fix any length 2 zero-dimensional scheme $Z$ and any $O \in D\backslash\{h(P), h(Q)\}$. Fix any length 2 zero-dimensional scheme $Z$ and any $O \in D\backslash\{h(P), h(Q)\}$. Let $W$ be the only length 4 zero-dimensional scheme $W \subset X$ with $Z$ as its residual with respect to $\{P, Q\}$. The linear projection from $O$ does not send isomorphically $Z$ into $\mathbb{P}^{g-1}$ if and only if $O \in \langle h(Z)\rangle$. We have $f(P) \in \langle Z\rangle$ with $P \notin Z_{reg}$ if and only if $h^0(X, \mathcal{O}_X(Z + P)) \ge 2$; here $\mathcal{O}_X(Z + P)$ is a coherent sheaf on $X$ with depth 1, pure rank 1 and degree 3. A similar statement holds for $Q$. In this way we get part (b) and that $h^{-1}(D) = \{P, Q\}$ (scheme-theoretically) in part (c). We immediately get parts (a) and (c). □

**Lemma 6.** *Assume $n \geq 5$ and that $X$ is 3-connected. Then $h_{\omega_Y}$ is an embedding and for each point $O \in \langle h_{\omega_Y}(P_1), h_{\omega_Y}(P_2)\rangle \backslash \{h_{\omega_Y}(P_1), h_{\omega_Y}(P_2)\}$ the composition of $h_{\omega_Y}$ with the linear projection from $O$ is unramified and $P_1, P_2$ are the only distinct points of $X$ contained in the same fiber of $\psi$.*

*Proof.* Lemma 4 gives that $h_{\omega_Y}$ is an embedding. As in the proof of Lemma 5 it is suffient to show the non-existence of a length 4 zero-dimensional scheme $W \subset X$ such that $h^1(X, \mathcal{I}_W \otimes \omega_X(P_1 + \cdots + P_n)) > 0$, i.e. (talking the residual with respect to the effective Cartier divisor $\{P_1, P_2\}$) the non-existence of a length 2 zero-dimensional scheme $Z \subset X$ such that $h^1(X, \mathcal{I}_Z \otimes \omega_X(P_3 + \cdots + P_n)) > 0$. Apply Lemma 4 to the pointed curve $(X; P_3, \dots, P_n)$. $\square$

**Example 1.** Assume $n = 0$ and $\sharp(\mathbb{B}(Y)) = 1$, say $\mathbb{B}(Y) = \{P, Q\}$. Let $B_1$ and $B_2$ be the closure in $X$ of the 2 connected components of $X \backslash \{P, Q\}$. Assume $\omega_{B_2}$ very ample and the existence of a length 3 scheme $E \subset B_2$ such that $\{P, Q\} \subset E$ and $h^0(B_2, \mathcal{O}_{B_2}(ZE)) \geq 2$ (in a certain sense $B_2$ is trigonal and $P$, $Q$ are in the same effective divisor of a trigonal pencil). Both condition may be satisfied taking as $B_2$ a smooth and connected curve of arbitrary genus. Then $h_{\omega_Y}$ contract the scheme $Z$ to the point $h_{\omega_Y}(P)$. Obviously, also the curve $h_{\omega_Y}(B_1)$ contains $h_{\omega_Y}(P)$. Similarly, if $p_a(B_2) \geq 5$, it is easy to give examples with $B_2$ without $E$ and satisfing or not satisfying (c) of Lemma 5

*Proof of Theorem 1.* Recall that $n \neq 1$ and that $\omega_Y$ is spanned.

(a) The definition of $\mathbb{B}(Y)$ gives that $X \backslash \{P, Q\}$ has two connected components and that one of them contains all marked points $P_1, \dots, P_n$. We have $\omega_Y | B_1 \cong \omega_{B_1}(P + Q)$ and $\omega_Y | B_2 \cong \omega_{B_2}(P + Q + P_1 + \cdots + P_n)$. Both $\omega_Y | B_1$ and $\omega_Y | B_2$ are spanned. Since the morphism induced by $|\omega_Y | B_1|$ sends $P$ and $Q$ into the same point, the same is done by the canonical map $h_{\omega_Y}$. Look at the Mayer-Vietoris exact sequence

$$0 \to \omega_Y \to \omega_{B_1}(P + Q) \oplus \omega_{B_2}(P + Q + P_1 + \cdots + P_n) \to \omega_Y | \{P, Q\} \to 0. \quad (1)$$

Since $n > 0$, $h^1(B_2, \omega_{B_2}(P_1 + \cdots + P_n)) = 0$. Hence the restriction map $H^0(B_2, \omega_{B_2}(P + Q + P_1 + \cdots + P_n)) \to H^0(\{P, Q\}, \omega_Y | \{P, Q\})$ is surjective. Hence the restriction map $\rho_1 : H^0(X, \omega_Y) \to H^0(B_1, \omega_Y | B_1))$ is surjective. Hence the linear system $|\omega_{B_1}(P + Q)|$ describes $h_{\omega_Y} | B_1$. Since $n \geq 2$ and $B_2$ is 3-connected, either $\omega_{B_2}(P + Q + P_1 + \cdots + P_n)$ is very ample or $n = 2$, $h_{\omega_{B_2}}$ is a degree 2 morphism onto a rational normal curve and $P, Q$ are in the same fiber of $B_2$ is $h_{\omega_{B_2}}$ (Lemmas 2, 3 and 4). In this case $h_{\omega_Y} | B_2$ is induced by the subsystem of $|\omega_{B_2}(P + Q + P_1 + \cdots + P_n)|$ which identifies $P$ and $Q$. $\Gamma$ is a hyperplane of $|\omega_{B_2}(P + Q + P_1 + \cdots + P_n)|$ and it corresponds to a linear projection from a point (neither $h_{\omega_Y}(P)$ nor $h_{\omega_Y}(Q)$) from the line of $\mathbb{P}^x$, $x = g+n-2$, spanned by $h_{\omega_Y}(P)$ and $h_{\omega_Y}(Q)$. Let $f : B_2 \to \mathbb{P}^y$, $y := p_a(B_2)+n$, be the morphism induced by $|\omega_{B_2}(P+Q+P_1+\cdots+P_n)|$ and $f' := h_{\omega_Y}|B_2 : B_2 \to \mathbb{P}^{y-1}$ the morphism induced by $\Gamma$. Let $D \subset \mathbb{P}^y$ be the line spanned by $f(P)$ and $f(Q)$.

(b) Consider the following set $\diamond$ of conditions:

$\diamond$: " $f_1|B_2\backslash\{P,Q\}$ is injective, $f^{-1}(f(\{P,Q\}) = \{P,Q\}$ and $f_1$ is unramified at $P$ and $Q$."

Since $f_2$ is obtained from the embedding $f$ taking a linear projection from a point of $D\backslash\{f(P),f(Q)\}$, all these condition are true if and only if $\{P,Q\}$ is the scheme-theoretic counterimage $f_2^{-1}(f_2(P))$. Hence one of these conditions fails if and only there is a length 3 zero-dimensional scheme $A \subset B_2$ such that $\{P,Q\} \subset A$ and $h^1(B_2, \mathcal{I}_A \otimes \omega_{B_2}(P+Q+P_1+\cdots+P_n)) > 0$. Since $P$ and $Q$ are Cartier divisors of $B_2$ there is $A$ as above if and only if $h^1(B_2, \mathcal{I}_O \otimes \omega_{B_2}(P_1+\cdots+P_n)) > 0$ for some $O \in B_2$. Assume the existence of such a point $O$. Duality gives $h^0(X, Hom(\mathcal{I}_{\{P_1,\cdots,P_n\}}, \mathcal{I}_O))) > 0$. Fix $b \in H^0(X, Hom(\mathcal{I}_{\{P_1,\cdots,P_n\}}, \mathcal{I}_O)))$ such that $b \neq 0$. Since $n \geq 2$, there is a maximal proper subcurve $C$ of $B_2$, $C \neq \emptyset$, such that $b|C \equiv 0$. Set $C' := \overline{B_2\backslash C}$. Obviously, $b|C'$ vanishes at at least $n \geq 2$ points, but it has only finitely many zeroes, contradiction.

(c) Here we assume the that $\diamond$ is true and that there is of a plane $N$ such that $D \subset N \subset \mathbb{P}^y$ and length$(f^{-1}(N) \cap B_2 \geq 4$, i.e. we assume the existence of a length 4 zero-dimensional subscheme $W$ of $B_2$ containing $\{P,Q\}$ and such that $h^1(B_2, \mathcal{I}_W \otimes \omega_{B_2}(P+Q+P_1+\cdots+P_n)) > 0$, i.e. (remember that $P$ and $Q$ are smooth points of $B_2$) the existence of a zero-dimensional scheme $Z \subset B_2$ such that length$(Z) = 2$ and $h^0(X, Hom(\mathcal{I}_{\{P_1,\cdots,P_n\}}, \mathcal{I}_Z))) > 0$. Take $b \in H^0(X, Hom(\mathcal{I}_{\{P_1,\cdots,P_n\}}, \mathcal{I}_Z)))$ such that $b \neq 0$. Now (as in part (vi) of Theorem 1) we assume $n \geq 3$. Since $n >$ length$(Z)$, there is a maximal proper subcurve $C$ of $B_2$, $C \neq \emptyset$, such that $b|C \equiv 0$. Set $C' := \overline{B_2\backslash C}$. Obviously, $b|C'$ vanisheas at at least $n \geq 3$ points, but it has only finitely many zeroes. Since $B_2$ is 3-connected, $\sharp(C' \cap C) \geq 3 >$ length$(Z)$, contradiction. Hence if $n \geq 3$, then no plane $N$ as above exists.

(c) Here we assume the non-existence of a plane $N$ such that $D \subset N \subset \mathbb{P}^y$ and length$(f^{-1}(N) \cap B_2)) \geq 4$. The non-existence of $N$ gives that $h_{\omega_Y}|B_2$ is unramified and that $\{P,Q\}$ are the only distinct points in a fiber of it. Hence in this case $h_{\omega_Y}|B_2$ is as described in the statement of Theorem 1.

Now assume $P_i \notin \langle Z \rangle$, $i = 1,2$. Riemann-Roch gives that $\langle Z \cup \{P_1,P_2\}\rangle$ is a plane. Hence $\langle D \rangle \neq \emptyset$. Since we assumed $P_i \notin \langle Z \rangle$ for all $i$, $D \cap \langle Z \rangle$ is a unique point, $O$, and $O \notin \{P,Q\}$. $f'|Z$ is not an isomorphism if and only if $f'$ is the composition of $f$ with the linear projection from $O$. $h_{\omega_Y}(P) = h_{\omega_Y}(Q)$, $\langle h_{\omega_Y}(B_1)\rangle \cap \langle h_{\omega_Y}(B_2) = \{h_{\omega_Y}(P)\}$ and $h_{\omega_Y}|B_1$ is induced by the complete linear system $|\omega_{B_1}(P+Q)|$. Hence (i) and (ii) are true, (iii) follows from (a), while (iv) follows from Lemmas 1 and 3. We checked (v) in (b). Set $Z = (B_2; P, Q, P_1, \dots, P_n)$. Applying Lemma 6 to $Z$ we get (vi). □

*Proof of Theorem 2.* We have $\omega_Y|B_1 \cong \omega_{B_1}(P+Q)$ and $\omega_Y|B_2 \cong \omega_{B_2}(P+Q)$. Since $\omega_Y$ is spanned, the line bundles $\omega_Y|B_1$ and $\omega_Y|B_2$ are spanned. Set $f_i := h_{\omega_Y}|B_i$. Since the morphism induced by $|\omega_Y|B_1|$ sends $P$ and $Q$ into the same point, the same is done by the canonical map $h_{\omega_Y}$. Look at the Mayer-Vietoris exact sequence (1) with $n = 0$. From it we get $H^0(X, \mathcal{I}_{\{P,Q\}} \otimes \omega_X) = H^0(B_1, \omega_{B_1}) \oplus H^0(B_2, \omega_{B_2})$ and that $h^0(X, \omega_X) = h^0(B_i, \omega|B_i) + h^0(B_2, \omega_X|B_2) - 1$ (for the $\leq$ inequality in the last assertion use the first equality, for the $\geq$ inequality in the last assertion use that $\omega_Y$ is spanned). Since $h_{\omega_Y}(P) = h_{\omega_Y}(Q)$, the restriction map $H^0(X, \omega_X) \to H^0(B_i, \omega_X|B_i)$ is surjective, and $\langle f_1(B_1)\rangle \cap \langle f_2(B_2)\rangle = \{h_{\omega_X}(P)\}$. We just saw that $\dim(\langle f_i(B_i)\rangle) = p_a(B_i) - 2$ and that $\{h_{\omega_Y}(P)\} = \langle f_1(B_1)\rangle \cap \langle f_1(B_1)\rangle$. Hence to describe $Y_\omega$ it is sufficient to describe $f_1(B_1)$ and $f_2(B_2)$ and then glue these curves at $h_{\omega_Y}(P)$. For instance, $Y_\omega$ has a seminornal point with multiplicity 4 at $h_{omega_Y}(P)$ if and only if each $f_i(B_i)$ has an ordinary node at $h_{\omega_Y}(P)$. In the set-up of Theorem 2 the non-exisyence of the scheme $W_i$ implies the very ampleness of $\omega_{B_i}$. Apply Lemma 5 to $B_1$ and to $B_2$ and conclude. □

*Proof of Theorem 3.* The definition of $\mathbb{B}(Y)$ gives that $Q$ is a disconnecting node of $X$. We have $P_i \in B_2$ for all $i \geq 2$ by the definition of $\mathbb{B}(Y)$. We have $\omega_Y|B_1 \cong \omega_{B_1}(Q+P_1)$ and $\omega_Y|B_2 \cong \omega_{B_2}(Q+P_2+\cdots+P_n)$. Since $\omega_Y$ is locally free, we have the Mayer-Vietoris exact sequence on $X$:

$$0 \to \omega_Y \to \omega_{B_1}(P+Q) \oplus \omega_{B_2}(Q+P_1+\cdots+P_n) \to \omega_Y|\{Q\} \to 0. \quad (2)$$

Since both $\omega_{B_1}(P+Q)$ and $\omega_{B_2}(Q+P_1+\cdots+P_n)$ are spanned at $Q$, (2) gives the surjectivity of the restriction maps $\rho_i : H^0(X, \omega_Y) \to H^0(B_i, \omega_Y|B_i))$, $i = 1, 2$. Hence $Y_\omega$ and the morphism induced by $|\omega_Y|$ is described in the following way. Fix $O \in \mathbb{P}^x$, $x = g+n-2$, and two linear subspaces $M_i$, $i = 1, 2$, of $\mathbb{P}^x$ such that $\dim(M_1) = p_a(B_1)$, $\dim(B_2) = p_a(B_2) + n - 2$ and $M_1 \cap M_2 = \{O\}$. Since $p_a(X) = p_a(B_1) + p_a(B_2)$, $M_1 \cup M_2$ spans $\mathbb{P}^x$. Notice that $\dim(M_i) = h^0(B_i, \omega_Y|B_i)$, $i = 1, 2$. Up to a projective transformation we may assume that the morphisms $h_i : B_i \to M_i$ induced by the complete linear system $|\omega_Y|B_i|$ satifies $h_i(Q) = O$. The surjectivity of $\rho_i$ is equivalent to $h_{\omega_Y}|B_i = h_i$ (up to a projective transformation). With this convention the canonical morphism $h_{\omega_Y}$ is uniquely determined by the rule $h_{\omega_Y}|B_i = h_i$, $i = 1, 2$. For the structure of the morphism $h_1$, see Lemmas 1 and 3. For the structure of $h_2$ see Lemmas 1 and 3 (case $n = 2$) and Lemma 4 (case $n \geq 3$). □

*Proof of Theorem 4.* Since $X$ is connected, the definition of $\mathbb{B}(Y)$ gives $n = 2$

and $\mathbb{B}(Y) = \{P_1, P_2\}$. Since $X$ is 3-connected, we may quote [1], Theorem 1.2, or [3], Theorem 3.6, and then apply Lemmas 1 and 2. □

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